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A NOTE ON JACOBSON RINGS AND GENERALIZATIONS

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Abstract

In this paper we extend results about Jacobson rings originally proved by J. Watters in [8] for polynomial rings to some graded subrings of polynomial rings.

Introduction

Throughout this paper rings are associative with an identity element. In this paper we consider graded subrings of a polynomial ring $R[x]$ of the following type:

$$T = S_0 + S_1x + ... + S_{n-1}x^{n-1} + R[x]x^n,$$

where S_0 is a subring and $S_1, ..., S_{n-1}$ are additive subgroups of R. We call them admissible subring of $R[x]$. In the rest of the paper we denote by T an admissible subring of $R[x]$. In [3] we described prime, maximal and primitive ideals of T.

A ring R is said to be a Jacobson ring if every prime ideal of R is an intersection of (either left or right) primitive ideals of R. In [8], Watters proved that if R is a Jacobson ring, the polynomial ring $R[x]$ is also a Jacobson ring. A similar result was also proved for Hilbert rings [7], where a ring is said to be a Hilbert ring if every strongly prime ideal is an intersection of maximal ideals.

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A more general approach was followed in [4]. Denote by $\mathcal{F}$ a class of prime rings. An ideal I of R is said to be an $\mathcal{F}$-ideal if $R/I \in \mathcal{F}$. A ring R is called an $\mathcal{F}$-Jacobson ring if every prime ideal of R is an intersection of $\mathcal{F}$-ideals. A class of prime rings $\mathcal{F}$ is said to satisfy condition (A) if the following holds:

(A) If $R \in \mathcal{F}$, then every non-zero R-disjoint prime ideal of $R[x]$ is an $\mathcal{F}$-ideal.

It was shown in [4] that if $\mathcal{F}$ is a class of prime rings which satisfies condition (A) and R is an $\mathcal{F}$-Jacobson ring, then $R[x]$ is also an $\mathcal{F}$-Jacobson ring. This result includes as a particular case the result in [8].

In the present paper we consider classes $\mathcal{G}$ and $\mathcal{F}$ of prime rings such that $\mathcal{F}$ is contained in $\mathcal{G}$. We prove that, under some conditions, every $\mathcal{G}$-ideal of R is an intersection of $\mathcal{F}$-ideals if and only if every $\mathcal{G}$-ideal of $R[x]$ is an intersection of $\mathcal{F}$-ideals of $R[x]$. This result includes, as particular cases, all the above results. Moreover, we extend the result to admissible subrings of polynomial rings. Finally we give several examples to which our result can be applied.

Results

We begin with the following

Definition 1. *Let $\mathcal{G}$ a class of prime rings and $\mathcal{F}$ a subclass of $\mathcal{G}$. A ring R is said to be $(\mathcal{G}, \mathcal{F})$-Jacobson if every $\mathcal{G}$-ideal of R is a intersection of $\mathcal{F}$-ideals.*

In the sequel $\mathcal{G}$ denotes a class of prime rings and $\mathcal{F}$ a subclass of $\mathcal{G}$.

Lemma 2. *Homomorphic images of $(\mathcal{G}, \mathcal{F})$-Jacobson rings are also $(\mathcal{G}, \mathcal{F})$-Jacobson rings.*

Proof. Assume that R is a $(\mathcal{G}, \mathcal{F})$-Jacobson ring and I is an ideal of R. Let $\overline{J}$ be a $\mathcal{G}$-ideal of $\overline{R} = R/I$. Then there exists a $\mathcal{G}$-ideal J of R containing I such that $\overline{J} = J/I$. By assumption there exists a family $(J_i)_{i\in\Omega}$ of $\mathcal{F}$-ideals of R with $J = \cap_{i\in\Omega} J_i$. Thus, $\overline{J} = \cap_{i\in\Omega}(J_i/I)$ is an intersection of $\mathcal{F}$-ideals of R/I. $\square$ The following is an immediate consequence of Lemma 2.

Corollary 3. *If $R[x]$ is a $(\mathcal{G}, \mathcal{F})$-Jacobson ring, then R is a $(\mathcal{G}, \mathcal{F})$-Jacobson ring.*

A class $\mathcal{G}$ of prime rings is said to satisfy condition (B) if the following holds:

(B) If L is an ideal of $R[x]$ and $R[x]/L \in \mathcal{G}$, then $R/(L \cap R) \in \mathcal{G}$.

In the following when we write strongly prime we mean either right of left strongly prime. Similar remark holds for non-singular rings, artinian rings and so on.

Proposition 4. *The following classes of primes rings satisfy condition* (B)*:*
(i) *Strongly prime rings.*
(ii) *Prime non-singular rings.*
(iii) *Unitary strongly prime rings.*
(iv) *Unitary strongly prime rings with non-zero pseudo-radical.*

Proof. (i) and (ii) follow easily from Corollary 4.3 and Corollary 4.6 of [2]. Also (iii) and (iv) follow from Lemma 2.6, (i), and Lemma 4.2 of [5]. □

Proposition 5. *The following classes of prime rings satisfy condition* (A)*:*
(i) Unitary strongly prime rings.
(ii) Unitary strongly prime rings with non-zero (resp. zero) pseudo-radical.

Proof. It is a direct consequence of Lemma 2.6, (ii), and Lemma 4.2 of [5]. □

The next result extends ([4], Theorem 5) and ([7], Theorem 3.1). The proof is similar to the one of Theorem 5 of [4].

Theorem 6. *Let* $\mathcal{G}$ *be a class of prime rings which satisfies condition* (B) *and* $\mathcal{F}$ *a subclass of* $\mathcal{G}$ *which satisfies condition* (A)*. Then* R *is a* $(\mathcal{G}, \mathcal{F})$*-Jacobson ring if and only if* $R[x]$ *is a* $(\mathcal{G}, \mathcal{F})$*-Jacobson ring.*

Now we will extend the above result to an admissible subring T of a polynomial ring. First we begin with the following.

A class $\mathcal{F}$ of prime rings is said to satisfy conditions (C) and (D) with respect to T if the following holds:

(C) For every ideal L of $R[x]$ with $R[x]x \nsubseteq L$ such that $R[x]/L \in \mathcal{F}$ we have $T/(L \cap T) \in \mathcal{F}$.

(D) If P is an ideal of T with $R[x]x^n \nsubseteq P$ and $T/P \in \mathcal{F}$, then there exists an ideal L of $R[x]$ with $L \cap T = P$ such that $R[x]/L \in \mathcal{F}$.

Proposition 7. *For any admissible subring* T *of* $R[x]$ *the following classes of prime rings satisfy conditions* (C) *and* (D)*:*
(i) *Simple rings (which do not necessarily have an identity element).*

(ii) *Simple artinian rings.*

(iii) *Strongly prime rings.*

Proof. (i) and (ii) follow easily from Propositions 3 and 4 of [3].
(iii) Assume that L is a right strongly prime ideal of $R[x]$ such that $R[x]x \nsubseteq L$. Let $\overline{U}$ a non-zero ideal of $T/(L \cap T)$. Then there exists an ideal U of T containing $L \cap T$ such that $\overline{U} = U/(L \cap T)$. Thus, $\overline{V} = (L + R[x]x^n U R[x]x^n)/L$ is a non-zero ideal of $R[x]/L$.

By assumption there exists a finite set $\overline{H} = \{\overline{h}_1, ..., \overline{h}_n\} \subseteq \overline{V}$ such that the right annihilator of $\overline{H}$ in $R[x]/L$ is zero. We have $\overline{h}_i = h_i + L$, where $h_i \in R[x]x^n U R[x]x^n$, for any $i \in \{1, ..., n\}$.

Put $\overline{J} = \{h_1 + (L \cap T), ..., h_n + (L \cap T)\} \subseteq U/(L \cap T)$. Then the right annihilator of $\overline{J}$ in $T/(L \cap T)$ is zero. Thus $T/(L \cap T)$ is a right strongly prime ring.

Using similar arguments as above we can show that the class of right strongly prime rings satisfies condition (D). □

Recall that a class $\mathcal{H}$ of rings is said to be a special class if it satisfies the following three conditions ([1], Chapter 7):
(i) Every ring in the class $\mathcal{H}$ is a prime ring;
(ii) every non-zero ideal of a ring in $\mathcal{H}$ is itself a ring in $\mathcal{H}$;
(iii) if I is a ring in $\mathcal{H}$ and I is an ideal of a ring R, then R/I^* is in $\mathcal{H}$, where I^* is the annihilator of I in R.

The classes of all prime rings, simple rings with identity and right (left) primitive rings are special classes of rings ([1], Chapter 7).

Theorem 8. *If $\mathcal{F}$ is a special class of prime rings, then $\mathcal{F}$ satisfy conditions (C) and (D) with respect to any admissible subring of $R[x]$:*

Proof. It follows easily from Lemma 1 of [3]. □

The following proposition is well-known. In particular parts (iv) and (v) follow from Proposition 2.2 and Lemma 4.4 of [5].

Proposition 9. *The following classes of rings are special classes of prime rings. In particular they satisfy conditions (C) and (D) with respect to any admissible subring of $R[x]$.*
(i) *Prime nonsingular rings.*
(ii) *Prime rings with finite Goldie dimension.*
(iii) *Prime Goldie rings.*
(iv) *Unitary strongly prime rings.*
(v) *Unitary strongly prime rings with non-zero (resp. zero) pseudo-radical.*

The next result extends Theorem 6 to any admissible subring of a polynomial ring.

Theorem 10. *Let $\mathcal{G}$ be a class of prime rings which satisfies conditions (B) and (D) and let $\mathcal{F}$ be a subclass of $\mathcal{G}$ which satisfies conditions (A) and (C). Then S_0 and R are $(\mathcal{G}, \mathcal{F})$-Jacobson rings if and only if T is a $(\mathcal{G}, \mathcal{F})$-Jacobson ring.*

Proof. If T is a $(\mathcal{G}, \mathcal{F})$-Jacobson ring, then R is a $(\mathcal{G}, \mathcal{F})$-Jacobson ring by Lemma 2. In fact, the map $\Phi : T \to R$ given by $\Phi(a_0 + a_1x + ... + a_kx^k) = a_0 + a_1 + ... + a_k$ is an epimorphism of rings. Similarly for S_0.

Conversely, assume that S_0 and R are $(\mathcal{G}, \mathcal{F})$-Jacobson rings and let P be an ideal of T such that $T/P \in \mathcal{G}$. If $R[x]x^n \subseteq P$, then $P \cap S_0 = \cap_{i\in\Omega} J_i$, where J_i is an $\mathcal{F}$-ideal of S_0, for every $i \in \Omega$. We have

$$P = \cap_{i\in\Omega}(J_i + S_1x + ... + S_{n-1}x^{n-1} + R[x]x^n)$$

is an intersection of $\mathcal{F}$-ideals of T.

If $R[x]x^n \nsubseteq P$, condition (D) implies that there exists an ideal L of $R[x]$ with $L \cap T = P$ and $R[x]/L \in \mathcal{G}$. By Theorem 6 $R[x]$ is a $(\mathcal{G}, \mathcal{F})$-Jacobson ring. Then there exists a family $(L_i)_{i\in\Gamma}$ of $\mathcal{F}$-ideals of $R[x]$ such that $L = \cap_{i\in\Gamma} L_i$. It follows that there is a subfamily $(L_j)_{j\in\Omega} \subseteq (L_i)_{i\in\Gamma}$ such that $R[x]x \nsubseteq L_j$, for every $j \in \Omega$, and $L = \cap_{j\in\Omega} L_j$.

Therefore $L_j \cap T$ is an $\mathcal{F}$-ideal of T, for every $j \in \Omega$, since $\mathcal{F}$ satisfies condition (C). Moreover, $P = L \cap T = (\cap_{j\in\Omega} L_j) \cap T = \cap_{j\in\Omega}(L_j \cap T)$. The proof is complete. □

As an immediate consequence of the above theorem we have:

Corollary 11. *S_0 and R are Jacobson (Hilbert) rings if and only if T is a Jacobson (Hilbert) ring.*

Lemma 12. *(i) If R is a unitary strongly prime ring, then R is a right strongly prime ring.*
(ii) If R is a prime Goldie ring, then R is a non-singular ring.

Proof. (i) Let I a non-zero ideal of R. Proposition 2.4 of [5] implies that there exists $H = \{a_1, ..., a_n\} \subseteq I$ such that $a_1c_1 + ... + a_nc_n = 1$, where $c_1, ..., c_n \in C(R)$, the extended centroid of R. Let $a \in A_{nn_r}(H) = \{r \in R \mid Hr = 0\}$. Thus $a = (a_1c_1 + ... + a_nc_n)a = 0$ and therefore $A_{nn_r}(H) = 0$. Consequently R is a right strongly prime ring.
(ii) It is well-known (Corollary 3.32 of [6]). □

Using the results of [4] and the above results we have the following classes of prime rings $\mathcal{F}$ satisfying the conditions (A), (C) and $\mathcal{G}$ satisfying the conditions (B), (D). Moreover $\mathcal{F} \subseteq \mathcal{G}$. Hence Theorem 10 can be applied to all the examples given bellow.

$a)$ $\mathcal{G}$ is the class of prime rings and $\mathcal{F}$ is the class of rings satisfying one of the conditions given in Propositions 4, 5, 7 and 9.
$b)$ $\mathcal{G}$ is the class of strongly prime rings and $\mathcal{F}$ is the class of simple prime rings.
$c)$ $\mathcal{G}$ is the class of strongly prime rings and $\mathcal{F}$ is the class of unitary strongly prime rings (Lemma 12).
$d)$ $\mathcal{G}$ is the class of strongly prime rings and $\mathcal{F}$ is the class of simple artinian rings.
$e)$ $\mathcal{G}$ is the class of unitary strongly prime rings and $\mathcal{F}$ is the class of simple rings.
$f)$ $\mathcal{G}$ is the class of unitary strongly prime rings and $\mathcal{F}$ is the class of unitary strongly prime rings with non-zero (resp. zero) pseudo-radical.
$g)$ $\mathcal{G}$ is the class of unitary strongly prime rings and $\mathcal{F}$ is the class of simple artinian rings.
$h)$ $\mathcal{G}$ is the class of prime nonsingular rings $\mathcal{F}$ is the class of prime Goldie rings.

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