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SOME FURTHER OBSERVATIONS CONCERNING B-SPLINES AND THEIR HILBERT TRANSFORM

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Abstract

In this paper we review properties of B-splines which are shared by their Hilbert transform and then present some extensions of these results.

1 Introduction

Given any positive integer k and any set of *distinct points* $\{t_j : j \in \mathbb{Z}_{k+1}\}$ in some interval I, where $\mathbb{Z}_{k+1} := \{0, \ldots, k\}$, we denote the k-th order divided difference of a real-valued function f on the set of points $\{t_j : j \in \mathbb{Z}_{k+1}\}$ by $[t_0, \ldots, t_k]f$. The k-th divided difference of f is defined to be

$$[t_0, \ldots, t_k]f := \sum_{i \in \mathbb{Z}_{k+1}} \frac{f(t_i)}{w'(t_i)}, \tag{1.1}$$

where w is the polynomial defined at $t \in \mathbb{R}$ as $w(t) = \prod_{j \in \mathbb{Z}_{k+1}} (t - t_j)$. Corresponding to the divided difference there is the *k-th order of B-spline* defined

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at $t \in \mathbb{R}$ by

$$B_{0,k}(t) = (t_k - t_0)[t_0 - t, \dots, t_k - t]T_k, \tag{1.2}$$

where T_k is the *truncated power function* defined at $t \in \mathbb{R}$ by

$$T_k(t) := \begin{cases} t^{k-1}, & t > 0, \\ 0, & t \le 0. \end{cases}$$

Notice, with this definition, we have that

$$B_{0,1}(t) = \begin{cases} 1, & t_0 \le t < t_1, \\ 0, & \text{otherwise}. \end{cases} \tag{1.3}$$

There are *three* important facts about B-spline which are of interest to us in this paper. The first one is the *Peano kernel representation* of the B-spline which states, for any real-valued function $f \in C^k(\mathbb{R})$, that

$$\int_{\mathbb{R}} B_{0,k}(t) f^{(k)}(t) dt = (k-1)!(t_k - t_0)[t_0, \dots, t_k]f. \tag{1.4}$$

The second issue which concerns us pertains to the structure of the *linear span* of a *biinfinite collection* of B-splines. To describe what we have in mind, we begin with a biinfinite set of *distinct points (knots)* $\mathbb{T} := \{t_j : j \in \mathbb{Z}\}$ ordered so that $t_j < t_{j+1}$, for each $j \in \mathbb{Z}$, and satisfying the condition that $\lim_{j\to\pm\infty} t_j = \pm\infty$. Corresponding to $\mathbb{T}$ there is a set of *consecutive* B-splines, denoted by $\mathcal{B} = \{B_{j,k} : j \in \mathbb{Z}, k \in \mathbb{N}\}$, where the j-th B-spline is defined at $t \in \mathbb{R}$ as

$$B_{j,k}(t) = (t_{j+k} - t_j)[t_j - t, \dots, t_{j+k} - t]T_k. \tag{1.5}$$

We use $\mathcal{S}_k(\mathbb{T})$ to denote the *algebraic span* of all such B-splines, that is,

$$\mathcal{S}_k(\mathbb{T}) = \left\{ \sum_{j\in\mathbb{Z}} c_j B_{j,k} : c = (c_j : j \in \mathbb{Z}) \in \mathbb{R}^{\mathbb{Z}} \right\}. \tag{1.6}$$

Note that, because $B_{j,k}$ is *compactly supported*, every element in $\mathcal{S}_k(\mathbb{T})$ is well defined on $\mathbb{R}$ for any biinfinite vector $c = (c_j : j \in \mathbb{Z}) \in \mathbb{R}^{\mathbb{Z}}$. The Curry-Schoenberg theorem states, for $k \ge 2$, that a function $f \in \mathcal{S}_k(\mathbb{T})$ if and only if $f \in C^{k-2}(\mathbb{R})$ and between every two consecutive knots it is a polynomial of degree at most $k-1$.

The third fact is the *de Boor recurrence relation* for the consecutive B-spline basis which states for $t \in \mathbb{R}, j \in \mathbb{Z}, k \in \mathbb{N}$ that

$$B_{j,k+1}(t) = p_{j,k}(t)B_{j,k}(t) + q_{j+1,k}(t)B_{j+1,k}(t), \tag{1.7}$$

where the linear functions $p_{j,k}$ and $q_{j,k}$ are defined at $t \in \mathbb{R}$ by

$$p_{j,k}(t) := \frac{t - t_j}{t_{j+k} - t_j}, \qquad q_{j,k}(t) := \frac{t_{j+k} - t}{t_{j+k} - t_j}. \tag{1.8}$$

Now, we turn to the *Hilbert transform* of B-splines. It is indeed *remarkable* that the Hilbert transform of B-spline also satisfies equations analogous to (1.4), (1.5) and (1.7). To describe these facts, we first recall that the Hilbert transform is defined, for each function $f \in L^p(\mathbb{R}), 1 \leq p < \infty$, at $t \in \mathbb{R}$ as

$$(H_{\mathbb{R}}f)(t) := \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \frac{f(s)}{t-s}ds := \lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int_{|s-t|\geq\varepsilon}\frac{f(s)}{t-s}ds, \tag{1.9}$$

whenever the Cauchy principal value of the above singular integral exists. It is well known that if $f \in L^p(\mathbb{R}), 1 \leq p < \infty$ then $H_{\mathbb{R}}f$ exists almost everywhere on $\mathbb{R}$, see for example, [2]. Among other things, we obtained in [3] an *explicit formula* for the Hilbert transform of the B-spline $B_{j,k}$, which we denote by $H_{j,k}$. To explain our formula, we introduce the function L_k which is defined at $t \in \mathbb{R}$ by the equation $L_k(t) = -\frac{1}{\pi}t^{k-1}\ln|t|$. In this definition, we see that $L_1(0) = -\infty$ while for $k \geq 2$ we have that $L_k(0) = 0$. Moreover, it was proved in [3], for any $t \in \mathbb{R}$, that

$$H_{j,k}(t) = (t_{j+k} - t_j)[t_j - t, \dots, t_{j+k} - t]L_k, \tag{1.10}$$

When $k \geq 2$, both sides of this equation are finite for all $t \in \mathbb{R}$. However, in the case $k = 1$, this equation remains valid for all $t \in \mathbb{R}$ but both sides may be $\pm\infty$. Indeed, for all $t \in \mathbb{R}$ we have that

$$H_{j,1}(t) = \frac{1}{\pi}\ln\left|\frac{t_j - t}{t_{j+1} - t}\right|. \tag{1.11}$$

To present the analog of the Peano kernel representation for the HB-functions we recall the definition of the Sobolev space $L_r^2(\mathbb{R})$, where r is some positive integer. Specifically, we define

$$L_r^2(\mathbb{R}) := \{f : f^{(m)} \in L^2(\mathbb{R}), m \in \mathbb{Z}_r\} \tag{1.12}$$

and recall the following fact from [3]. If $f \in L_k^2(\mathbb{R})$, $j \in \mathbb{Z}, k \in \mathbb{N}$ then

$$\int_{\mathbb{R}} H_{j,k}(t)f^{(k)}(t)dt = (-1)^{k+1}(k-1)!(t_{j+k} - t_j)\text{p.v.}\int_{\mathbb{R}}\frac{f(t)}{\prod_{l\in\mathbb{Z}_{k+1}}(t - t_{j+l})}dt. \tag{1.13}$$

We view this equation as the Peano kernel representation for the HB-functions.

We remark in passing that the structure of the subspace generated by the set of functions $\{H_{j,k} : j \in \mathbb{Z}, k \in \mathbb{N}\}$ is *unclear to us.* Unlike the B-splines, these functions are *not* compactly supported. Therefore, we must resort to their *closed linear span* as a subset of $L^p(\mathbb{R}), 1 \leq p < \infty$. Currently, we cannot provide a result for the $L^p(\mathbb{R})$-closure of a linear span of the set of functions $\{H_{j,k} : j \in \mathbb{Z}, k \in \mathbb{N}\}$ which is analogous to the Curry-Schoenberg theorem for the space $\mathcal{S}_k(\mathbb{T})$. We leave this issue as an interesting *open problem.*

Finally, for the third issue, we refer to [1] where it was shown that the functions in the set $\{H_{j,k} : j \in \mathbb{Z}, k \in \mathbb{N}\}$ satisfy for $t \in \mathbb{R}$ the recurrence relation

$$H_{j,k+1}(t) = p_{j,k}(t)H_{j,k}(t) + q_{j+1,k}(t)H_{j+1,k}(t), \tag{1.14}$$

which is identical to the recurrence relation for the B-splines (1.7). However, it is important to realize that this recurrence relation has an initialization given in (1.11), which is *different* from the initialization for the B-spline recurrence relation given in equation (1.3). This subject was investigated in the *multivariate* case in [4].

Since the B-splines and their Hilbert transform satisfy the same recurrence relation with different initialization, we ask whether or not there are other sets of functions which satisfy the same recurrence relation? To this end, we begin with a function $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ which satisfies the condition that $\lim_{t\to 0} tg(t) = a$ for some $a \in \mathbb{R}$ and define the function $G_{j,1}$ for $t \in \mathbb{R}\setminus\{t_j, t_{j+1}\}$ by the equation $G_{j,1}(t) = g(t_{j+1} - t) - g(t_j - t)$. Now we generate a set of functions $\{G_{j,k} : j \in \mathbb{Z}, k \in \mathbb{N}\}$ by the same recurrence relation. That is, for $t \in \mathbb{R}, j \in \mathbb{Z}, k \in \mathbb{N}$, we define

$$G_{j,k+1}(t) = p_{j,k}(t)G_{j,k}(t) + q_{j+1,k}(t)G_{j+1,k}(t), \tag{1.15}$$

where $p_{j,k}, q_{j,k}$ are defined as in equation (1.8).

Although the function $G_{j,1}$ may be not well defined at the points $\{t_j, t_{j+1}\}$, the function $G_{j,k}$ *is* well defined on $\mathbb{R}$ for all $k \geq 2$. To see this, we only need to pay particular attention to the definition of $G_{j,2}$ at the points $\{t_j, t_{j+1}, t_{j+2}\}$. Actually, an easy computation shows that

$$\begin{aligned} G_{j,2}(t) = \frac{t_j - t}{t_{j+1} - t_j} g(t_j - t) + \left(\frac{t - t_j}{t_{j+1} - t_j} - \frac{t_{j+2} - t}{t_{j+2} - t_{j+1}} \right) g(t_{j+1} - t) \\ + \frac{t_{j+2} - t}{t_{j+2} - t_{j+1}} g(t_{j+2} - t). \end{aligned}$$

Since $\lim_{t\to 0} tg(t) = a$ for some $a \in \mathbb{R}$, the limit of $G_{j,2}$ always exist when t tends to t_j, t_{j+1} and t_{j+2} respectively. Therefore, as asserted, $G_{j,k}$ are well defined on $\mathbb{R}$ for all $j \in \mathbb{Z}, k \geq 2$. We now provide an explicit formula for the function in the set $\{G_{j,k} : j \in \mathbb{Z}, k \in \mathbb{N}\}$.

2 Main results

Our first task is to give a divided difference representation for the function $G_{j,k}$. For this purpose, we define functions g_1 to be g at $\mathbb{R} \setminus \{0\}$ and g_k to be $g_k(t) = t^{k-1}g(t)$ at $t \in \mathbb{R}$ for $k \geq 2$. For each $k \in \mathbb{N}$, the function g_k yields a biinfinite set of functions $\{D_{j,k} : j \in \mathbb{Z}, k \in \mathbb{N}\}$ defined as

$$D_{j,k}(t) = (t_{j+k} - t_j)[t_j - t, \ldots, t_{j+k} - t]g_k. \tag{2.1}$$

Notice, although the function $D_{j,1}$ may be not well defined at $\{t_j, t_{j+1}\}$, nonetheless, for $k \geq 2$ the function $D_{j,k}$ *is* well defined on $\mathbb{R}$ because $\lim_{t\to 0} tg(t) = a$ for some $a \in \mathbb{R}$. Moreover, it is interesting to observe that $D_{j,1}(t) = G_{j,1}(t)$ for all $t \in \mathbb{R} \setminus \{t_j, t_{j+1}\}$ and both sides of this equation are finite. Furthermore, when $t \in \{t_j, t_{j+1}\}$, this equation remains valid, with the caveat, that the function values can be $\pm\infty$. This fact leads us to the following result.

Theorem 2.1. *For $t \in \mathbb{R}, j \in \mathbb{Z}, k \in \mathbb{N}$, we have that*

$$G_{j,k}(t) = D_{j,k}(t). \tag{2.2}$$

Proof. The initialization of $G_{j,1} = D_{j,1}$ has been discussed in the first paragraph of this section. Now, we are going to prove the set of functions $\{D_{j,k} : j \in \mathbb{Z}, k \in \mathbb{N}\}$ satisfies the same recurrence relation as in definition of $G_{j,k}$. Since the elements in $\mathbb{T}$ are distinct, the definition of the divided difference ensures for any function h in $C(\mathbb{R})$ that

$$[t_j - t, \ldots, t_{j+k+1} - t]h = \frac{[t_{j+1} - t, \ldots, t_{j+k+1} - t]h - [t_j - t, \ldots, t_{j+k} - t]h}{t_{j+k+1} - t_j}. \tag{2.3}$$

We specialize the above formula by choosing the function h to be g_{k+1} to obtain that

$$D_{j,k+1}(t) = [t_{j+1} - t, \ldots, t_{j+k+1} - t]g_{k+1} - [t_j - t, \ldots, t_{j+k} - t]g_{k+1}. \tag{2.4}$$

Next, we appeal to the Leibniz formula for the divided difference of a product of two functions to observe that the first expression on the right hand side of equation (2.4) is given as

$$[t_{j+1} - t, \ldots, t_{j+k+1} - t]g_{k+1} = q_{j+1,k}(t)D_{j+1,k}(t) + [t_{j+1} - t, \ldots, t_{j+k} - t]g_k.$$

Similarly, for the second expression on the right hand side of equation (2.4) we have that

$$[t_j - t, \ldots, t_{j+k} - t]g_{k+1} = -p_{j,k}(t)D_{j,k}(t) + [t_{j+1} - t, \ldots, t_{j+k} - t]g_k.$$

Using these two formulas we rewrite the equation (2.4) to obtain the recurrence relation for the sequence of functions $\{D_{j,k} : j \in \mathbb{Z}\}$, which is the same as the sequence of functions $\{G_{j,k} : j \in \mathbb{Z}\}$ with the same initialization. These remarks complete the proof. $\square$

Remark 2.2. *The recurrence relation in Theorem 2.1 is initialized with the function $G_{j,1}$ defined at $t \in \mathbb{R}\setminus\{t_j, t_{j+1}\}$ by the equation $G_{j,1}(t) = g(t_{j+1} - t) - g(t_j - t)$. Therefore, if we choose g to be the function T_+ which is defined at $t \in \mathbb{R}$ as*

$$T_+ = \begin{cases} 1, & t > 0 \\ 0, & t \leq 0 \end{cases},$$

then we get the B-spline recurrence relation with initialization (1.3). On the other hand, if g is chosen to be the function L_1 then we get the same recurrence relation for HB-functions with initialization (1.11). In both cases we have that $\lim_{t\to 0} tg(t) = 0$. *Moreover, since $G_{j,k}$ satisfies the recurrence relation (1.15), it is easy to develop a recursive algorithm for the evaluation of the function $\sum_{j\in\mathbb{Z}} c_j G_{j,k}$ when the cardinality of the set $\{j : c_j \neq 0\}$ is finite.*

Next, we will develop an integral representation for $G_{j,k}$. To this end, we first focus on the functions which are smooth, except the origin. Namely, we assume $g \in C^k(\mathbb{R}\backslash\{0\})$. For the description of the integral representation of $G_{j,k}$ it is convenient to introduce the following linear operator.

Definition 2.3. *If $g \in C^k(\mathbb{R}\backslash\{0\})$ then we define a linear operator V_k : $C^k(\mathbb{R}\backslash\{0\}) \to C(\mathbb{R})$ on g at $t \in \mathbb{R}\backslash\{0\}$ by the equation*

$$(V_k g)(t) = \frac{1}{(k-1)!}[g_k(t)]^{(k)}. \tag{2.5}$$

Note that the null space of the operator V_k consists of the constant functions. This linear operator is useful to us because it leads us to the following integral representation of $G_{j,k}$.

Proposition 2.4. *If $g_k \in C^k(\mathbb{R}\backslash\{0\})$ then $G_{j,k}$ at $t \in \mathbb{R}\backslash\mathbb{T}$ can be expressed as*

$$G_{j,k}(t) = \int_{\mathbb{R}} B_{j,k}(x)(V_k g_k)(x-t)dx. \tag{2.6}$$

Proof. By Theorem 2.1, we only need to show for $t \in \mathbb{R}$ that

$$D_{j,k}(t) = \int_{\mathbb{R}} B_{j,k}(x)(V_k g_k)(x-t)dx.$$

This equation is equivalent to the formula

$$(k-1)!(t_{j+k} - t_j)[t_j - t, \ldots, t_{j+k} - t]g_k = \int_{\mathbb{R}} B_{j,k}(x) g_k^{(k)}(x-t)dx.$$

But by the Peano kernel representation for the B-spline, (1.4), the above equation is readily verified. □

Remark 2.5. *There are two cases of Proposition 2.4 which are of special interest. The first one concerns the B-spline itself. To explain what we have in mind, we first extend the range of the validity of equation (2.6). Specifically, for functions $g_k \in C^{k-l}(\mathbb{R} \setminus \{0\})$ with $l < k-1$ we have that*

$$G_{j,k}(t) = \frac{(-1)^l}{(k-1)!} \int_{\mathbb{R}} B_{j,k}^{(l)}(x) g_k^{(k-l)}(x-t)dx. \tag{2.7}$$

For example, when $l = 1$ and $g_k := T_k$ we see in this case that $G_{j,k}$ is given at $t \in \mathbb{R}$ as

$$G_{j,k}(t) = -\int_t^\infty B'_{j,k}(x)dx.$$

Hence, we have established that $G_{j,k} = B_{j,k}$. The second example is concerned with the HB-functions. In this case, we choose $g_k = L_k$ and conclude for any $k \in \mathbb{N}$ that $G_{j,k}$ is the Hilbert transform of B-splines, namely,

$$H_{j,k}(t) = -p.v.\frac{1}{\pi}\int_{\mathbb{R}} \frac{B_{j,k}(x)}{x-t}dx. \tag{2.8}$$

Remark 2.6. *If $g \in C^k(\mathbb{R}\setminus\{0\})$ then Leibniz formula provides, for $t \in \mathbb{R}\setminus\{0\}$, the equation*

$$(V_k g_k)(t) = \sum_{l\in\mathbb{Z}_k} \frac{k!}{l!(k-l)!(k-1-l)!} t^{k-l-1} g^{(k-l)}(t). \tag{2.9}$$

Corresponding to the function $V_k g_k$ is the linear operator $\mathcal{V}_k$ defined for all real-valued function f on $\mathbb{R}$ at $t \in \mathbb{R}$ as

$$(\mathcal{V}_k f)(t) = \int_{\mathbb{R}} f(x)(V_k g_k)(x-t)dx,$$

whenever the integral exists. Therefore, if our numerical task is to compute the function $\mathcal{V}_k f$ we may proceed by *approximating* f by a linear combination of B-splines $\sum_{j\in\mathbb{Z}} c_j B_{j,k}$ and observe that its image under $\mathcal{V}_k$ is $\sum_{j\in\mathbb{Z}} c_j G_{j,k}$. We can then efficiently compute this function by the recurrence relation (1.15). To make best use of this observation we *start with* a function h and seek a function g_k such that $V_k g_k = h$. In doing so, the linear operator $\mathcal{V}_k$ simplifies to the equation

$$(\mathcal{V}_k f)(t) = \int_{\mathbb{R}} f(x)h(x-t)dx.$$

So, to compute the function $\mathcal{V}_k f$, we choose g_k such that $V_k g_k = h$ and use our previous remarks. Given the function h, the corresponding choice of g is answered in the following theorem.

Theorem 2.7. *If $h \in L^1(\mathbb{R})$ then the function g_k defined at $t \in \mathbb{R}$ by*

$$g_k(t) = t\int_0^1 (1-s)^{k-1} h(ts)ds + c, \tag{2.10}$$

where c is a constant, satisfies equation $V_k g_k = h$.

Proof. The proof easily follows by induction on k. □

Remark 2.8. *Obviously, in general* h *depends on* k*. However, for the special choice that* $g_k = L_k$ *we get, for* $t \in \mathbb{R} \setminus \{0\}$*, that* $(V_k L_k)(t) = -\frac{1}{\pi} t^{-1}$*.*

Remark 2.9. *An interesting special choice for Theorem 2.7 is the function* h_α *defined for* $t \in \mathbb{R} \setminus \{0\}$ *as* $h_\alpha(t) = t^{-\alpha}$ *where* $0 < \alpha < 1$*. According to Theorem 2.7, the corresponding* g_k *is given at* $t \in \mathbb{R}$ *as* $g_k(t) = t^{1-\alpha} B(k, 1-\alpha)$*, where* $B(\cdot,\cdot)$ *is the Beta function. Therefore, our method for this special case gives a procedure for computing the convolution transform of a function* f *with the function* h_α*, where* $0 < \alpha < 1$*, by using a B-spline approximation to* f*.*

Finally, we end this paper with some comments about the special case of our previous discussion when the knots of the B-spline are *equally spaced*, that is, $\mathbb{T} = \mathbb{Z}$. In other words, for each $j \in \mathbb{Z}$ we have that $t_j = j$. In this important case all the B-splines $B_{j,k}$ are the integer translates of the forward B-spline $B_{0,k}$. For notational simplicity we merely denote this B-spline by B. A direct computation confirms that its Fourier transform is given at $w \in \mathbb{R}$ by the equation

$$\hat{B}(w) := \int_{\mathbb{R}} e^{-iwt} B(t) dt = \left(\frac{1 - e^{-iw}}{iw} \right)^k . \tag{2.11}$$

Another direct computation yields the Fourier transform of $G_{0,k}$ given at $w \in \mathbb{R}$ by the equation

$$\hat{G}_{0,k}(w) = \frac{(-1)^k}{(k-1)!} (1 - e^{-iw})^k \int_{\mathbb{R}} t^{k-1} g(t) e^{iwt} dt, \tag{2.12}$$

which reduces to (2.11) when $g = T_+$. From equation (2.11) we get the *refinement equation* for the B-spline,

$$B = 2^{-k+1} \sum_{j \in \mathbb{Z}_k} \binom{k}{j} B(2 \cdot - j). \tag{2.13}$$

Likewise, when the function g_k satisfies a refinement equation of the form

$$g_k = \sum_{l \in \mathbb{Z}} b_l g_k(2 \cdot - l)$$

for some biinfinite vector $(b_l : l \in \mathbb{Z})$ where the cardinality of the set $\{l : l \in \mathbb{Z}, b_l \neq 0\}$ is finite then $G_{0,k}$ satisfies the refinement equation

$$G_{0,k} = \sum_{l \in \mathbb{Z}} d_l G_{0,k}(2 \cdot - l),$$

where the coefficient vector $(d_l : l \in \mathbb{Z})$ is defined by the equation, valid for $w \in \mathbb{R}$,

$$\sum_{l \in \mathbb{Z}} d_l e^{-ilw} = (1 + e^{-iw})^k \sum_{l \in \mathbb{Z}} b_l e^{ilw}. \tag{2.14}$$

For example, when $g_k = T_k$, then $b_l = \begin{cases} 2^{-k+1}, & l = 0 \\ 0, & l \in \mathbb{Z} \setminus \{0\} \end{cases}$, and equation (2.14) yields equation (2.13).

Moreover, it is easy to see, but nonetheless surprising, that the Hilbert transform of B-spline satisfies the *same* refinement equation. That is, for $C := H_{\mathbb{R}} B$, we conclude that

$$C = 2^{-k+1} \sum_{j \in \mathbb{Z}_k} \binom{k}{j} C(2 \cdot - j). \tag{2.15}$$

For example, it is amusing to note, when $k = 1$ and $t \in \mathbb{R} \setminus \{0, \frac{1}{2}, 1\}$, that

$$\ln \left| \frac{t}{t-1} \right| = \ln \left| \frac{2t}{2t-1} \right| + \ln \left| \frac{2t-1}{2t-2} \right|.$$

The proof of equation (2.15) is straightforward because the Cauchy kernel defined at $x, t \in \mathbb{R}, x \neq t$ as $K(x,t) = \frac{1}{x-t}$, appearing in the definition of the Hilbert transform, has the property that

$$K(x, \frac{t}{2}) = 2K(2x, t).$$

Similarly, whenever ϕ is refinable, that is, satisfies the equation

$$\phi = \sum_{j \in \mathbb{Z}} a_j \phi(2 \cdot - j), \tag{2.16}$$

where $\{j \in \mathbb{Z} : a_j \neq 0\}$ is finite and ϕ is in the domain of the Hilbert transform, then $H\phi$ satisfies the same equation. Indeed, the Hilbert transform is the unique integral operator with this property.

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