East-West J. of Mathematics: Vol. 15, No 2 (2013) pp. 170-181

SOME RESULTS ON GEODESIC CURVES IN MANIFOLDS WITH DENSITY

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Abstract

In this article, we propose some useful notions in manifold with density: weighted Levi-Civita connection, weighted covariant derivative, weighted geodesic curve, totally weighted geodesic submanifold, weighted energy functional and minimal weighted geodesic curves. We proved that a constant speed curve is a weighted geodesic curve if and only if it is a critical point of the weighted length functional, a curve γ minimizes the weighted energy functional if and only if γ is a minimal weighted geodesic with constant weighted velocity.

1 Introduction

Manifold with density is an n-dimensional Riemannian manifold with a smooth positive density $e^{\varphi(x)}$ used to weight both n-dimensional volume and $(n-1)$-dimensional area. In terms of the underlying Riemannian volume dV, area dA and length ds, the weighted volume, weighted area and weighted length are given by

$$dV_\varphi = e^\varphi dV,\ dP_\varphi = e^\varphi dA,\ ds_\varphi = e^\varphi ds.$$

Such a density is not equivalent to scaling the metric conformally by a factor λ, since volume and area would be scaled by different powers of λ.

Authors are supported in part by the National Foundation for Science and Technology Development, Vietnam (Grant No. 101.01-2011.26).
Key words: Manifold with density, weighted geodesic, weighted minimal geodesic, totally weighted geodesic, the shortest weighted length.
2010 AMS Subject Classification: Primary 49Q20; Secondary 53A04, 53A40.

Manifolds with density long have arisen naturally in mathematics. For example, the area of a surface of revolution equals to the weighted length of its generating curve, with weight or density $2\pi|x|$. So when we study areas and volumes of surfaces of revolution, we, in fact, study the weighted lengths and the weighted areas on upper half-plane with density $2\pi x$. We can consider the upper half-plane surface with density $2\pi x$ as the quotient of $\mathbb{R}^3$ modulo rotation about the y-axis. In general, quotients of Riemannian manifolds are manifolds with density. Another example of manifold with density is the Gauss space $\mathbb{G}^n$, $\mathbb{R}^n$ with Gaussian probability density $(2\pi)^{-n/2}e^{-|x|^2/2}$, that has many applications to probability and statistics. For more details about manifolds with density, we refer the reader to [3], [5], [7], [9].

In section 3, we propose some useful notions in manifolds with density such as weighted derivative, weighted length, weighted Levi-Civita connection, weighted geodesic, totally weighted geodesic submanifold. We proved that

(i) A submanifold K is a totally weighted geodesic submanifold if and only if any geodesic curve on the submanifold K with induced weighted connection is also a geodesic curve on the Riemannian manifold M;

(ii) If K is a totally weighted geodesic submanifold, then K is a totally umbilical submanifold;

(iii) A curve with constant velocity in a manifold with density is a weighted geodesic curve if and only if it is a critical point of the weighted length functional.

In the special case of $n = 2$, a curve is a weighted geodesic curve if and only if it has zero weighted curvature. We give some well-known examples of weighted geodesic curves and prove that in the plane with density that has nonpositive weighted Gauss curvature, if there exists a weighted geodesic curve joining two points, then it is unique.

In section 4, we introduce the notions of the minimal weighted geodesics, the weighted energy functional of a curve. Proposition 12 states that among all curves joining p to q, γ_0 minimizes the weighted energy functional if and only if γ_0 is of constant weighted velocity and minimal weighted geodesic. Theorem 13 gives us a system of equations of minimal weighted geodesic curve.

2 Preliminaries

Following Gromov [5], a generalized mean curvature of a hypersurface Σ on an n-dimensional manifold with density e^φ, called *weighted mean curvature*, is defined as

$$H_\varphi = H - \frac{1}{n-1}\frac{d\varphi}{d\mathbf{n}}. \tag{1}$$

where H is the Riemannian mean curvature and $\mathbf{n}$ is the outward normal vector field of Σ.
Therefore, in plane with density e^{φ}, the *weighted curvature* k_{φ} of a curve with unit normal $\mathbf{n}$ is given by

$$k_{\varphi} = k - \frac{d\varphi}{d\mathbf{n}}, \tag{2}$$

where k is the Riemannian curvature of the curve.

Definition 1. ([3]) The *weighted Gauss curvature* G_{φ} of a Riemannian surface with density e^{φ} is given by

$$G_{\varphi} = G - \Delta_{\varphi} \tag{3}$$

where G is the Riemannian Gauss curvature.

Proposition 1. ([3]) *Given a piecewise-smooth curve enclosing a topological disc R in a Riemannian surface with density e^{φ} and inward pointing unit normal $\mathbf{n}$, then the weighted Gauss curvature G_{φ} satisfies*

$$\int_R G_{\varphi} dA + \int_{\partial R} k_{\varphi} ds + \sum (\pi - \alpha_i) = 2\pi, \tag{4}$$

where α_i are interior angles and the integrals are with respect to Riemannian area and arc length.

Theorem 2. ([4], [11]) *Let $\tau : [a, b] \longrightarrow [a, b]$ be a smooth monotone map taking the endpoints of $[a, b]$ to the endpoints of $[a, b]$. Then,*

$$\int_a^b \left(\frac{d\tau}{dt} \right)^2 dt \geq b - a, \tag{5}$$

with equality holding if and only if $\frac{d\tau}{dt} = 1$.

3 The geodesic curves in manifolds with density

Definition 2. Let M be a manifold with smooth density e^{φ} and $f : (a, b) \subset \mathbb{R} \longrightarrow M$ be a smooth function. We define the *weighted derivative* $\frac{d_{\varphi} f}{dt}$ of function f as follows

$$\frac{d_{\varphi} f}{dt} := e^{\varphi \circ f} \frac{df}{dt}, \tag{6}$$

where $\frac{df}{dt}$ is the derivative with respect to the variable t of the function f in M.

From Definition 2, we immediately obtain

Corollary 3. *Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}^n,\ t \longrightarrow f(t)$ and $\tau : J \subset \mathbb{R} \to I,\ s \to \tau(s)$ be smooth function. Then,*

$$\frac{d_\varphi\,(f \circ \tau)}{ds} = \left(\left(\frac{d_\varphi f}{dt} \right) \circ \tau \right) . \frac{d\tau}{ds}. \tag{7}$$

Definition 3. Let (M, g) be a Riemannian manifold with density e^φ and $\gamma : (a, b) \longrightarrow M$ be a smooth piecewise curve. *The weighted length* of γ is defined by

$$\ell_\varphi(\gamma) := \int_a^b \sqrt{g_\gamma\Big(\frac{d_\varphi\gamma}{dt}, \frac{d_\varphi\gamma}{dt}\Big)}dt = \int_a^b \mathrm{e}^{\varphi\circ\gamma}\sqrt{g_\gamma\Big(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\Big)}dt. \tag{8}$$

Definition 4. Let (M, g) be a Riemannian manifold with density e^φ. We define the *weighted connection* ∇_φ in M by

$$(\nabla_\varphi)_Y X := \nabla_Y(e^\varphi X) - g(Y, Y)e^\varphi\,\nabla\varphi, \tag{9}$$

where X, Y are two smooth vector fields on M, ∇ is Levi-Civita connection on M and $\nabla\varphi$ is the gradient of the function f.

From Definition 4, we obtain

$$(\nabla_\varphi)_Y X = e^\varphi \nabla_Y X + Y(e^\varphi)X - g(Y, Y)e^\varphi \nabla\varphi. \tag{10}$$

Let $\alpha : I \subset \mathbb{R} \longrightarrow M$ be a smooth curve. The *weighted covariant derivative* $\dfrac{D_\varphi X}{dt}$ of a vector field X along α is defined by

$$\frac{D_\varphi X}{dt} = e^{\varphi\circ\alpha} \left(\frac{DX}{dt} + g\left(\nabla\varphi, \alpha'\right) X - g\left(\alpha', \alpha'\right) \nabla\varphi \right), \tag{11}$$

where $\dfrac{DX}{dt}$ is covariant derivative of X along α.
The curve α is called a *weighted geodesic curve* if the weighted covariant derivative of α' along α is equal to 0. This is equivalent to

$$\alpha''(t) + g\big(\nabla\varphi\big(\alpha\,(t)\big), \alpha'\,(t)\big)\alpha'\,(t) - g\big(\alpha'\,(t)\,, \alpha'\,(t)\big)\nabla\varphi\big(\alpha\,(t)\big) = 0, \tag{12}$$

for all t.

Definition 5. Let (M, g) be a Riemannian manifold with density e^φ and $K \subset M$ be a submanifold of M. For all vector fields X, Y on K, let $\big((\nabla_\varphi)_Y\,X\big)^T$ and $\big((\nabla_\varphi)_Y\,X\big)^N$ denote the tangential and normal components of $(\nabla_\varphi)_Y\,X$ to K respectively.
The map $\overline{\nabla}_\varphi : \mathcal{X}(K) \times \mathcal{X}(K) \longrightarrow \mathcal{X}(K), (X, Y) \longmapsto \big((\nabla_\varphi)_Y\,X\big)^T$ is called the *induced weighted connection* on K.

The map $II_\varphi : \mathcal{X}(K) \times \mathcal{X}(K) \longrightarrow \mathcal{B}(K), (X,Y) \longmapsto \left((\nabla_\varphi)_Y X \right)^N$ is called the *weighted second fundamental form* of K.
The submanifold K is called *totally weighted geodesic* if its weighted second fundamental form is equal to 0.

Let α be a smooth curve on a submanifold $K \subset M$ and Y be a vector field along α. Then, by Definition 5

$$\frac{D_\varphi Y}{dt} = \frac{\overline{D}_\varphi Y}{dt} + II_\varphi \left(\alpha', Y\right),$$
$$II_\varphi \left(\alpha', Y\right) = e^{\varphi \circ \alpha} \left(II \left(\alpha', Y\right) - g \left(\alpha', \alpha'\right) \alpha \nabla \varphi^N \right),$$

where $\dfrac{\overline{D}_\varphi Y}{dt}$ is the covariant derivative of Y along α and II is the second fundamental form of K. Therefore,

Proposition 4.

(i) Let K be a submanifold of a Riemannian manifold M. K is totally weighted geodesic if and only if any geodesic curve on K with induced weighted connection ∇_φ is also a geodesic curve on M.

(ii) If K is totally weighted geodesic, then K is totally umbilical.

We have known that geodesics with constant velocities are locally length-minimizing, Studying this property for weighted geodesic curves on a manifold with density led us to the following.

Lemma 5. *A curve with constant velocity in manifold with density M is a weighted geodesic curve if and only if it is a critical point of the weighted length functional.*

Proof. Without loss of generality, we can assume that $\alpha : (a,b) \longrightarrow M$ has a natural parametrization. We consider a family of parameter curves $\alpha_\lambda : (a,b) \longmapsto M$, where $\lambda \in (-\varepsilon, \varepsilon), \varepsilon > 0$, satisfying $\alpha_0(t) = \alpha(t)$ for all $t \in (a,b)$, $\alpha_\lambda(a) = \alpha(a)$ and $\alpha_\lambda(b) = \alpha(b)$ for all $\lambda \in (-\varepsilon, \varepsilon)$. We get,

$$\ell_\varphi(\alpha_\lambda) = \int_a^b |\alpha'_\lambda(s)| e^{\varphi(\alpha_\lambda(s))} ds.$$

Therefore,

$$
\begin{aligned}
\frac{\ell_\varphi(\alpha_\lambda)}{d\lambda}\Big|_{\lambda=0} &= \int_a^b \frac{\frac{d\alpha'_\lambda(s)}{d\lambda}\Big|_{\lambda=0}\alpha'_0(s)}{|\alpha'_0(s)|} e^{\varphi(\alpha_0(s))}ds + \int_a^b |\alpha'_\lambda(s)| \frac{d\big(e^{\varphi(\alpha_\lambda(s))}\big)}{d\lambda}\Big|_{\lambda=0} ds \\
&= -\int_a^b \frac{d\alpha_\lambda(s)}{d\lambda}\Big|_{\lambda=0}\Big(\alpha'_0(s)e^{\varphi(\alpha_0(s))}\Big)' ds + \int_a^b \frac{d\big(e^{\varphi(\alpha_\lambda(s))}\big)}{d\lambda}\Big|_{\lambda=0} ds \\
&= -\int_a^b \frac{d\alpha_\lambda(s)}{d\lambda}\Big|_{\lambda=0}\Big(\alpha''_0(s) + \alpha'_0(s)g\big(\nabla\varphi(\alpha_0(s)), \alpha'_0(s)\big) \\
&\qquad\qquad - \nabla\varphi\big(\alpha_0(s)\big)\Big)e^{\varphi(\alpha_0(s))}ds \\
&= -\int_a^b \frac{d\alpha_\lambda(s)}{d\lambda}\Big|_{\lambda=0} D_\varphi\alpha'(s)ds. \qquad (13)
\end{aligned}
$$

By the Bois-Reymond lemma, α is a critical point of the weighted length functional if and only if $D_\varphi\alpha'(s) = 0$. This is equivalent to that α is weighted geodesic.

Now, we study some properties of weighted geodesic curves in the case of $n = 2$.

Lemma 6. *Let α be a curve in the plan $\mathbb{R}^2$ with density e^φ. Then, α is weighted geodesic if and only if its weighted curvature is equal to 0.*

Proof. Let $\alpha : [a, b] \longrightarrow \mathbb{R}^2$ be a parametric curve with arc-length parameter. For any smooth function $\eta : [a, b] \longrightarrow \mathbb{R}^2$ satisfying $\eta(a) = \eta(b) = 0$, we obtain, for all $\varepsilon > 0$,

$$
\begin{aligned}
\frac{d\ell_\varphi(\alpha+\varepsilon\eta)}{d\varepsilon}\Big|_{\varepsilon=0} &= \int_a^b (\alpha'.\eta')e^\varphi ds + \int_a^b \big(\eta.\nabla\varphi\big)e^\varphi ds \\
&= -\int_a^b \eta.\big(\alpha'' + \nabla\varphi - (\nabla\varphi.\mathbf{n})\mathbf{n}\big)e^\varphi ds + \int_a^b \big(\eta.\nabla\varphi\big)e^\varphi ds \\
&= -\int_a^b \eta.\big(\alpha'' - (\nabla\varphi.\mathbf{n})\mathbf{n}\big)e^\varphi ds. \qquad (14)
\end{aligned}
$$

where $\mathbf{n}$ is normal vector of α. From equations (13) and (14), we conclude that

$$
\int_a^b \eta.D_\varphi\alpha' ds = \int_a^b k_\varphi(\eta.\mathbf{n})e^\varphi ds. \qquad (15)
$$

Applying the Bois-Reymond lemma, we obtain the desired result.

The following examples show us some weighted geodesic curves by using Lemma 6.

Example 7. ([1]) In the plane $\mathbb{R}^2$ with density $e^{\frac{-x^2-y^2}{2}}$.

(i) The unit circle is the unique closed geodesic curve.

(ii) The straight lines passing through the origin are geodesic curves.

Theorem 8. ([1]) *In the plane $\mathbb{R}^2$ with density e^{φ} and $G_\varphi \leq 0$, if there exists a weighted geodesic α joining two points, then it is unique.*

Proof.
Let α and β be two distinct geodesics joining two points p and q. Begin at one point, which we will call p, let q_1 be the first intersection point of α and β. Now since we are in the plane and the bounded region is a disc, we can apply Gauss-Bonnet to the region bounded by α and β from p to q_1, with θ_1 and θ_2 the angles formed where the two geodesics meet. Thus,

$$\iint_R K_\varphi + \int_\alpha k_\varphi(\alpha) + \int_\beta k_\varphi(\beta) + \sum(\pi - \theta_i) = 2\pi.$$

Because α and β are both geodesics, the $\int k_\varphi$ terms vanish so that,

$$\iint_R K_\varphi = \theta_1 + \theta_2.$$

Since $G_\varphi \leq 0$, $\theta_1 = \theta_2$, and the geodesics must coincide.

Theorem 9. *In the plane $\mathbb{R}^2$ with density e^x, the weighted geodesic curves are either a straight line, parallel to the x-axis, or the Grim-Reaper curve whose equation is*

$$x = -\ln(\cos y),\ y \in \mathbb{R}. \tag{16}$$

Proof. We can suppose that α has the parametrization $\alpha(s) = \big(x(s), y(s)\big)$ with

$$\begin{cases} x' = \cos(2\xi), \\ y' = \sin(2\xi), \end{cases}$$

where ξ is a function of variable s. Then,

$$k_\varphi = k - \frac{d\varphi}{d\mathbf{n}} = 2\xi' + \sin(2\xi). \tag{17}$$

Thus, α is weighted geodesic if and only if

$$2\xi' + \sin(2\xi) = 0. \tag{18}$$

If there exists s_0 satisfying $\sin\big(2\xi(s_0)\big) = 0$, then $\xi(s) = \pi/2$ is the unique solution of the ODE (17) in the interval $(0;\pi)$, since the sin function is Lipschitz. In this case, (α) is a straight line parallel to the x-axis.

Consider Equation (17) in the interval $(0,\pi/2)$. In this case, $\sin(2\xi) > 0$, for all s. Solving Equation (5), we obtain

$$-s+b = \ln\big|\tan\xi\big| = \ln\tan\xi \ \ (\text{since } \tan\xi > 0).$$

Thus, $e^{-s+b} = \tan\xi$. We can suppose that $b = 0$ because two curves $\alpha(-s)$ and $\alpha(-s+b)$ have the same trace. Therefore,

$$\begin{cases} x\left(s\right) = \displaystyle\int \frac{1-e^{-2s}}{1+e^{-2s}}ds = \ln\left(e^{s}+e^{-s}\right)+c_1, \\ y\left(s\right) = \displaystyle\int \frac{2e^{-s}}{1+e^{-2s}}ds = 2\arctan\left(e^{s}\right)+c_2. \end{cases} \tag{19}$$

Similarly, the equation (17) in the interval $(\pi/2,\pi)$ has the solution

$$\begin{cases} x\left(s\right) = \displaystyle\int \frac{1-e^{-2s}}{1+e^{-2s}}ds = \ln\left(e^{s}+e^{-s}\right)+c_1, \\ y\left(s\right) = \displaystyle\int \frac{-2e^{-s}}{1+e^{-2s}}ds = -2\arctan\left(e^{s}\right)+c_2. \end{cases} \tag{20}$$

The curve defined by (20) is just the image of the one defined by (19) under the reflection across the x-axis. From equation (19), we have

$$x = -\ln\big(\sin(y-c_2)\big)+c_1+\ln 2. \tag{21}$$

By changing the coordinate system

$$\begin{cases} \overline{x} = x - \ln 2 - c_1 \\ \overline{y} = -y + \dfrac{\pi}{2} + c_2 \end{cases}$$

equation (21) become the equation

$$\overline{x} = -\ln(\cos\overline{y}).$$

We can realize that the curve α is the Grim-Reaper curve.

4 The minimal geodesic in manifolds with density

Definition 6. Let (M,g) be a manifold with density e^{φ}, the *weighted distance* $d_{\varphi}(x,y)$ between two points x and y of M is the infimum of the set of weighted lengths of all smooth piecewise curves joining x to y.
If there exists a curve γ joining x to y such that its weighted length equals to $d_{\varphi}(x,y)$, then γ is called a *minimal weighted geodesic* on manifold with density.

Sharing some interesting properties with its counterpart in Riemannian manifolds, manifolds with density allow us to generalize some results from the classical setting. In particular, the following lemma is true.

Lemma 10. *Let (M, g) be a manifold with density e^{φ}, $\gamma : [a, b] \longrightarrow M$ be a smooth curve.*

(i) The weighted length of γ is independent on the parametrization of γ, i.e. if $\tau : [a', b'] \longrightarrow [a, b]$ is a change of parameter, then $\overline{\gamma} = \gamma \circ \tau$ and γ have the same weighted length.

(ii) If γ is regular, then there is a change of parameter $\tau : [a, b] \longrightarrow [a, b]$ such that $\left|\dfrac{d_{\varphi}(\gamma \circ \tau)}{dt}(t)\right|$ is independent on t.

Definition 7. In a Riemannian manifold M with density e^{φ} let $\gamma : [a, b] \longrightarrow M$ be a smooth curve. The functional $\mathcal{A}(\gamma) := \displaystyle\int_a^b \left|\frac{d_{\varphi}\gamma}{dt}\right|^2 dt$ is called the *weighted energy functional* of γ.

We will study some properties of the weighted energy functional.

Proposition 11. *Let M be a Riemannian manifold with density e^{φ} and $\gamma : [a, b] \longrightarrow M$ be a smooth curve. Suppose that, as s goes from a to b, its image $\gamma(s)$ moves at constant weighted velocity. Let $\overline{\gamma} = \gamma \circ \tau : [a, b] \longrightarrow M$ be a reparametrization of γ. Then $\mathcal{A}(\overline{\gamma}) \geq \mathcal{A}(\gamma)$, the equality holds if and only if $\tau(t) = t$ for all $t \in [a, b]$.*

Proof. Suppose that $\left|\dfrac{d_{\varphi}\gamma}{ds}(s)\right| = c$, for all $s \in [a, b]$. Then,

$$\mathcal{A}(\gamma) = \int_a^b \left|\frac{d_{\varphi}\gamma}{ds}\right|^2 ds = c^2(b - a).$$

Otherwise,

$$\mathcal{A}(\overline{\gamma}) = \int_a^b \left|\frac{d_{\varphi}(\gamma \circ \tau)}{dt}\right|^2 dt = c^2 \int_a^b \left(\frac{d\tau}{dt}\right)^2 dt.$$

Hence $\mathcal{A}(\overline{\gamma}) \geq \mathcal{A}(\gamma)$ by Theorem 2. Moveover, it is clear that $\mathcal{A}(\overline{\gamma}) = \mathcal{A}(\gamma)$ if and only if $\tau(t) = t$ for all $t \in [a, b]$.

Proposition 12. *Let p, q be two points on a Riemannian manifold with density e^{φ}. Then, among all curves joining p to q, γ_0 minimizes the weighted energy functional if and only if γ_0 is a minimal weighted geodesic with constant weighted velocity.*

Proof. Suppose that γ_0 minimizes weighted energy functional. Let $\overline{\gamma}_0$ be a reparametrization of γ_0 with constant weighted velocity. Then, by Proposition 11 we obtain

$$\mathcal{A}(\gamma_0) \leq \mathcal{A}(\overline{\gamma}_0) \leq \mathcal{A}(\gamma_0).$$

This implies that $\mathcal{A}(\gamma_0) = \mathcal{A}(\overline{\gamma}_0)$ and $\gamma_0 \equiv \overline{\gamma}_0$.
For any smooth curve γ on M joins two points p and q with constant weighted velocity,

$$\left|\frac{d_\varphi \gamma}{dt}\right|^2 (b-a) = \mathcal{A}(\gamma) \geq \mathcal{A}(\gamma_0) = \left|\frac{d_\varphi \gamma_0}{dt}\right|^2 (b-a).$$

Therefore,

$$\left|\frac{d_\varphi \gamma}{dt}\right| \geq \left|\frac{d_\varphi \gamma_0}{dt}\right|. \tag{22}$$

Otherwise,

$$\ell_\varphi(\gamma) = \left|\frac{d_\varphi \gamma}{dt}\right| (b-a), \tag{23}$$

$$\ell_\varphi(\gamma_0) = \left|\frac{d_\varphi \gamma_0}{dt}\right| (b-a). \tag{24}$$

By (22), (23), (24), we conclude that γ_0 is a minimal weighted geodesic.

Conversely, if γ_0 has a constant weighted velocity and is a minimizing weighted length, we easily check that γ_0 minimizes the weighted energy functional by Proposition 11 and equations (23), (24).

Theorem 13. *Let $\gamma : (a, b) \longrightarrow M$ be a smooth curve with constant weighted velocity on a Riemannian manifold with density e^φ. If γ is a minimal weighted geodesic, then on a coordinate chart $(\mathcal{U}, x_1, \ldots, x_n)$*

$$\frac{d^2\gamma^k}{dt^2} + \sum_{i,j=1}^{n} \left(\Gamma_{ij}^k + \Gamma_{\varphi;ij}^k\right) \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt} = 0, \quad k = 1, \ldots, n, \tag{25}$$

where $\gamma = (\gamma_1, \ldots, \gamma_n)$, Γ_{ij}^k are Christoffel symbols and $\Gamma_{\varphi;ij}^k$ are defined in terms of the coefficients of the Riemannian metric by

$$\Gamma_{ij}^k = \frac{1}{2}\sum_{l=1}^{n} g^{lk}\left(\frac{\partial g_{li}}{\partial x_j} + \frac{\partial g_{lj}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l}\right)$$

$$\Gamma_{\varphi;ij}^k = \sum_{l=1}^{n} g^{lk}\left(g_{li}\frac{\partial \varphi}{\partial x_j} + g_{lj}\frac{\partial \varphi}{\partial x_i} - g_{ij}\frac{\partial \varphi}{\partial x_l}\right),$$

here (g^{ij}) is the inverse matrix of (g_{ij}).

Proof. We consider the functional F defined by

$$F\Big(\gamma(t),\frac{d\gamma}{dt}(t)\Big)=\sum_{i,j=1}^{n}e^{2\varphi\big(\gamma(t)\big)}g_{ij}\big(\gamma(t)\big)\frac{d\gamma^i}{dt}(t)\frac{d\gamma^j}{dt}(t).$$

Then, $\mathcal{A}(\gamma)=\int_a^b F\Big(\gamma(t),\frac{d\gamma}{dt}(t)\Big)dt$. Hence if γ is a minimal weighted geodesic, then it must satisfy the Euler-Lagrange equations for functional F. Therefore,

$$\frac{\partial F}{\partial x_k}\Big(\gamma(t),\frac{d\gamma}{dt}(t)\Big)=\frac{d}{dt}\frac{\partial F}{\partial v_k}\Big(\gamma(t),\frac{d\gamma}{dt}(t)\Big),\ k=1,\dots,n. \tag{26}$$

Now, we compute left-hand side and right-hand side of equations (26).

$$\frac{\partial F}{\partial x_k}\left(\gamma,\frac{d\gamma}{dt}\right)=e^{2\varphi}\sum_{i,j=1}^{n}\left(\frac{\partial g_{ij}}{\partial x_k}+2g_{ij}\frac{\partial\varphi}{\partial x_k}\right)\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt},$$

$$\frac{d}{dt}\frac{\partial F}{\partial v_k}\left(\gamma,\frac{d\gamma}{dt}\right)=2e^{2\varphi}\left[\sum_{i=1}^{n}g_{ik}\frac{d^2\gamma^i}{dt^2}+\left(\sum_{i,j=1}^{n}\frac{\partial g_{ik}}{\partial x_j}+2\sum_{i,j=1}^{n}g_{ik}\frac{\partial\varphi}{\partial x_j}\right)\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\right].$$

Therefore, we derive

$$\sum_{i=1}^{n}g_{ik}\frac{d^2\gamma^i}{dt^2}+\sum_{i,j=1}^{n}\frac{\partial g_{ik}}{\partial x_j}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}-\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial g_{ij}}{\partial x_k}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}$$
$$+\left(\sum_{i,j=1}^{n}2g_{ik}\frac{\partial\varphi}{\partial x_j}-g_{ij}\frac{\partial\varphi}{\partial x_k}\right)\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}=0.$$

Thus,

$$\frac{d^2\gamma^k}{dt^2}+\sum_{i,j=1}^{n}\left(\Gamma_{ij}^k+\Gamma_{\varphi}{}_{ij}^{k}\right)\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}=0,\ k=1,\dots,n.$$

Example 14. In the plane $\mathbb{R}^2$ with density $e^{\frac{-x^2-y^2}{2}}$, the functional F is defined by

$$F(x,y,x',y')=e^{-x^2-y^2}(x'^2+y'^2),$$

and therefore system of equations (25) become

$$\begin{cases} x''-x{x'}^2-2yx'y'+x{y'}^2=0,\\ y''+y{x'}^2-2xx'y'-y{y'}^2=0.\end{cases} \tag{27}$$

It is easy to see that circle arcs with parametrization $c(t)=(\cos t,\sin t)$, $t\in(a,b)\subseteq(0,2\pi)$ satisfying equation system (27).

Moreover, with the x, y variables are fixed, the function $(x', y') \longmapsto F(x, y, x', y')$ is strictly convex since its hessian matrix is

$$e^{-x^2-y^2}\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}, \forall\, x, y \in (a, b)\,.$$

Hence, circle arcs (C) locally minimizes the weighted energy functional.

However, globally they do not minimize the weighted energy functional. In fact, consider the semicircle $(C) : c(t) = (\cos t, \sin t),\ t \in [0, \pi]$ and the segment $(d) : \alpha(t) = (t, 0),\ t \in [-1, 1]$, joining two points $p(-1, 0)$, $q(1, 0)$. We have

$$\mathcal{A}(C) = \frac{\pi}{e} > \frac{2}{e} = \mathcal{A}(d).$$

Therefore (C) is not the shortest weighted length joining two points p, q.

References

[1] C. Carroll, A. Jacob, C. Quinn, R. Walters , *The isoperimetric problem on planes with density*, Bull. Australia Math. Soc., **78**(2) (2008), pp 177-197.

[2] M. P. D. Carmo, “Differential geometry of curves and surfaces”, Prentice - Hall, Englewood Cliffs, NJ, 1976.

[3] R. Corwin, N. Hoffman, S. Hurder, V. Sesum, and Y. Xu , *Differential geometry of manifolds with density*, Rose - Hulman Und. Math. J., **7**(1)(2006), pp 1-15, https://www.rose-hulman.edu/mathjournal/v7n1.php.

[4] M. Gromov, “Partial Differential Relations”, Springer-Verlag, Berlin-New York, 1986.

[5] M. Gromov (2003), *Isoperimetry of waists and concentration of maps*, Geom. Funct. Anal., No. 13, pp 178-215.

[6] Q. Maurmann and F. Morgan, *Isoperimetric comparison theorems for manifolds with density*, Calc. Var. PDE **36**(1) (2009), pp 1 - 5.

[7] F. Morgan, *Manifolds with density*, Notices Amer. Math. Soc., **52** (2005), pp 853 - 858.

[8] F. Morgan, *Myers theorem with density*, Kodai Math. J. **29** (2006), pp 454 - 460.

[9] F. Morgan, *Manifolds with density and Perelman’s proof of the Poincare conjecture*, Amer. Math. Monthly **116** (2009), pp 134 - 142.

[10] C. Rosales, A. Cañete, V. Bayle and F. Morgan, *On the isoperimetric problem in Euclidean space with density*, Calc. Var. PDE **31**(1)(2008), pp 27 - 46.

[11] V. S. Varadarajan, “Lectures on Symplectic Geometry”, Springer - Verlag - New York - Berlin - Heidelberg - Tokyo, 2001.