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LINEAR PROGRAMMING MODELS TO SUPPORT INVENTORY DECISION-MAKING IN THE CASE OF INCOMPLETE INFORMATION ON DEMAND DURING LEAD-TIME

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Abstract

Most logistics managers face uncertainty in demand which forces them to hold safety stock to provide high levels of service to their customers. The level of safety stock depends on what the company's targets are on some performance characteristics of an inventory management decision problem like the expected number of units short or the stock-out probability. In the case only incomplete information is available on the demand distribution during the lead-time, which is relevant in inventory decision-making, preset service levels do not lead to a unique value of the safety stock to be hold, but rather to a range of values. Incomplete information refers to the fact that the full functional form of the distribution is not known, but some knowledge is available like the range or the mode or a few moments of the demand size distribution. In this way, upper and lower bounds may be determined for the safety stock in the

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inventory management problem. It is shown how the bounds of both performance measures can be obtained through a numerical approximation using linear programming. Results are obtained for demand distributions for which the range, and first and second moments are known.

1 Introduction

A primary objective of inventory management is to ensure that a product is available at the time and in the quantities desired ((2)). But almost every inventory system contains uncertainty. Some uncertainty is attributable to customers, especially demand. If insufficient inventory is hold, a stock-out may occur which leads to shortage costs or a decreased level of service. Shortage costs are usually high in relation to holding costs, i.e. the costs of keeping the goods during some time period in the warehouse. Companies are willing to hold additional inventory, above their forecasted needs, to add a margin of safety.

Inventory management decisions are made making use of optimisation models taking a performance measure into consideration which might be cost-oriented or service-oriented. Performance measures of the service-oriented type may be expressed relatively as a probability of stock-out during a certain replenishment period, or may be expressed absolutely in terms of the number of units short, which is a direct indication for lost sales.

In this study, both performance measures are taken into consideration and special attention is paid to feasible combinations of company's objectives regarding both performance measures. For a definition of both measures, reference is made to (9). The authors, Silver, Pyke and Peterson, define the measures as follows. The expected shortage per replenishment cycle (ESPRC) is defined as ((9), p. 258):

$$ESPRC = \int_{t}^{+\infty} (x-t)f(x)dx \tag{1}$$

where $f(x)$ is the probability density function of the demand in the replenishment lead time and t is the level of the inventory position at which an order is placed. In many other sources this performance measure is referred to as the 'expected number of units short'. If ordered per fixed quantity Q the fraction backordered is equal to ESPRC/Q and a performance measure, indicated as P_2, is defined as ((9), p. 299):

$$P_2 = 1 - ESPRC/Q \tag{2}$$

The other performance measure is the probability of a stock-out during replenishment lead time ((9), p. 258), defined as:

$$1 - P_1 = Pr\{x \geq t\} = \int_t^{+\infty} f(x)dx \tag{3}$$

P_1 is defined as the specified probability of no stockout per replenishment cycle.

From a production or trading company's point of view, a decision might be formulated to answer the following question: *Given a maximum expected number of units short and/or a maximum stock-out probability the company wants to face, what should be the safety inventory at least?* In case the distribution of demand is known, determining the inventory level, given a maximum expected shortage or a maximum stock-out probability, reduces to the calculation of the inverse cumulative probability function. The decision problem becomes more difficult if incomplete information exists on the distribution of demand during lead time, for example if only the range of demand or the first and second moments are known. In such a case, no single value for this amount of safety inventory can be determined but rather an interval.

This research deals with the case where the demand distribution during lead time is not completely known. This situation is realistic either with products which have been introduced recently to the market or with slow moving products. In both cases no sufficient data are available to decide on the functional form of the demand distribution function. Some but not complete information might exist like the range of the demand, its expected value, its variance and maybe some knowledge about unimodality of the distribution.

In case incomplete information is available regarding the demand distribution the integrals of the performance measures P_1 and P_2 cannot be evaluated in an analytical manner. This means that also the inverse problem of determining the safety stock level to satisfy the performance measures cannot be obtained analytically. However, the integrals can be approximated by a linear programming formulation with a large set of constraints.

The next section introduces the methodology of how a linear program can be constructed for the evaluation of the performance measures P_1 and P_2 in the case the range, first and second moments of the demand distribution during lead time (DDLT) are known. Subsequently, the level of safety stock is determined based on the inverse problem. The problem is formulated first for the case where an inventory manager considers P_1 and P_2 separately and second for the case where P_1 and P_2 are considered simultaneously. Finally, conclusions are formulated in the last section.

2 Demand distributions with known range, expected value and variance

2.1 Bounds on the expected number of units short

When a company holds t units of a specific product in inventory at the start of a period between order and delivery, any demand less than t during this period is satisfied while any demand X greater than t results in a shortage of $X - t$ units. The expected shortage or the expected number of units short in a replenishment period can be written as:

$$\int_0^\infty (x-t)_+ \; dF(x) \tag{4}$$

Let the size of the demand X for a specific product in a finite period have a distribution F with first two moments $m_1 = E(X)$ and $m_2 = E(X^2)$.

From a mathematical point of view, the problem is to find the following bounds:

$$\sup_{F\in\phi} \int_0^\infty (x-t)_+ \; dF(x) \tag{5}$$

and

$$\inf_{F\in\phi} \int_0^\infty (x-t)_+ \; dF(x) \tag{6}$$

where ϕ is the class of all distribution functions F which have moments m_1 and m_2, and which have support in $\Re^+$. Let further $\sigma^2 = m_2 - m_1^2$. We assume t to be strictly positive.

For any polynomial $P(x)$ of degree 2 or less, the integral $\int_0^\infty P(x) \; dF(x)$ only depends on m_1 and m_2, so it takes the same value for all distributions in ϕ. There exists some distribution G in ϕ for which the equality holds:

$$\int_0^\infty P(x)dG(x) = \int_0^\infty (x-t)_+ dG(x) \tag{7}$$

As distribution G a two-point or three-point distribution is used. The equality (7) is attained when $P(x)$ and $(x-t)_+$ are equal in the two points of G. The best upper and lower bounds on this term with given moments m_1 and m_2 are derived. The method is inspired by a paper of (4). In the following we assume the known range of the distribution to be a finite interval $[a,b]$.

In this section we consider, of the relevant integral, the supremum version. Let $f_1, f_2, ..., f_n$ be functions on $\Re$. For any $z' = (z_1, ..., z_{n-1}) \in \Re^{n-1}$, we consider the primal maximisation problem:

$$P(z') = \sup_{F \in \phi} \left[\int_0^\infty (x-t)_+ \; dF(x) | I(F) \right] \tag{8}$$

where $I(F)$ is a set of integral equality constraints of the type $\int f_i(x)dF(x) = z_i$, $(i = 1, ..., n-1)$ and $f_n = (x-t)_+$. In our application, the constraints are moment constraints, i.e. the first and second moment equalities and the obvious constraint because any member of ϕ is a probability distribution.

$$\int dF(x) = 1, \int x dF(x) = m_1, \int x^2 dF(x) = m_2 \tag{9}$$

which means that:
$n = 3$
$f_1(x) = x$
$f_2(x) = x^2$
$f_3(x) = (x-t)_+$
$z_1 = m_1$
$z_2 = m_2$

The integral may be approximated by a sum making use of finite masses p_i in a large number of points x_i. Its formulation looks like:

$$Max \sum_i (x_i - t)_+ * p_i \tag{10}$$

Subject to

$$\sum_i p_i = 1 \tag{11}$$

$$\sum_i x_i * p_i = m_1 \tag{12}$$

$$\sum_i x_i^2 * p_i = m_2 \tag{13}$$

And $p_i \geq 0, 0 \leq x_i \leq b$.

However, any refinement in granularity leads to an increased number of variables both in the objective function and in the three constraints. This phenomenon does not guarantee any convergence towards the exact upper bound.

The optimisation problem (8)-(9) has a dual program of the type:

$$Q(z') = \inf \left(y_1 m_1 + y_2 m_2 + y_3 | y_1 \widehat{f}_1(\theta) + y_2 \widehat{f}_2(\theta) + y_3 \geq \widehat{f}_3(\theta) \right), (\theta \in J) \tag{14}$$

where the infimum is over all $y = (y_1, y_2, y_3) \in \Re^3$ satisfying the constraints indicated after the slash. The functions $\widehat{f}_i(\theta)$ $(i = 1, ..., n)$ are defined on J by

$$\widehat{f}_i(\theta) = \int f_i dH_\theta(x) \text{ with } \theta \in J, \tag{15}$$

where H_θ is a subset of functions depending on the family of distributions under consideration.

The family of distributions considered in this study concerns distributions on a finite interval $[0, b]$, where b is a fixed positive number. In this case ((8)):

$$H_\theta(x) = 1_{\theta \leq x} \qquad (0 \leq \theta \leq b), \tag{16}$$

which means in our case that

$$\widehat{f}_1(\theta) = \theta \qquad (0 \leq \theta \leq b) \tag{17}$$

$$\widehat{f}_2(\theta) = \theta^2 \qquad (0 \leq \theta \leq b) \tag{18}$$

Mostly the set J is infinite, so the number of linear constraints on y is infinite. In (8) an idea is launched to replace J by a large finite subset of J and then to solve the so obtained linear program.

The optimisation problem (14) can thus be approximated by the problem Q^A:

$$Q^A(z') = \inf_{y \in \Re^3} \left(y_1 m_1 + y_2 m_2 + y_3 | y_1 \theta_i + y_2 \theta_i^2 + y_3 \geq (\theta_i - t)_+ \right), \left(\theta_i = i * \frac{b}{k}, i = 0, ..., k \right) \tag{19}$$

This leads to the following dual problem:

$$Min \sum_{j=1}^{2} y_j m_j + y_3 * 1 \tag{20}$$

Subject to

$$\sum_{j=1}^{2} y_j \hat{f}_j(x_i) + y_3 \geq \hat{f}_3(x_i) \qquad (x_i = i * \frac{b}{k}, i = 0, ..., k) \tag{21}$$

With

$$\hat{f}_1(x_i) = x_i \quad (22)$$
$$\hat{f}_2(x_i) = x_i^2 \quad (23)$$
$$\hat{f}_3(x_i) = (x_i - t)_+ \quad (24)$$
$$(25)$$

And y_j (j = 1,2,3) unbounded.

To obtain the lower bound (6), only the objective function type needs to be replaced from Max to Min in the primal problem and from Min to Max in the dual problem while constraints remain the same.

A numerical example demonstrates the use of bounds on the expected number of units short. In this example, the following information on demand during the replenishment period is known: the first moment $m_1 = 25$, the second moment $m_2 = 725$ and the range of demand is $[0, b]$ with $b = 50$. The upper and lower bounds on the number of stock-out units are presented in Tables 1 and 2 for different values of t and varying sizes k of the evaluation point set. The exact values for the upper bounds are 16.37931 for $t = 10$, 5.0 for $t = 25$ and 1.37931 for $t = 40$. The exact values for the lower bounds are 15.0 for $t = 10$, 2.0 for $t = 25$ and 0.0 for $t = 40$.

Table 1: Upper bounds on the expected number of units short

	k = **10**	k = **20**	k = **40**	k = **80**
t = **10**	16.3333	16.3636	16.3768	16.3784
t = **25**	5.0000	5.0000	5.0000	5.0000
t = **40**	1.3333	1.3636	1.3768	1.3784

Table 2: Lower bounds on the expected number of units short

	k = **10**	k = **20**	k = **40**	k = **80**
t = **10**	15.0000	15.0000	15.0000	15.0000
t = **25**	2.0000	2.0000	2.0000	2.0000
t = **40**	0.0000	0.0000	0.0000	0.0000

2.2 Bounds on the probability of a shortage

(author?) (9, p. 245) define the concept of a cycle service level as the specified probability (P_1) of no stock-out per replenishment cycle. This means that P_1 is the fraction of cycles in which a stock-out does not occur.

If an order to the supplier is released whenever the inventory level drops below a level t and assuming that X represents the demand during lead time, then P_1 can be written as:

$$P_1 = 1 - P(X \geq t) = 1 - E[1_{[t,+\infty[}(X)] \tag{26}$$

In this section upper and lower bounds are determined for the tail probabilities $E[1_{[t,+\infty[}(X)]$, i.e. the values are to be found for

$$\sup_{F \in \Phi} \int_0^b 1_{[t,+\infty[}(x) dF(x) \tag{27}$$

and

$$\inf_{F \in \Phi} \int_0^b 1_{[t,+\infty[}(x) dF(x) \tag{28}$$

where Φ is the class of all distribution functions with range $[0,b]$ and moments m_1 and m_2 known.

For reasons similar to the ones in section 2.1, this section concentrates on the dual problems only to find the extreme values. The dual problem for obtaining the upper bound is formulated as:

$$Min \sum_{j=1}^{2} y_j m_j + y_3 * 1 \tag{29}$$

subject to

$$\sum_{j=1}^{2} y_j \hat{f}_j(x_i) + y_3 \geq \hat{f}_3(x_i) \qquad (x_i = i * \frac{b}{k}, i = 0, ..., k) \tag{30}$$

with

$$\hat{f}_1(x_i) = x_i \tag{31}$$

$$\hat{f}_2(x_i) = x_i^2 \tag{32}$$

$$\hat{f}_3(x_i) = 1_{[t,x_i]} \tag{33}$$

The dual problem for obtaining the lower bound uses

$$Max \sum_{j=1}^{2} y_j m_j + y_3 * 1 \tag{34}$$

as its objective function and the constraints are identical to those of the upper bound.

A numerical example demonstrates the use of bounds on the probability of a shortage. In this example, the following information on demand during the replenishment period is known: the first moment $m_1 = 25$, the second moment $m_2 = 725$ and the range of demand is $[0,\ b]$ with $b = 50$. The upper and lower bounds on the stock-out probability are presented in Tables 3 and 4 for different values of t and varying sizes k of the evaluation point set. The exact values for the upper bounds are 1.0 for $t = 10$, 0.92 for $t = 25$ and 0.30769 for $t = 40$. The exact values for the lower bounds are 0.692308 for $t = 10$, 0.08 for $t = 25$ and 0.0 for $t = 40$.

Table 3: Upper bounds on the stock-out probability

	k = **10**	k = **20**	k = **40**	k = **80**
t = **10**	1.0000	1.0000	1.0000	1.0000
t = **25**	0.8000	0.8818	0.9000	0.9098
t = **40**	0.2000	0.2444	0.2745	0.2905

Table 4: Lower bounds on the stock-out probability

	k = **10**	k = **20**	k = **40**	k = **80**
t = **10**	0.7000	0.6944	0.6928	0.6924
t = **25**	0.0800	0.0800	0.0800	0.0800
t = **40**	0.0000	0.0000	0.0000	0.0000

2.3 Inverse problem to determine the level of safety stock

From a production or trading company's point of view, a decision might be formulated to answer the following question: '*Given a maximum expected number of units short and/or a maximum stock-out probability the company wants to face, what should be the safety inventory at least (or at most)?*'. The question with the 'at most' option might be only of academic nature, as it reflects the most optimistic viewpoint. In human terms, this question would be interpreted as: 'Would there exist any probability distribution so that I can still reach my preset performance criteria given a specific level of safety inventory?'. This type of question is not relevant for a manager facing a real-life situation.

In (6) a method is proposed in which the integral constraints are transformed into a sequence, with increasing number of evaluation points, of optimisation problems and where the integral is replaced by an infinite sum. Instead of evaluating the objective function on a continuous interval [low,high], the functions are evaluated in a discrete number of points x_i ($i = 1,...,k$). This approach leads to the solution of the continuous problem if $k \rightarrow \infty$. The objective function contains only one decision variable, i.e. the reorder level. This leads to the following optimisation problem, where:

t = level of inventory
p_i = the mass in point x_i
z_1 = the expected value
z_2 = the absolute second moment
z_3 = the maximum expected number of items short

The optimisation problem might be formulated as:

$$Mint \tag{35}$$

Subject to

$$\sum_i p_i = 1 \tag{36}$$

$$\sum_i x_i * p_i = z_1 \qquad (x_i = i * \frac{b}{k}, i = 0, ..., k) \tag{37}$$

$$\sum_i x_i^2 * p_i = z_2 \qquad (x_i = i * \frac{b}{k}, i = 0, ..., k) \tag{38}$$

$$\sum_i (x_i - t)_+ * p_i \leq z_3 \qquad (x_i = i * \frac{b}{k}, i = 0, ..., k) \tag{39}$$

where $(x_i - t)_+$ stands for $max(x_i - t, 0)$.

The problem formulated above gives an answer to the following question: 'What is the minimal amount of inventory so that a distribution with given characteristics exists in which the expected number of units short maximally equals the value z_3?'.

The non-linear constraint may be approximated by letting the value of t coincide with one of the x_i values (so as $k \rightarrow \infty$, the approximation takes the correct value). In such a way the constraint is linearized.

In the case t coincides with one of the x_i values, say x_j, then:

$$\sum_{i=1}^{n} p_i(x_i - x_j)_+ \leq z_3 \tag{40}$$

A binary variable needs to be introduced to indicate the condition '$t = x_j$'. In the case t does not coincide with a point x_j, a general truth should be indicated, for example 'the expected number of units short cannot be larger than the expected demand', expressed by a binary variable y_j.

$$\begin{aligned} y_j &= 1 \text{ if } t = x_j \\ \text{else} &= 0 \end{aligned}$$

As t can coincide with only one x_j value, the additional constraint is introduced:

$$\sum_{j=1}^{n} y_j = 1 \tag{41}$$

The y variable is introduced in the last constraint as:

$$\sum_{i=1}^{n} p_i(x_i - x_j)_+ \leq z_3 y_j + z_1(1 - y_j) \tag{42}$$

Finally, a link should be made between t and the value of x with which t coincides:

$$t \geq x_j y_j \qquad \forall j \tag{43}$$

If $y_j = 0$, a universal truth is mentioned.

The elaboration above will be illustrated by means of a numerical example similar to the one used in the previous sections. With a range of demand [0,50], an expected value $z_1 = 25$ and a second moment $z_2 = 725$, Table 5 presents the minimal amount of inventory for different values of z_3 (maximum number of items short) and varying sizes k of the evaluation point set. The exact values for the minimal inventory level are 25 for $z_3 = 2$, 21 for $z_3 = 4$ and 19 for $z_3 = 6$.

Table 5: Minimal required inventory level based on the maximum expected number of units short

	k = **10**	k = **20**	k = **40**	k = **80**
z_3 = **2**	25.0000	25.0000	25.0000	25.0000
z_3 = **4**	25.0000	22.5000	21.2500	21.2500
z_3 = **6**	20.0000	20.0000	20.0000	19.3750

In case the company, given a maximum stock-out probability, is looking for the least level of safety inventory to satisfy that constraint, a similar optimisation problem (44)-(50) might be formulated, where:

t = level of inventory
p_i = the mass in point x_i
z_1 = the expected value
z_2 = the absolute second moment
z_4 = the maximum stock-out probability

This leads to the following optimisation problem:

$$Mint \tag{44}$$

Subject to

$$\begin{aligned}
\sum_i p_i &= 1 && (45)\\
\sum_i x_i * p_i &= z_1 \qquad (x_i = i * \frac{b}{k}, i = 0, ..., k) && (46)\\
\sum_i x_i^2 * p_i &= z_2 \qquad (x_i = i * \frac{b}{k}, i = 0, ..., k) && (47)\\
\sum_{j=1}^{n} y_j &= 1 && (48)\\
\sum_{i=1}^{n} p_i * 1_{[x_j, x_i]} &\leq z_4 y_j + (1 - y_j) && (49)\\
t &\geq x_j y_j && (50)
\end{aligned}$$

Constraints (45) till (50) are equal to those in the previous optimisation problem. Constraint (49) makes use of the stock-out probability objective in the left-hand side and uses the value 1 instead of z_1 in the right-hand side to make the universal truth come true.

Using the same numerical example Table 6 presents the minimal amount of inventory for different values of z_4 (maximum stock-out probability) and varying amounts of measuring values. The exact values for the minimal inventory level are 23.75 for $z_4 = 0.1$, 20 for $z_4 = 0.2$ and 15 for $z_4 = 0.5$.

Table 6: Minimal required inventory level based on the maximum stock-out probability

	$k = \mathbf{10}$	$k = \mathbf{20}$	$k = \mathbf{40}$	$k = \mathbf{80}$
$z_4 = \mathbf{0.1}$	25.0000	25.0000	23.7500	23.7500
$z_4 = \mathbf{0.2}$	20.0000	20.0000	20.0000	20.0000
$z_4 = \mathbf{0.5}$	15.0000	15.0000	15.0000	15.0000

In the company, the inventory manager might have target values for both performance measures (number of stock-out units and stock-out probability) in mind, looking for a safety inventory level satisfying both his targets. The linear programs formulated in this section smoothly allow the manager to make use of his combination in mind.

The combination will be illustrated by means of the same numerical example in which all three illustrative values for the maximum expected number of units short (z_3) from Table 5 are combined with all three values for the maximum stock-out probability (z_4) in Table 6. The formulation of the linear program combining constraints both on expected number of units short and on stock-out probability is given below. The results for the nine combinations are shown in Table 7. In most cases one of both constraints will be the binding constraint. Take the example of $z_4 = 0.1$, which leads to a value $t = 23.75$. When combined with $z_3 = 2$, the value is increased to $t = 25$. But when combined with $z_3 = 4$ or $z_3 = 6$, it remains unchanged because these z_3-values impose less strict constraints.

$$Mint \tag{51}$$

Subject to

$$\sum_i p_i = 1 \tag{52}$$

$$\sum_i x_i * p_i = z_1 \qquad (x_i = i * \frac{b}{k}, i = 0, ..., k) \tag{53}$$

$$\sum_i x_i^2 * p_i = z_2 \qquad (x_i = i * \frac{b}{k}, i = 0, ..., k) \tag{54}$$

$$\sum_{j=1}^{n} y_j = 1 \tag{55}$$

$$\sum_{i=1}^{n} p_i (x_i - x_j)_+ \leq z_3 y_j + z_1 (1 - y_j) \tag{56}$$

$$\sum_{i=1}^{n} p_i * 1_{[x_j, x_i]} \leq z_4 y_j + (1 - y_j) \tag{57}$$

$$t \geq x_j y_j \tag{58}$$

Table 7: Minimal required inventory level based on the maximum expected number of units short (z_3) and the maximum stock-out probability (z_4)

	k = **10**	k = **20**	k = **40**	k = **80**
z_3 = **2 and** z_4 = **0.1**	25.0000	25.0000	25.0000	25.0000
z_3 = **2 and** z_4 = **0.2**	25.0000	25.0000	25.0000	25.0000
z_3 = **2 and** z_4 = **0.5**	25.0000	25.0000	25.0000	25.0000
z_3 = **4 and** z_4 = **0.1**	25.0000	25.0000	23.7500	23.7500
z_3 = **4 and** z_4 = **0.2**	25.0000	22.5000	21.2500	21.2500
z_3 = **4 and** z_4 = **0.5**	25.0000	22.5000	21.2500	21.2500
z_3 = **6 and** z_4 = **0.1**	25.0000	25.0000	23.7500	23.7500
z_3 = **6 and** z_4 = **0.2**	20.0000	20.0000	20.0000	20.0000
z_3 = **6 and** z_4 = **0.5**	20.0000	20.0000	20.0000	19.3750

Figure 1 shows the minimal required re-order point (called 'Inventory level') on the x-axis for obtaining either a maximum expected number of units short (called 'Max number of units short') on the left y-axis or a maximum shortage probability (called 'Max shortage probability') on the right y-axis. The manager can easily obtain, from this figure, the required inventory level selecting the lower values of 'Inventory level' corresponding to both of his targets. The detailed Lingo code is written in appendix for a combination of $z_3 = 2$ and $z_4 = 0.1$ for the numerical example under study throughout the whole paper.

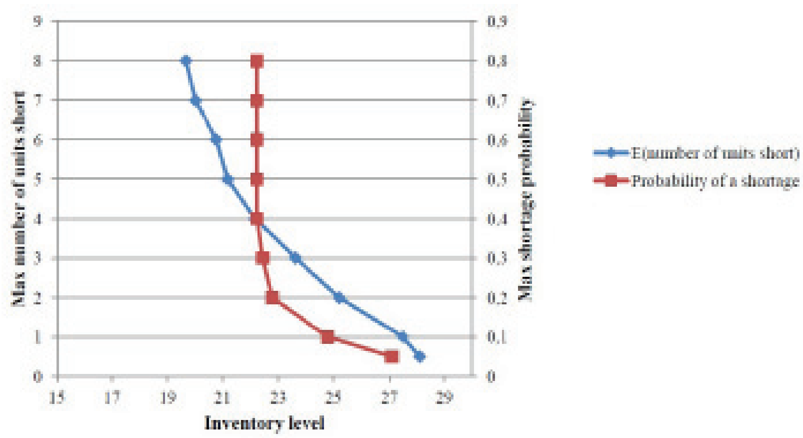

Figure 1: Minimal required inventory level based on the maximum expected number of units short and the maximum stock-out probability

Conclusions

A logistics manager should determine the level of safety stock for goods in inventory based on service level criteria like the expected number of units short or the stock-out probability per replenishment cycle. The decision makes use of the probability distribution of demand during lead time, but if that distribution is not fully specified, the decision maker is confronted with an additional uncertainty. In that case, he might choose to find the least value of safety stock that leads to the required service level criteria values in the worst cases. This paper offers a tool to support the manager in his decision. The answer to his decision question is formulated as a linear program. The paper shows how such a linear program can be used in the case the information on the distribution of demand during lead time is limited to the range, the expected value and the absolute second moment. The method is illustrated both for the criteria of 'expected number of units short' and 'stock-out probability', and the combination of both performance criteria.

Appendix

Lingo code for the combination model with $z_3 = 2$ and $z_4 = 0.1$:

(1) $MIN = D;$
(2) $PROB_1 + PROB_2 + PROB_3 + PROB_4 + PROB_5 + PROB_6 + PROB_7 + PROB_8 + PROB_9 + PROB_{10} + PROB_{11} = 1;$
(3) $5 * PROB_2 + 10 * PROB_3 + 15 * PROB_4 + 20 * PROB_5 + 25 * PROB_6 + 30 * PROB_7 + 35 * PROB_8 + 40 * PROB_9 + 45 * PROB_{10} + 50 * PROB_{11} = 25;$
(4) $25 * PROB_2 + 100 * PROB_3 + 225 * PROB_4 + 400 * PROB_5 + 625 * PROB_6 + 900 * PROB_7 + 1225 * PROB_8 + 1600 * PROB_9 + 2025 * PROB_{10} + 2500 * PROB_{11} = 725;$
(5) $Y_1 + Y_2 + Y_3 + Y_4 + Y_5 + Y_6 + Y_7 + Y_8 + Y_9 + Y_{10} + Y_{11} = 1;$
(6) $PROB_1 >= 0;$
(7) $PROB_2 >= 0;$
(8) $PROB_3 >= 0;$
(9) $PROB_4 >= 0;$
(10) $PROB_5 >= 0;$
(11) $PROB_6 >= 0;$
(12) $PROB_7 >= 0;$
(13) $PROB_8 >= 0;$
(14) $PROB_9 >= 0;$
(15) $PROB_{10} >= 0;$
(16) $PROB_{11} >= 0;$
(17) $-D <= 0;$
(18) $-D + 5 * Y_2 <= 0;$
(19) $-D + 10 * Y_3 <= 0;$
(20) $-D + 15 * Y_4 <= 0;$
(21) $-D + 20 * Y_5 <= 0;$
(22) $-D + 25 * Y_6 <= 0;$
(23) $-D + 30 * Y_7 <= 0;$
(24) $-D + 35 * Y_8 <= 0;$
(25) $-D + 40 * Y_9 <= 0;$
(26) $-D + 45 * Y_{10} <= 0;$
(27) $-D + 50 * Y_{11} <= 0;$
(28) $23 * Y_1 + 5 * PROB_2 + 10 * PROB_3 + 15 * PROB_4 + 20 * PROB_5 + 25 * PROB_6 + 30 * PROB_7 + 35 * PROB_8 + 40 * PROB_9 + 45 * PROB_{10} + 50 * PROB_{11} <= 25;$

(29) $23 * Y_2 + 5 * PROB_3 + 10 * PROB_4 + 15 * PROB_5 + 20 * PROB_6 + 25 * PROB_7 + 30 * PROB_8 + 35 * PROB_9 + 40 * PROB_{10} + 45 * PROB_{11} <= 25;$
(30) $23 * Y_3 + 5 * PROB_4 + 10 * PROB_5 + 15 * PROB_6 + 20 * PROB_7 + 25 * PROB_8 + 30 * PROB_9 + 35 * PROB_{10} + 40 * PROB_{11} <= 25;$
(31) $23 * Y_4 + 5 * PROB_5 + 10 * PROB_6 + 15 * PROB_7 + 20 * PROB_8 + 25 * PROB_9 + 30 * PROB_{10} + 35 * PROB_{11} <= 25;$
(32) $23 * Y_5 + 5 * PROB_6 + 10 * PROB_7 + 15 * PROB_8 + 20 * PROB_9 + 25 * PROB_{10} + 30 * PROB_{11} <= 25;$
(33) $23 * Y_6 + 5 * PROB_7 + 10 * PROB_8 + 15 * PROB_9 + 20 * PROB_{10} + 25 * PROB_{11} <= 25;$
(34) $23 * Y_7 + 5 * PROB_8 + 10 * PROB_9 + 15 * PROB_{10} + 20 * PROB_{11} <= 25;$
(35) $23 * Y_8 + 5 * PROB_9 + 10 * PROB_{10} + 15 * PROB_{11} <= 25;$
(36) $23 * Y_9 + 5 * PROB_{10} + 10 * PROB_{11} <= 25;$
(37) $23 * Y_{10} + 5 * PROB_{11} <= 25;$
(38) $23 * Y_{11} <= 25;$
(39) $0.9 * Y_1 + PROB_2 + PROB_3 + PROB_4 + PROB_5 + PROB_6 + PROB_7 + PROB_8 + PROB_9 + PROB_{10} + PROB_{11} <= 1;$
(40) $0.9 * Y_2 + PROB_3 + PROB_4 + PROB_5 + PROB_6 + PROB_7 + PROB_8 + PROB_9 + PROB_{10} + PROB_{11} <= 1;$
(41) $0.9 * Y_3 + PROB_4 + PROB_5 + PROB_6 + PROB_7 + PROB_8 + PROB_9 + PROB_{10} + PROB_{11} <= 1;$
(42) $0.9 * Y_4 + PROB_5 + PROB_6 + PROB_7 + PROB_8 + PROB_9 + PROB_{10} + PROB_{11} <= 1;$
(43) $0.9 * Y_5 + PROB_6 + PROB_7 + PROB_8 + PROB_9 + PROB_{10} + PROB_{11} <= 1;$
(44) $0.9 * Y_6 + PROB_7 + PROB_8 + PROB_9 + PROB_{10} + PROB_{11} <= 1;$
(45) $0.9 * Y_7 + PROB_8 + PROB_9 + PROB_{10} + PROB_{11} <= 1;$
(46) $0.9 * Y_8 + PROB_9 + PROB_{10} + PROB_{11} <= 1;$
(47) $0.9 * Y_9 + PROB_{10} + PROB_{11} <= 1;$
(48) $0.9 * Y_{10} + PROB_{11} <= 1;$
(49) $0.9 * Y_{11} <= 1;$
$@FREE(D);$
$@BIN(Y_1); @BIN(Y_2); @BIN(Y_3); @BIN(Y_4); @BIN(Y_5); @BIN(Y_6);$
$@BIN(Y_7); @BIN(Y_8); @BIN(Y_9); @BIN(Y_{10}); @BIN(Y_{11});$
$@FREE(PROB_1); @FREE(PROB_2); @FREE(PROB_3); @FREE(PROB_4);$
$@FREE(PROB_5); @FREE(PROB_6); @FREE(PROB_7); @FREE(PROB_8);$
$@FREE(PROB_9); @FREE(PROB_{10}); @FREE(PROB_{11});$
END

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