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ON SOME CLASSES OF FRACTIONAL STOCHASTIC DYNAMICAL SYSTEMS

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Abstract

A simple method for investigating two classes of fractional stochastic dynamical systems previously studied by other authors is proposed. Based on a practical approach to fractional SDE's, the method gives an effective and easy way for practioners to solve fractional stochastic problems related to these classes. Some well-known particular cases are discussed using this approach.

1. Introduction

It is known that there exist phenomena where the occurrence of an event may influence upon what that happens long time later. This fact requires studies on stochastic dynamical systems of long-range dependence. And those on systems driven by a fractional Brownian motion (fBm) have expressed to be a good reply.

A fBm W_t^H with Hurst index $H \in (0,1)$ is a centered Gaussian process such that its covariance function $R(t,s) = EW_t^H W_s^H$ is given by

$$R(t,s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

For $H = \frac{1}{2}$, W_t^H is the usual standard Brownian motion W_t.
For $H \neq \frac{1}{2}$, W_t^H is not a martingale as W_t; it is a process having memory. That is why the mechanism of Ito calculus cannot be applied.

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Many achievements have been made for fractional stochastic dynamics driven by a fBm from various approaches. A short story of the development can be found in the Introduction of papers [1,2] from there we can cite some best contributions by T. E. Duncan, D.Nualart, L. Decreusefond, M. Z*ä*hle, M. Taqqu, R. J. Elliott, ...[3-10].

In this paper we discuss about the two classes of dynamical systems expressed by fractional stochastic differential equations:

$$dX_t = f(t, X_t)dt + c(t)X_t dB_t^H, \ X(t=0) = X_0 \tag{1.1}$$

and

$$dX_t = f(t, X_t)dt + c(t)dB_t^H, \ X(t=0) = X_0 \tag{1.2}$$

where B_t^H is a fractional Brownian motion.
We would like to propose a simple method for proving the solution's existence and uniqueness and solving these classes of equations under some assumptions that are not difficult to be checked.
From a practical approach to the theory, T. H. Thao and his colleagues have studied on fractional stochastics driven by a fBm of Liouville form (LfBm) based on a crucial fact that any LfBm can be approached in the space $L^2(\Omega, \mathcal{F}, P)$ by semimartingales ([11-19]).
A LfBm B_t^H is defined by

$$B_t^H = \int_0^t (t-s)^\alpha dW_s$$

where $\alpha = H - 1/2$ and W_t is a standard Brownian motion. It is related to the Mandelbrot form W_t^H mentioned above by

$$W_t^H = C_H(Z_t + B_t^H)$$

where Z_t is a process of absolutely continuous sample paths and C_H is a constant depending only to $H \in (0,1)$ and therefore the long memory property focuses at the process B_t^H. We will work with B_t^H instead of W_t^H in our study.

D. Nualart and al. in [4] have introduced the semimartingale

$$B_t^{H,\epsilon} = \int_0^t (t-s+\epsilon)^\alpha dW_s,$$

with

$$dB_t^{H,\epsilon} = \alpha\varphi_t^\epsilon dt + \epsilon^\alpha dW_t \ \text{ for every } \epsilon > 0$$

where

$$\varphi_t^\epsilon = \int_0^t (t-s+\epsilon)^{\alpha-1} dW_s.$$

We have proved in [11] that $B_t^{H,\epsilon}$ converges uniformly in $t \in [0,T]$ to B_t^H. So after constructing an approximation theory for fractional stochastic integration and differential equations ([1, 11, 12, 13]) we see that a common scheme for solving a fractional model can be described as follows:

1. Replace B_t^H by $B_t^{H,\epsilon}$ in the model to obtain an approximation model driven by a semimartingale.

2. Solve the approximation model by the traditional stochastic calculus.

3. Prove that approximate solutions converges to the solution of the initial model as $\epsilon \to 0$.

The organization of this paper is as follows: After the introduction and recalling some of our previous results on fractional stochastic integration and differential equations, we show in Section 3 and Section 4 how to prove the existence and uniqueness of two classes of equations (1.1) and (1.2) and how to solve them in a simple way. Some of well-known particular cases are considered by this method. An application of local linearization method and a fractional Merton's model for credit risk are discussed as well in Sections 5 and 6.

Throughout the paper,except for Section 2, the Hurst index is supposed to be large than 1/2: $1/2 < H < 1$.

2. Recall on fractional stochastic integration and differential equations

We would like to make a brief recall on our previous results [1,11] that are needed for using later.

2.1. Integration

(a) If $\{f(t,\omega), 0 \leq t \leq T\}$ is a process having trajectories of bounded variation such that

$$E[\int_0^T f^2(t,\omega)dt] < \infty \tag{2.1}$$

then we can define the integral I_t as follows:

$$I_t = \int_0^t f(s,\omega)dB_t^H = f(t,\omega)B_t^H - \int_0^t B_s^H df(s,\omega) - [f, B^H]_t \tag{2.2}$$

where $[f,B]_t$ is the quadratic variation of $f(t,\omega)$ and B_t^H.
(b) Now we consider the usual stochastic integral I_t^ϵ given by

$$I_t^\epsilon = \int_0^t f(s,\omega)dB_t^{H,\epsilon},$$

where $f(s,\omega)$ is the function mentioned above and $B_t^{H,\epsilon}$ is the semimartingale $B_t^{H,\epsilon} = \int_0^t (t-s+\epsilon)^\alpha dW_s,\ \alpha = H - 1/2,\ 0 < H < 1$.
(c) We can prove that I_t^ϵ converges to I_t defined by (2.1) in the space $L^2(\Omega, \mathcal{F}, P)$ as $\epsilon \to 0$ and this convergence is uniform with respect to $t \in [0,T]$:

$$\int_0^t f(s,\omega) dB_s^H = L^2 - \lim_{\epsilon \to 0} \int_0^t f(s,\omega) dB_t^{H,\epsilon}. \tag{2.3}$$

(d) Motivated by the fact (c), we can define the integral for a general stochastic process $f(t,\omega)$ satisfying (2.1) by the relation (2.3) provided the L^2-limit exists and is uniform with respect to $t \in [0,T]$.

2.2. Fractional stochastic differential equation.

(a) Consider a formal equation

$$dX_t = b(t,X_t)dt + \sigma(t,X_t)dB_t^H \tag{2.4}$$

$$X_t|_{t=0} = X_0,\ \ 0 \le t \le T$$

where $b(t,x)$ and $\sigma(t,x)$ are two continuous functions and X_0 is a given random variable such that $E[X_0^2] < \infty$.
(b) The relation (2.4) means that

$$X_t = X_0 + \int_)^t b(s,X_s)ds + \int_0^t \sigma(s,X_s)dB_s^H,\ \ t \in [0,T] \tag{2.5}$$

if the stochastic integral in (2.5) exists.
(c) A solution of equation (2.4) is defined as a stochastic process $(X_t, 0 \le t \le T)$ adapted to the $\sigma-$algebra $\mathcal{F}_t = \sigma(X_0, B_s^H, 0 \le s \le t \le T)$ and satisfying (2.5).
(d) Existence and Uniqueness for solution.
Theorem 3.1 in [1] says that if $b(t,x)$ and $\sigma(t,x)$ satisfies the following conditions:

(i) they are Lipschitzian with respect to x,

(ii) if $\int_0^t \sigma(s,X_s)dB_s^H$ and $\int_0^t \sigma(s,Y_s)dB_s^H$ exist so does $\int_0^t \sigma(s,\alpha X_s + \beta Y_s)dB_s^H$, where α and β are any real constants and the fractional integrals are defined in the sense of (2.3),

then there exists a unique solution of (2.4).

3. On the equation of form

$$dX_t = f(t, X_t)dt + c(t)X_t dB_t^H \tag{3.1}$$

Some authors have studied this class of equations among them N. T. Dung ([2]) has used the technique of "integrating factor" by B. Oksendal ([20]) and method of R. Carmona ([21]) to prove a theorem of existence and uniqueness for solution in his sense by convoking Malliavin calculus although he has been starting from our approach of semimartingale L^2-approximation.

In order to facilitate application works in practice, we would like to discuss here how to investigate this kind of equation by using our simple method of fractional stochastics mentioned in Section 2.

Proposition 3.1 *Suppose that $f(t, x)$ is a function satisfying Lipschitz condition with respect to x and $c(t)$ is a deterministic continuous function on $[0, T]$. Then there exists a unique solution X_t for Equation (3.1) given by*

$$X_t = L^2 - \lim_{\epsilon \to 0} X_t^\epsilon \tag{3.2}$$

where

$$X_t^\epsilon = yF_t^{-1},$$

$$F_t = \exp\,[- \int_0^t c(s)\epsilon^\alpha dW_s + \frac{1}{2}\int_0^t \epsilon^{2\alpha} c^2(s)ds] \tag{3.4}$$

and $y = y(t)$ is the solution of the following ordinary differential equation:

$$\frac{dy}{dt} = F_t[f(t, yF_t^{-1}) + \alpha\varphi_t^\epsilon yF_t^{-1}] \tag{3.5}$$

Proof The existence and uniqueness for the solution X_t of (3.1) is clear because it is easy to verify that coefficients $b(t, x) = f(t, x)$ and $\sigma(t, x) = c(t)x$ satisfy all conditions of theorem 3.1 of [1] that is recalled in 2.3.c.
The existence and uniqueness of the solution y of (3.5) is resulted from a simple technique of "integrating factor" as presented in [20]
Finally, the convergence (3.2) is assured by Theorem 4.1 in [1]. □

A linear equation.

In this subsection, we consider an important particular case of (3.1). We will see how the mechanism given in Proposition 3.1 works for this case. Also the problem of convergence $X_t^\epsilon \longrightarrow X_t$ in L^2 can be treated by some remark that is much simpler than the method used in [2].

Proposition 3.2. *Consider the equation*

$$dX_t = [a(t)X_t + b(t)]dt + c(t)X_t dB_t^H,\ 0 \le t \le T \tag{3.2.1}$$

$$X_t|_{t=0} = X_0$$

where $a(t)$, $b(t)$ and $c(t)$ are real continuous function on $[0, T]$. Then the unique solution of (3.2.1) is given by

$$X_t = \exp\,[\int_0^t a(s)ds + \int_0^t c(s)dB_s^H][X_0 + \int_0^t b(s)e^{-\int_0^s c(u)dB_u^H - \int_0^s a(u)du}ds] \tag{3.2.2}$$

Proof Firstly we can say that equation (3.2.1) satisfies all conditions of Proposition 3.1 and it has a unique solution which can be found explicitly by our method of approximation.
The corresponding approximate equation to (3.2.1) is

$$dX_t^\epsilon = [a(t)X_t^\epsilon + b(t)]dt + c(t)X_t^\epsilon dB_t^{H,\epsilon}$$

and set $X_0^\epsilon = X_0$.

$$dX_t^\epsilon = [(a(t) + \alpha\varphi_t^\epsilon c(t))X_t^\epsilon + b(t)]dt + \epsilon^\alpha c(t)X_t^\epsilon dW_t \tag{3.2.3}$$

Put

$$\bar{f}(t, X_t^\epsilon) = (a(t) + \alpha\varphi_t^\epsilon c(t))X_t^\epsilon + b(t)$$

$$F_t = \exp(-\int_0^t \epsilon^\alpha c(s)dW_s + \frac{1}{2}\int_0^t \epsilon^{2\alpha}c^2(s)ds)$$

and

$$y = y(t, \omega) = F_t X_t^\epsilon.$$

Equation (3.1.5)now becomes

$$\begin{aligned}\frac{dy}{dt} &= F_t\bar{f}(t, F_t^{-1}y)\\ &= F_t[(a(t) = \alpha\varphi_t^\epsilon c(t))F^{-1}y + b(t)]\end{aligned}$$

or

$$\frac{dy}{dt} - (a(t) + \alpha\varphi_t^\epsilon c(t))y = b(t)F_t \tag{3.2.4}$$

This is an ordinary linear differential equation with continuous coefficients on $[0, T]$ so it has a unique solution expressed by

$$\begin{aligned}y = \exp\big(&\int_0^t a(s)ds + \int_0^t c(s)dB_s^{H,\epsilon} - \epsilon^\alpha \int_0^t c(s)dW_s\big)\times\\ &\times \big\{y_0 + \int_0^t b(s)\exp[-\int_0^s c(u)dB_u^{H,\epsilon} - \int_0^s (a(u) + \frac{1}{2}\epsilon^{2\alpha}c^2(u))du]ds\big\}\end{aligned} \tag{3.2.5}$$

where $y_0 = F_0 X_0^\epsilon = X_0^\epsilon = X_0$.
And we have the solution X_t^ϵ of (3.2.3) as

$$X_t^\epsilon = F_t^{-1} y = \exp(\int_0^t \epsilon^\alpha c(s) dW_s - \frac{1}{2}\int_0^t \epsilon^{2\alpha} c^2(s)ds).y$$

$$\begin{aligned} X_t^\epsilon =& \exp(\int_0^t a(s)ds + \int_0^t c(s)dB_s^{H,\epsilon} - \frac{\epsilon^{2\alpha}}{2}\int_0^t c^2(s)ds)\times \\ &\times \{y_0 + \int_0^t b(s)\exp[-\int_0^s c(u)dB_u^{H,\epsilon} - \int_0^s (a(u) + \frac{1}{2}\epsilon^{2\alpha}c^2(u))du]ds\} \end{aligned} \tag{3.2.6}$$

In order to find L^2-limit of X_t^ϵ as $\epsilon \to 0$ from its expression (3.2.6) we notice a simple remark as follows:
Remark 3.2: Suppose f and g are two functions $\in L^2$. We can say

1. If f is Lipschitzian then $L^2-\lim f(g) = f(L^2-\lim g)$
2. $L^2-\lim f.g = (L^2-\lim f).(L^2-\lim g)$ if f and L^2-lim$g$ or g and L^2-lim$f$ are Lipschitzian.
3. $L^2-\lim(f+g) = (L^2-\lim f) + (L^2-\lim g)$

In taking account of this remark and noticing that the function e^x is Lipschitzian in any finite interval of x, we can conclude that X_t^ϵ converges in $L^2(\Omega, \mathcal{F}, P)$ to the solution X_t of the equation (3.2.1) given by

$$X_t = e^{\int_0^t a(s)ds + \int_0^t c(s)dB_s^H}(X_0 + \int_0^t b(s)e^{-\int_0^s c(u)dB_u^H - \int_0^s a(u)du}ds) \tag{3.2.7}$$

or in a brief form

$$X_t = e^{\psi(t)}(X_0 + \int_0^t b(s)e^{-\psi(s)ds}) \tag{3.2.7'}$$

where

$$\psi(t) = \exp(\int_0^t a(s)ds + \int_0^t c(s)dB_s^H). \tag{3.2.8}$$

□

In the case where $a(t) = \mu = const, b(t) \equiv 0$ and $c(t) = \sigma = const > 0$ we have the equation

$$dX_t = \mu X_t dt + \sigma X_t dB_t^H \tag{3.2.9}$$

whose solution is a fractional geometric Brownian motion

$$X_t = X_0 \exp(B_t^H + \mu t) \tag{3.2.10}$$

Equation (3.2.9) is used also for a fractional Black-Scholes model. It expresses the fractional dynamics of a stock price.

4. On the equation of form

$$dX_t = f(t, X_t)dt + c(t)dB_t^H, 0 \leq t \leq T \tag{4.1}$$

Dealing with this equation, the author of [2] announced that the existence and uniqueness for its solution was proved by Y. Mishura in her excellent book [22]. In fact there may be some confusions: Firstly, the driving fractional Brownian motion used in Mishura's book is of Mandelbrot form, not in Liouville form. Secondly, the theorem of existence (Section 3.5.1 in [22]) is for a weak solution that was defined by Mishura. It is not a theorem of existence and uniqueness for a solution that may be obtained by the approximation approach used by N. T. Dung.

And with our Theorem 3.1 and Theorem 4.1 in [1], we can see that if $f(t, x)$ and $c(t)$ are supposed to be as in Section 3.1, the existence and uniqueness for solution of (4.1) is obviously assured.

The approximation equation corresponding to (4.1) is

$$dX_t^\epsilon = [f(t, X_t^\epsilon) + \alpha\varphi_t^\epsilon c(t)]dt + \epsilon^\alpha c(t)dW_t \tag{4.1.1}$$

This is an Ito equation and its solution X_t^ϵ converges to the solution X_t of (4.1).

In general there is no explicit form for X_t^ϵ, so not for X_t as well, except for some particular cases for example when $f(t, x)$ is a linear function and $c(t)$ is constant.

4.1. Another linear equation

Consider the equation

$$dX_t = (a(t)X_t + b(t))dt + c(t)dB_t^H, 0 \leq t \leq T \tag{4.1.2}$$

where $a(t), b(t)$ and $c(t)$ are continuous functions on $[0, T]$.

Proposition 4.1 *The solution of (4.1.2) is given by*

$$X_t = e^{\int_0^t a(s)ds}[\int_0^t b(s)e^{-\int_0^s a(u)du}ds + \int_0^t c(s)e^{-\int_0^s a(u)du}dB_s^H + X_0] \tag{4.1.3}$$

Proof The way of finding the solution X_t^ϵ of (4.1.1) can be found in [19] by using a technique of "splitting equation" firstly introduced by T. H. Thao and T. T. Nguyen ([23]). X_t is obtained immediately from X_t^ϵ in taking account of Remark 3.2. □

4.2 Fractional Brownian bridge.

Consider the equation

$$dX_t = (-\frac{1}{1-t}X_t + \frac{b}{1-t})dt + dB_t^H, \tag{4.2.1}$$

$$X_t|_{t=0} = a$$

Its corresponding approximation equation is

$$dX_t^\epsilon = (-\frac{1}{1-t}X_t^\epsilon + \frac{b}{1-t} + \alpha\varphi_t^\epsilon)dt + \epsilon^\alpha dW_t, \tag{4.2.2}$$

$$X_t^\epsilon|_{t=0} = a$$

Using the technique of "splitting equation" we put

$$X_t^\epsilon = X_1^\epsilon(t) + X_2^\epsilon(t),\ X_1^\epsilon(0) = a, X_2^\epsilon(0) = 0$$

where $X_1^\epsilon(t)$ and $X_2^\epsilon(t)$ are solutions of two following equations, respectively,

$$dX_1^\epsilon(t) = -\frac{1}{1-t}X_1^\epsilon(t)dt + \epsilon^\alpha dW_t, \tag{4.2.3}$$

$$X_1^\epsilon(0) = a$$

$$dX_2^\epsilon(t) = (-\frac{1}{1-t}X_2^\epsilon(t) + \frac{b}{1-t} + \alpha\varphi_t^\epsilon)dt, \tag{4.2.4}$$

$$X_2^\epsilon(0) = 0$$

The first equation is a stochastic Langevin equation and the second is an ordinary linear differential equation of order 1. Combining their solutions $X_1^\epsilon(t)$ and $X_2^\epsilon(t)$ and noticing that $\alpha\varphi_t^\epsilon dt + \epsilon^\alpha dW_t = dB_t^{H,\epsilon}$ then taking the L^2-limit of $X_t^* = X_1^*(t) + X_2^*(t)$ we have the fractional Brownian bridge X_t from a to b given by

$$X_t = a(1-t) + bt + (1-t)\int_0^t \frac{dB_s^H}{1-s} \tag{4.2.5}$$

We have to prove that the fractional noise $\xi_t := (1-t)\int_0^t \frac{dB_s^H}{1-s}$ appeared in (4.2.5) should converges almost surely to 0 as $t \to 1$.
Indeed, it suffices to show that

$$\xi_t^\epsilon := (1-t)\int_0^t \frac{dB_s^{H,\epsilon}}{1-s} \to 0 \text{ a.s. when } t \to 1, \tag{4.2.6}$$

where $dB_s^{H,\epsilon} = \alpha\varphi_s^\epsilon ds + \epsilon^\alpha dW_s,\ \alpha = H - 1/2 > 0$.
We have

$$\xi_t^\epsilon := \alpha(1-t)\int_0^t \frac{\varphi_s^\epsilon}{1-s}ds + \epsilon^\alpha(1-t)\int_0^t \frac{dW_s}{1-s} \tag{4.2.7}$$

For the second term of (4.2.7) where W_t is an usual standard Brownian motion, it is known already that

$$(1-t)\int_0^t \frac{dW_s}{1-s} \to 0 \text{ a. s. } \text{ when } t \to 1 \text{ ([20])}$$

This follows from an application of the Doob martingale inequality and that of the Borel-Cantelli lemma. It is easy to extend this result as

$$(1-t)\int_0^t \frac{dW_s}{(1-s)^\alpha} \to 0 \text{ a. s. } \text{ when } t \to 1 \text{ for } \alpha > 0. \tag{4.2.8}$$

Now we have

$$\begin{aligned}
\int_0^t \frac{\varphi_s^\epsilon}{1-s} ds &= \int_0^t \Big(\frac{1}{1-s}\int_0^s (s-u+\epsilon)^{\alpha-1} dW_u\Big) ds \\
&= \int_0^t \Big(\int_u^t \frac{1}{1-s}(s-u+\epsilon)^{\alpha-1} ds\Big) dW_u \\
&= \int_0^t \Big(\int_u^t \frac{ds}{(1-s)(s-u+\epsilon)^{1-\alpha}}\Big) dW_u \\
&\le \int_0^t \Big(\int_u^t \frac{ds}{(1-s)(s-u)^{1-\alpha}}\Big) dW_u \\
&\le \int_0^t \Big(\int_u^t \frac{ds}{(1-u)(1-u)^{1-\alpha}}\Big) dW_u \text{ since } 0 < u < s < t < 1.
\end{aligned}$$

Noticing that $t-u<1$ and

$$\int_u^t \frac{ds}{(1-u)^\alpha} = \frac{t-u}{(1-u)^\alpha} < \frac{1}{(1-u)^\alpha}$$

we get almost surely

$$(1-t)\int_0^t \frac{\varphi_s^\epsilon}{1-s} \le (1-t)\int_0^t \frac{dW_u}{(1-u)^\alpha} \to 0 \text{ as } t \to 1$$

in taking account of (4.2.8).
Therefore $\xi_t^\epsilon \to 0$ a.s. when $t \to 1$.

4.3 Fractional Ornstein-Uhlenbeck processes

We know that in the classical stochastic Langevin dynamics described by the equation

$$dX_t = -bX_t dt + \sigma dW_t \tag{4.3.1}$$

$$X_t|_{t=0} = X_0, 0 \le t \le T,$$

and $b, \sigma : const > 0$, the solution is an Ornstein-Uhlenbeck process having the form

$$X_t = X_0 e^{-bt} + \sigma \int_0^t e^{-b(t-s)} dW_s \tag{4.3.2}$$

It has useful for study on volatility in various problems on stochastic dynamical systems.

The fractional partner of (4.3.1) is the equation

$$dX_t = -bX_t dt + \sigma dB_t^H \tag{4.3.3}$$

where B_t^H is a Liouville fractional Brownian motion.

Some authors such as P. Cheridito, H. Kawaguchi, M. Maejima [24,25] have results on fractional Langevin equation driven by a fBm of Mandelbrot form. Turning our attention to (4.3.3) we see that it is only a particular case of (4.1.3) and it has been solved explicitly by the technique of "splitting equation" in [23,2]. Its solution is

$$X_t = X_0 e^{-bt} + \sigma \int_0^t e^{-b(t-s)} dB_s^H \tag{4.3.4}$$

It is called a fractional Ornstein-Uhlenbeck process that has many applications in practice, for example in the study on fractional volatility in a financial evolution dynamics.

We can see again here an advantage of our approximate approach: it supplies a simple method for solving the problem and the analogous form of (4.3.4) in comparison of (4.3.3) can facilitate some similar calculations.

5. Local Linearization Method

The local linearization method was introduced by T. Ozaki [26] in considering nonlinear time series and dynamical system. This method firstly was an attempt to obtain a numerical scheme in the form of a linear multivariate autogressive time series with state-dependent coefficient. R. Biscay and al. [27] has extended to general scalar SDE's and to non-autonomous multidimensional SDE's with additive noise.

In this section we consider how to apply this method for one-dimensional scalar non- autonomous fractional stochastic differential equations (3.1) and (4.1) studied above.

For both cases we make the approximation

$$f(t,x) \approx f(t_0,x_0) + \frac{\partial f}{\partial x}(t_0,x_0)(x-x_0) + \frac{\partial f}{\partial t}(t_0,x_0)(t-t_0) \tag{5.1}$$

where $x_0 = X(t_0)$ is a known deterministic value of X at a moment t_0. In some practical problems one needs only to receive values of the solution X_t

near from a known value x_0. And such an approximation may be useful to avoid complicated calculations.

5.1 Local Linearization for Equation (3.1).

The approximation for (3.1) is the linear equation

$$dX_t = [aX_t + b(t) + d]dt + c(t)X_t dB_t^H \tag{5.1.1}$$

where

$$a = \frac{\partial f}{\partial x}(t_0, x_0) = const,$$

$$b(t) = \frac{\partial f}{\partial t}(t_0, x_0)t$$

,

$$d = f(t_0, x_0) - x_0\frac{\partial f}{\partial x}(t_0, x_0) - t_0\frac{\partial f}{\partial t}(t_0, x_0) = const.$$

Equation (5.1.1) is a particular case of (3.2.1)that can be solved explicitly by (3.2.7).

5.2. Local Linearization for Equation (4.1)

The approximation equation is

$$dX_t = [aX_t + b(t) + d]dt + c(t)dB_t^H \tag{5.2.1}$$

It is a particular case of (4.1.2) with the explicit solution (4.1.3).

6. An application for Finance:

A fractional Merton model for default risk.

In the classical Merton's model (1974) for valuing corporate liabilities, the firm's asset value is assumed to follow a diffusion process given by

$$dV_t = rV_t dt + \sigma_v V_t dW_t, \tag{6.1}$$

where σ_v is the asset volatility, r is the risk-free rate and W_t is a Brownian motion. In this model, all probabilities and expectations are taken under the risk neutral measure.
Now we consider a fractional analogous version of Merton model given by

$$dV_t = rV_t dt + \sigma V_t dB_t^H \tag{6.2}$$

where B_t^H is a Liouville fractional Brownian motion. If at some time t the asset's value of a firm is less than its total debt L_t that should be paid at that time and the company has not ability to pay for this, it will jump into default risky state.
The problem is how to estimate the possibility for the occurrence of this risky state. In other word, we have to consider the probability

$$P(V_t < L_t)$$

where P is a risk neutral probability.

It follows from (3.2.10) that the firm's value V_t satisfying (6.2) has the form

$$V_t = V_0 \exp(rt + \sigma B_t^H) \tag{6.3}$$

Then $V_t < L_t$ means that

$$\ln V_0 + rt + \sigma B_t^H < L_t.$$

And we have to calculate the probability

$$P(V_t < L_t) = P(B_t^H < x_t), \tag{6.4}$$

where $x_t := \frac{\ln \frac{V_0}{L_t} + rt}{-\sigma}$.

For each t, $B_t^H = \int_0^t (t-s)^\alpha dW_s$ is a Gaussian random variable with

$$E(B_t^H) = 0$$

$$\begin{aligned} \sigma_H^2 = Var(B_t^H) = E(B_t^H)^2 &= E(\int_0^t (t-s)^\alpha dW_s)^2 \\ = \int_0^t (t-s)^{2\alpha} ds &= \frac{t^{2\alpha+1}}{2\alpha+1} = \frac{t^{2H}}{2H} \end{aligned}$$

So

$$B_t^H \sim \mathcal{N}(0, \frac{t^{2H}}{2H})$$

and

$$P_{default} = P(B_t^H < x_t) = \frac{1}{\sigma_H \sqrt{2\pi}} \int_{-\infty}^{x_t} \exp(-\frac{x^2}{2\sigma_H^2}) dx,$$

or

$$P_{default} = \frac{\sqrt{H}}{t^H \sqrt{\pi}} \int_{-\infty}^{\frac{\ln(V_0/L_t)+rt}{-\sigma}} \exp(-\frac{H}{t^{2H}} x^2) dx. \tag{6.5}$$

A numerical example

We try to calculate default probabilities for a company following our fractional model in Section 6. Data are taken from Global Trust Bank (India) during numerous fiscal years, since 1997-1998 up to 2000-2003 [28]. Each value such as the total market value V_0 of the firm, the risky free rate r, the asset returns volatility σ, the debt L, is given for each fiscal year and the default is considered for maturity $t = 1$ because credit risk models routinely assume one-year time horizon for debt maturity.

Table 1. Case: $H = 0.54, t = 1$

	$t = 1$	$t = 1$	$t = 1$	$t = 1$	$t = 1$	$t = 1$
Year	1997-98	98-99	99-2000	2000-01	2001-02	2002-03
V_0	684.11	678.88	1100.76	1625.37	661.93	713.12
r	0.089	0.095	0.096	0.093	0.069	0.057
σ	0.345	0.258	0.338	0.506	0.276	0.130
L	256.39	593.24	661.42	365.775	421.55	511.68
H	0.54	0.54	0.54	0.54	0.54	0.54
P	0.0004088	0.1679888	0.0265376	0.0003600	0.0208928	0.0006162

Table 2. Case: $H = 0.55, t = 1$

	$t = 1$	$t = 1$	$t = 1$	$t = 1$	$t = 1$	$t = 1$
Year	1997-98	98-99	99-2000	2000-01	2001-02	2002-03
V_0	684.11	678.88	1100.76	1625.37	661.93	713.12
r	0.089	0.095	0.096	0.093	0.069	0.057
σ	0.345	0.258	0.338	0.506	0.276	0.130
L	256.39	593.24	661.42	365.775	421.55	511.68
H	0.55	0.55	0.55	0.55	0.55	0.55
P	0.0002313	0.1635527	0.0244118	0.0002861	0.0190700	0.0004990

Table 3. Case: $H = 0.56, t = 1$

	$t = 1$	$t = 1$	$t = 1$	$t = 1$	$t = 1$	$t = 1$
Year	1997-98	98-99	99-2000	2000-01	2001-02	2002-03
V_0	684.11	678.88	1100.76	1625.37	661.93	713.12
r	0.089	0.095	0.096	0.093	0.069	0.057
σ	0.345	0.258	0.338	0.506	0.276	0.130
L	256.39	593.24	661.42	365.775	421.55	511.68
H	0.56	0.56	0.56	0.56	0.56	0.56
P	0.0002554	0.1591935	0.0224308	0.0002265	0.0173843	0.0004027

Tables 1,2 and 3 corresponding to $H = 0.54,\ 0.55$, and 0.56 respectively are assumed to be according to an inference result for H-index in some model of Dow-Jones return [29]. One can see, the financial crisis in 1997-1998 has left consequences of default risk for some subsequent years later. This example is only a simple illustration for the model with many limitations of course. An application of the model into practice requires much more rigorous studies.

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