

GENERALIZED IDEAL CO-TRANSFORMS

Tran Tuan Nam and My Vinh Quang

*Ho Chi Minh University of Pedagogy
280 An Duong Vuong, District 5, Ho Chi Minh City, Vietnam
Email: namtuantran@gmail.com*

Abstract

We introduce the definition of generalized ideal co-transforms $C_i^I(M, N)$ which is a generalization of the definition of ideal co-transforms $C_i^I(M)$. We also study some basic properties of ideal co-transforms of linearly compact modules.

1 Introduction

Throughout this paper, (R, m) is a local noetherian commutative ring and has a topological structure. Let I be an ideal of R , in [14] we defined the i -th ideal co-transform $C_i^I(M)$ of M with respect to I (or i -th I -co-transform of M) by

$$C_i^I(M) = \varprojlim_t \operatorname{Tor}_i^R(I^t; M).$$

Note that the definition of $C_i^I(M)$ at degree zero ($i=0$) is in some sense dual to Brodmann's definition of ideal transforms ([1]). Also in [14] we studied some basic properties of ideal co-transforms $C_i^I(M)$ and found out relations to local homology modules $H_i^I(M) = \varprojlim_t \operatorname{Tor}_i^R(R/I^t, M)$.

In the paper we will define the i -th generalized ideal co-transform $C_i^I(M, N)$ by

$$C_i^I(M, N) = \varprojlim_t \operatorname{Tor}_i^R(I^t M, N).$$

This definition is in fact a generalization of the definition of ideal co-transforms. The organization of the paper is as follows.

Key words: Linearly compact module, (generalized) ideal co-transform, (generalized) local homology, (generalized) local cohomology.

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In section 2 we recall briefly some properties of linearly compact modules and generalized local homology modules that we shall use.

In section 3 we introduce the definition of generalized ideal co-transforms $C_i^I(M, N)$. If M is a finitely generated R -module and N is a linearly compact, then $C_i^I(M, N)$ ($i \geq 0$) is also linearly compact and the sequence of functors $\{C_i^I(M, -)\}$ is a positive strongly connected sequence on the category of linearly compact modules and continuous homomorphisms. On the other hand, we have a long exact sequence of linearly compact R -modules

$$\begin{aligned} \cdots \rightarrow H_{i+1}^I(M, N) \rightarrow C_i^I(M, N) \rightarrow \text{Tor}_i^R(M, N) \rightarrow H_i^I(M, N) \rightarrow \cdots \\ \cdots \rightarrow H_1^I(M, N) \rightarrow C^I(M, N) \xrightarrow{\eta_{M,N}} M \otimes_R N \xrightarrow{\theta_{M,N}} \Lambda_I(M \otimes_R N) \rightarrow 0. \end{aligned}$$

Moreover, if $pd(M) = p < +\infty$, then $H_{p+1}^I(M, N) \cong C_p^I(M, N)$ (Theorem 3.4).

Section 4 is devoted to study the relations between generalized ideal co-transforms and co-localization of linearly compact modules. In Theorem 4.2 we show that $C^I(M \otimes_R N) \cong C^I(M, N)$ and $C^{aR}(M, N) \cong {}_a C^{aR}(M, N)$ for any $a \in R$. The section is close by Theorem 4.3 in which we prove the isomorphism $C^{aR}(M, N) \cong {}_a(M \otimes_R N)$.

2 PRELIMINARIES

Let us begin by recalling briefly the definition and basic properties of linearly compact modules by terminology of I. G. Macdonald ([10]) that we shall use.

Let M be a topological R -module. M is said to be *linearly topologized* if M has a base of neighborhoods of the zero element \mathcal{M} consisting of submodules. M is called *Hausdorff* if the intersection of all the neighborhoods of the zero element is 0. A Hausdorff linearly topologized R -module M is said to be *linearly compact* if \mathcal{F} is a family of closed cosets (i.e., cosets of closed submodules) in M which has the finite intersection property, then the cosets in \mathcal{F} have a non-empty intersection (see [10, 10]).

It is clear that Artinian R -modules are linearly compact and discrete. We have some following properties of linearly compact modules.

Lemma 2.1. (see [10, §3]) (i) *If M is a Hausdorff linearly topologized R -module and N a closed submodule of M , then M is linearly compact if and only if N and M/N are linearly compact.*

(ii) *Let $f : M \rightarrow N$ be a continuous homomorphism of Hausdorff linearly topologized R -modules. If M is linearly compact, then $f(M)$ is linearly compact and f is a closed map.*

(iii) *The inverse limit of a system of linearly compact R -modules with continuous homomorphisms is linearly compact.*

Lemma 2.2. (see [8, 7.1]) *Let $\{M_t\}$ be an inverse system of linearly compact modules with continuous homomorphisms. Then $\varprojlim_t^i M_t = 0$ for all $i > 0$.*

Therefore, if

$$0 \longrightarrow \{M_t\} \longrightarrow \{N_t\} \longrightarrow \{P_t\} \longrightarrow 0$$

is a short exact sequence of inverse systems of R -modules, then the sequence of inverse limits

$$0 \longrightarrow \varprojlim_t M_t \longrightarrow \varprojlim_t N_t \longrightarrow \varprojlim_t P_t \longrightarrow 0$$

is exact.

Lemma 2.3. (see [3, §2]) *Let N be a finitely generated R -module and M a linearly compact R -module. Then for all $i \geq 0$, $\text{Tor}_i^R(N; M)$ is a linearly compact R -module. Moreover,*

(i) *If $f : N \longrightarrow N'$ is a homomorphism of finitely generated R -modules, then the induced homomorphism $f_{i,M} : \text{Tor}_i^R(N; M) \longrightarrow \text{Tor}_i^R(N'; M)$ is continuous.*

(ii) *If $g : M \longrightarrow M'$ is a continuous homomorphism of linearly compact R -modules, then the induced homomorphism $g_{N,i} : \text{Tor}_i^R(N; M) \longrightarrow \text{Tor}_i^R(N; M')$ is also continuous.*

Let I be an ideal of the ring R and M, N R -modules. In [12], the i -th generalized local homology module $H_i^I(M, N)$ of M, N with respect to I is defined by

$$H_i^I(M, N) = \varprojlim_t \text{Tor}_i^R(M/I^t M, N).$$

In the special case $M = R$, $H_i^I(R, N) = H_i^I(N)$ the i -th local homology module $H_i^I(N)$ of N with respect to I ([2], [3]).

Lemma 2.4. ([13, 3.4]) *Let M be a finitely generated module and N a linearly compact R -module. If N is complete in I -adic topology (i.e., $\Lambda_I(N) \cong N$), then there is an isomorphism for all $i \geq 0$,*

$$\text{Tor}_i^R(M, N) \cong H_i^I(M, N).$$

The i -th ideal co-transform $C_i^I(M)$ of M with respect to I (or i -th I -co-transform of M) is defined by

$$C_i^I(M) = \varprojlim_t \text{Tor}_i^R(I^t; M) \text{ ([14])}.$$

$C_0^I(M)$ is called the I -co-transform of M and denoted by $C^I(M)$.

Lemma 2.5. ([14, 3.5]) *Let M be a linearly compact R -module. There are two short exact sequences*

$$0 \longrightarrow H_1^I(M) \longrightarrow C^I(M) \longrightarrow \bigcap_{t>0} I^t M \longrightarrow 0,$$

$$0 \longrightarrow \bigcap_{t>0} I^t M \longrightarrow M \xrightarrow{\theta_M} \Lambda_I(M) \longrightarrow 0.$$

3 Generalized ideal Co-transforms

Let (R, m) be a local noetherian commutative ring. We suggest the following definition.

Definition 3.1. Let I be an ideal of R and M, N R -modules. The i -th generalized ideal co-transform $C_i^I(M, N)$ of M, N with respect to I is defined by

$$C_i^I(M, N) = \varprojlim_t \text{Tor}_i^R(I^t M, N).$$

Especially, $C_0^I(M, N)$ is called the generalized I -co-transform of M, N and denoted by $C^I(M, N)$ for simplicity. When $M = R$, we have $C_i^I(R, N) = C_i^I(N)$ the i -th ideal co-transform $C_i^I(M)$ of M with respect to I ([14]).

Lemma 3.2. *Let M be a finitely generated R -module. If N is a linearly compact R -module, then $C_i^I(M, N)$ is also linearly compact for all $i \geq 0$.*

Proof It follows from 2.3 (i) that $\{\text{Tor}_i^R(I^t M, N)\}$ ($i \geq 0$) forms an inverse system of linearly compact modules which continuous homomorphisms. Therefore $C_i^I(M, N)$ is also linearly compact by 2.1 (iii). \square

Set $D_i^I(M, N) = \varinjlim_t \text{Ext}_R^i(I^t M, N)$. Note that

$$D_I^0(M, N) = \varinjlim_t \text{Hom}_R(I^t M, N) = D_I(M, N)$$

is the generalized I -transform of M, N ([15]).

Let $D(M) = \text{Hom}_R(M, E(R/m))$ the Matlis dual of M .

Proposition 3.3. *Let M be a finitely generated R -module and N an R -module. Then*

$$C_i^I(M, D(N)) \cong D(D_I^i(M, N))$$

for all $i \geq 0$.

Proof We first note that $\mathrm{Tor}_i^R(I^t M, D(N)) \cong D(\mathrm{Ext}_R^i(I^t M, N))$ by [17, 3.4.14] and $\varprojlim_t D(\mathrm{Ext}_R^i(I^t M, N)) \cong D(\varinjlim_t \mathrm{Ext}_R^i(I^t M, N))$ by [16, 2.27]. Therefore

$$\begin{aligned} C_i^I(M, D(N)) &\cong \varprojlim_t \mathrm{Tor}_i^R(I^t M, D(N)) \\ &\cong \varprojlim_t D(\mathrm{Ext}_R^i(I^t M, N)) \\ &\cong D(\varinjlim_t \mathrm{Ext}_R^i(I^t M, N)) = D(D_i^I(M, N)) \end{aligned}$$

as required. \square

Theorem 3.4. *Let M be a finitely generated R -module.*

(i) *If N is a linearly compact R -module, then we have a long exact sequence of linearly compact R -modules*

$$\begin{aligned} \cdots \rightarrow H_{i+1}^I(M, N) \rightarrow C_i^I(M, N) \rightarrow \mathrm{Tor}_i^R(M, N) \rightarrow H_i^I(M, N) \rightarrow \cdots \\ \cdots \rightarrow H_1^I(M, N) \rightarrow C^I(M, N) \xrightarrow{\eta_{M, N}} M \otimes_R N \xrightarrow{\theta_{M, N}} \Lambda_I(M \otimes_R N) \rightarrow 0. \end{aligned}$$

Moreover, if $\mathrm{pd}(M) = p < +\infty$, then $H_{p+1}^I(M, N) \cong C_p^I(M, N)$;

(ii) *If $0 \rightarrow N'' \xrightarrow{f} N \xrightarrow{g} N' \rightarrow 0$ is a short exact sequence of linearly compact R -modules with continuous homomorphisms, then we have a long exact sequence of linearly compact R -modules*

$$\begin{aligned} \cdots \rightarrow C_i^I(M, N'') \xrightarrow{f_i} C_i^I(M, N) \xrightarrow{g_i} C_i^I(M, N') \rightarrow \\ \cdots \rightarrow C^I(M, N'') \xrightarrow{f_0} C^I(M, N) \xrightarrow{g_0} C^I(M, N') \rightarrow 0 \end{aligned}$$

in which the homomorphisms f_i, g_i are continuous for all $i \geq 0$.

Proof (i). For any positive integer t the short exact sequence

$$0 \rightarrow I^t M \rightarrow M \rightarrow M/I^t M \rightarrow 0$$

gives rise a long exact sequence of linearly compact R -modules

$$\begin{aligned} \cdots \rightarrow \mathrm{Tor}_i^R(I^t M, N) \rightarrow \mathrm{Tor}_i^R(M, N) \rightarrow \mathrm{Tor}_i^R(M/I^t M, N) \rightarrow \cdots \\ \cdots \rightarrow I^t M \otimes_R N \rightarrow M \otimes_R N \rightarrow (M/I^t M) \otimes_R N \rightarrow 0. \end{aligned}$$

Since M is the finitely generated R -module and N is the linearly compact R -module, the modules of the long exact sequence above are also linearly

compact. Thus, passing inverse limits, we have the long exact sequence of linearly compact R -modules as required. Note that $\text{Tor}_i^R(M, N) = 0$ for all $i > p$, so $H_{p+1}^I(M, N) \cong C_p^I(M, N)$.

(ii). The short exact sequence of linearly compact modules $0 \longrightarrow N'' \xrightarrow{f} N \xrightarrow{g} N' \longrightarrow 0$ gives rise to a long exact sequence of linearly compact modules for all $t > 0$

$$\begin{aligned} \cdots \longrightarrow \text{Tor}_i^R(I^t M; N'') &\xrightarrow{f_{it}} \text{Tor}_i^R(I^t M; N) \xrightarrow{g_{it}} \text{Tor}_i^R(I^t M; N') \longrightarrow \\ \cdots \longrightarrow I^t M \otimes_R N'' &\xrightarrow{f_{0t}} I^t M \otimes_R N \xrightarrow{g_{0t}} I^t M \otimes_R N' \longrightarrow 0 \end{aligned}$$

in which homomorphisms f_{it}, g_{it} are continuous by 2.3, (ii). Then $\text{Im } f_{it}, \text{Ker } f_{it}$ are linearly compact. Passing inverse limits, we get the long exact sequence as required. Since the homomorphisms f_{it}, g_{it} are continuous, the homomorphisms f_i, g_i induced on inverse limits are also continuous. \square

Corollary 3.5. *Let M be a finitely generated R -module and N a linearly compact R -module. Then the homomorphism*

$$\eta_i(M, N) : C_i^I(M, N) \longrightarrow \text{Tor}_i^R(M, N)$$

is an isomorphism if $H_{i+1}^I(M, N) = H_i^I(M, N) = 0$.

Proof It is immediately induced from 3.4 (i). \square

Let $\Lambda_I(M) = \varprojlim_t M/I^t M$ the I -adic completion of M , we have the following lemma.

Lemma 3.6. *Let M be a finitely generated R -module and N a linearly compact R -module. If N is complete in I -adic topology (i.e., $\Lambda_I(N) \cong N$), then $C_i^I(M, N) = 0$ for all $i \geq 0$.*

Proof From 3.4 (i) we have a long exact sequence

$$\begin{aligned} \cdots \rightarrow H_{i+1}^I(M, N) \rightarrow C_i^I(M, N) \rightarrow \text{Tor}_i^R(M, N) \rightarrow H_i^I(M, N) \rightarrow \cdots \\ \cdots \rightarrow H_1^I(M, N) \rightarrow C^I(M, N) \rightarrow M \otimes_R N \rightarrow \Lambda_I(M \otimes_R N) \rightarrow 0. \end{aligned}$$

As N is complete in I -adic topology, we have $H_i^I(M, N) \cong \text{Tor}_i^R(M, N)$ for all $i \geq 0$ by 2.4. Thus $C_i^I(M, N) = 0$ for all $i \geq 0$. \square

Since $H_i^I(N)$ is complete in I -adic topology for all $i \geq 0$, we have the immediate consequence.

Corollary 3.7. *Let M be a finitely generated R -module and N a linearly compact R -module. Then $C_i^I(M, H_j^I(N)) = 0$ for all $i, j \geq 0$.*

Note that $C_i^I(R, N) = C_i^I(N)$, so we also have the following immediate consequence.

Corollary 3.8. *Let N be a linearly compact R -module. If N is complete in I -adic topology, then $C_i^I(N) = 0$ for all $i \geq 0$.*

Lemma 3.9. *Let M, N be R -modules. Then*

$$\Lambda_I(C^I(M, N)) \cong C^I(M, \Lambda_I(N)).$$

Proof We have

$$\begin{aligned} \Lambda_I(C^I(M, N)) &\cong \varprojlim_t (R/I^t \otimes_R C^I(M, N)) \\ &= \varprojlim_t (R/I^t \otimes_R \varprojlim_s (I^s M \otimes_R N)) \\ &\cong \varprojlim_t \varprojlim_s (R/I^t \otimes_R (I^s M \otimes_R N)) \\ &\cong \varprojlim_s \varprojlim_t (I^s M \otimes_R (R/I^t \otimes_R N)) \\ &\cong \varprojlim_s (I^s M \otimes_R \varprojlim_t (R/I^t \otimes_R N)) \\ &\cong C^I(M, \Lambda_I(N)) \end{aligned}$$

as required. \square

Corollary 3.10. *Let M be a finitely generated R -module and N a linearly compact R -module. Then $\Lambda_I(C^I(M, N)) = 0$.*

Proof It follows from 3.7 and 3.9. \square

Lemma 3.11. *Let M be a finitely generated R -module and N a linearly compact R -module. Then $C_i^I(M, C^I(N)) \cong C_i^I(M, N)$.*

Proof From 2.5 we have two short exact sequences

$$0 \rightarrow H_1^I(N) \rightarrow C^I(N) \rightarrow \bigcap_{t>0} I^t N \rightarrow 0,$$

$$0 \rightarrow \bigcap_{t>0} I^t N \rightarrow N \rightarrow \Lambda_I(N) \rightarrow 0.$$

By 3.4 (ii) we get two exact sequences

$$\dots C_i^I(M, C^I(N)) \rightarrow C_i^I(M, \bigcap_{t>0} I^t N) \rightarrow C_{i-1}^I(M, H_1^I(N)) \dots,$$

$$\dots C_i^I(M, \bigcap_{t>0} I^t N) \rightarrow C_i^I(M, N) \rightarrow C_i^I(M, \Lambda_I(N)) \dots$$

Then the conclusion follows from 3.7. \square

Lemma 3.12. *Let M be a finitely generated R -module and N a linearly compact R -module. Then $C^I(C^I(M, N)) \cong C^I(M, N)$.*

Proof We have

$$\begin{aligned} C^I(C^I(M, N)) &\cong \varprojlim_t (I^t \otimes_R \varprojlim_s (I^s M \otimes_R N)) \\ &\cong \varprojlim_t \varprojlim_s (I^t \otimes_R (I^s M \otimes_R N)) \\ &\cong \varprojlim_s \varprojlim_t (I^s M \otimes_R (I^t \otimes_R N)) \\ &\cong \varprojlim_s (I^s M \otimes_R \varprojlim_t (I^t \otimes_R N)) \\ &\cong C^I(M, C^I(N)). \end{aligned}$$

Now the conclusion follows from 3.12. \square

Proposition 3.13. *Let $f : N' \rightarrow N$ be a homomorphism of linearly compact R -modules such that $\text{Ker } f$ and $\text{coker } f$ are complete in I -adic topology. Let $\varphi : K \rightarrow N$ be a further homomorphism of linearly compact R -modules. Then*

(i) *The homomorphism $C_i^I(M, f) : C_i^I(M, N') \rightarrow C_i^I(M, N)$ is an isomorphism for all $i \geq 0$.*

(ii) *There is a homomorphism $\psi : C^I(M, K) \rightarrow M \otimes_R N'$ such that the diagram*

$$\begin{array}{ccc} M \otimes_R N' & \xrightarrow{M \otimes_R f} & M \otimes_R N \\ \uparrow \psi & & \uparrow M \otimes_R \varphi \\ C^I(M, K) & \xrightarrow{\eta_{M, K}} & M \otimes_R K, \end{array}$$

is commutative, i. e., $M \otimes_R f \circ \psi = M \otimes_R \varphi \circ \eta_{M, K}$.

Proof (i) We have short exact sequences of linearly compact modules

$$0 \rightarrow \text{Ker } f \rightarrow N' \xrightarrow{\alpha} \text{Im } f \rightarrow 0$$

and

$$0 \rightarrow \text{Im } f \xrightarrow{\beta} N \rightarrow \text{coker } f \rightarrow 0$$

in which $f = \beta\alpha$ and homomorphisms are continuous. It is therefore enough to show that $C_i^I(M, \alpha)$ and $C_i^I(M, \beta)$ are both isomorphisms.

The first short exact sequence above induces by 3.4 (ii) an exact sequence

$$\dots C_i^I(M, \text{Ker } f) \rightarrow C_i^I(M, N') \xrightarrow{C_i^I(M, \alpha)} C_i^I(M, \text{Im } f) \dots$$

From 3.7 and the hypothesis $\text{Ker } f \cong \Lambda_I(\text{Ker } f)$ we have $C_i^I(M, \text{Ker } f) = 0$. Hence $C_i^I(M, \alpha)$ is an isomorphism. Next, from the second short exact sequence we get an induced exact sequence

$$\dots C_i^I(M, \text{Im } f) \xrightarrow{C_i^I(M, \beta)} C_i^I(M, N) \rightarrow C_i^I(M, \text{coker } f) \dots$$

We have $C_i^I(M, \text{coker } f) \cong C_i^I(M, \Lambda_I(\text{coker } f)) = 0$ for all $i \geq 0$ by 3.6. Therefore $C_i^I(M, \beta)$ is an isomorphism.

(ii) We have a commutative diagram

$$\begin{array}{ccccccc} M \otimes_R N' & \xrightarrow{M \otimes_R f} & M \otimes_R N & \xleftarrow{M \otimes_R \varphi} & M \otimes_R K \\ \uparrow \eta_{M, N'} & & \uparrow \eta_{M, N} & & \uparrow \eta_{M, K} \\ C^I(M, N') & \xrightarrow{C^I(M, f)} & C^I(M, N) & \xleftarrow{C^I(M, \varphi)} & C^I(M, K). \end{array}$$

By (i), $C^I(M, f)$ is an isomorphism.

Set $\psi = \eta_{M, N'} \circ C^I(M, f)^{-1} \circ C^I(M, \varphi)$, we have

$$\begin{aligned} M \otimes_R f \circ \psi &= M \otimes_R f \circ \eta_{M, N'} \circ C^I(M, f)^{-1} \circ C^I(M, \varphi) \\ &= \eta_{M, N} \circ C^I(M, \varphi) = M \otimes_R \varphi \circ \eta_{M, K}. \end{aligned}$$

The proof is complete. \square

4 CO-LOCALIZATION

Let S be multiplicative set of R . For an R -module M the module $\text{Hom}_R(R_S; M)$ is called co-localization of M with respect to S (see [11]). We denote it briefly by ${}_S M$. If M is a linearly compact R -module, then ${}_S M$ is also a linearly compact R -module by [5, 2.4]. The following proposition says that the co-localization can "commute" to the generalized ideal co-transform of a linearly compact R -module.

Proposition 4.1. *Let M be a finitely generated R -module and N a linearly compact R -module. Then*

$${}_S C_i^I(M, N) \cong C_i^{I R_S}(M_S, {}_S N)$$

for all $i \geq 0$.

Proof We first note that

$${}_S(\mathrm{Tor}_i^R(I^t M, N)) \cong \mathrm{Tor}_i^{R_S}(I^t M_S; {}_S N)$$

by [4, 3.9]. Then

$$\begin{aligned} {}_S C_i^I(M, N) &= \mathrm{Hom}_R(R_S, \varprojlim_t \mathrm{Tor}_i^R(I^t M, N)) \\ &\cong \varprojlim_t \mathrm{Hom}_R(R_S, \mathrm{Tor}_i^R(I^t M, N)) = \varprojlim_t {}_S(\mathrm{Tor}_i^R(I^t M, N)) \\ &\cong C_i^{IR_S}(M_S, {}_S N) \end{aligned}$$

as required. \square

Let a be an element in R , the notation ${}_a M$ means that the co-localization of M with respect to the multiplicative set $S = \{1, a, a^2, \dots\}$.

Theorem 4.2. *Let M be a finitely generated R -module and N a linearly compact R -module. Then*

- (i) $C^I(M \otimes_R N) \cong C^I(M, N)$;
- (ii) $C^{aR}(M, N) \cong {}_a C^{aR}(M, N)$ for any element $a \in R$.

Proof (i). We have an exact sequence by 3.4 (i)

$$H_1^I(M, N) \xrightarrow{\varphi} C^I(M, N) \xrightarrow{\eta_{M,N}} M \otimes_R N \xrightarrow{\theta_{M,N}} \Lambda_I(M \otimes_R N) \rightarrow 0.$$

It induces two short exact sequences

$$0 \rightarrow \mathrm{Im} \varphi \rightarrow C^I(M, N) \rightarrow \mathrm{Im} \eta_{M,N} \rightarrow 0,$$

$$0 \rightarrow \mathrm{Im} \eta_{M,N} \rightarrow M \otimes_R N \xrightarrow{\theta_{M,N}} \Lambda_I(M \otimes_R N) \rightarrow 0.$$

Combining 3.4 (ii) with 3.8 yields induced exact sequences

$$0 \rightarrow C^I(\mathrm{Im} \varphi) \rightarrow C^I(C^I(M, N)) \rightarrow C^I(\mathrm{Im} \eta_{M,N}) \rightarrow 0,$$

$$0 \rightarrow C^I(\mathrm{Im} \eta_{M,N}) \rightarrow C^I(M \otimes_R N) \xrightarrow{\theta_{M,N}} C^I(\Lambda_I(M \otimes_R N)) \rightarrow 0.$$

Note that $\mathrm{Im} \varphi$ and $\Lambda_I(M \otimes_R N)$ are both complete in I -adic topology, so

$$C^I(M \otimes_R N) \cong C^I(C^I(M, N)) \cong C^I(M, N)$$

by 3.8 and 3.12.

(ii). It follows from [18, 10.8.3] that $a^t M \otimes_R \text{Hom}(S^{-1}R, N) \cong \text{Hom}_R(S^{-1}R, a^t M \otimes_R N)$. On the other hand, $C^{aR}(M, N) \cong C^{aR}(M, C^{aR}(N))$ by 3.12 and $C^{aR}(N) \cong {}_a N$ by [14, 4.4]. Therefore

$$\begin{aligned} C^{aR}(M, N) &\cong C^{aR}(M, C^{aR}(N)) \\ &\cong C^{aR}(M, {}_a N) \\ &\cong \varprojlim_t a^t M \otimes_R \text{Hom}(S^{-1}R, N) \\ &\cong \varprojlim_t \text{Hom}_R(S^{-1}R, a^t M \otimes_R N) \\ &\cong \text{Hom}_R(S^{-1}R, \varprojlim_t a^t M \otimes_R N) \\ &= {}_a C^{aR}(M, N) \end{aligned}$$

as required. \square

Theorem 4.3. *Let M be a finitely generated R -module and N an artinian. There is an isomorphism*

$$C^{aR}(M, N) \cong {}_a(M \otimes_R N)$$

for any element $a \in R$.

Proof It follows from [6, 2.7] that $D_{aR}(M, N) \cong \text{Hom}_R(M, N)_a$. Moreover, $C^{aR}(M, D(N)) \cong D(D_{aR}(M, N))$ by 3.3. Hence

$$\begin{aligned} C^{aR}(M, N) &\cong C^{aR}(M, DD(N)) \\ &\cong D(D_{aR}(M, D(N))) \\ &\cong D(\text{Hom}_R(M, D(N))_a) \\ &\cong {}_a D(\text{Hom}_R(M, D(N))) \\ &\cong {}_a(M \otimes_R DD(N)) \\ &\cong {}_a(M \otimes_R N) \end{aligned}$$

as required. \square

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