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THE COMMUTATIVITY OF SEMIPRIME RINGS WITH SYMMETRIC Bi-(α, α)-DERIVATIONS

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Abstract

Let R be a semiprime ring, I a nonzero ideal of R, $D : R \times R \to R$ a symmetric bi-(α, α)-derivation, d be the trace of D and α an automorphism. In the present paper, we shall prove that R contains a nonzero central ideal if any one of the following holds: i) $d([x, y]_{\alpha,\alpha}) \pm [x, y]_{\alpha,\alpha} \in C_{\alpha,\alpha}$, ii)$[d(x), d(y)]_{\alpha,\alpha} \pm [x, y]_{\alpha,\alpha} \in C_{\alpha,\alpha}$, iii)$d((x \circ y)_{\alpha,\alpha}) \pm (x \circ y)_{\alpha,\alpha} \in C_{\alpha,\alpha}$, iv)$(d(x) \circ d(y))_{\alpha,\alpha} \pm (xoy)_{\alpha,\alpha} \in C_{\alpha,\alpha}$, v)$d((x \circ y)_{\alpha,\alpha}) \pm [x, y]_{\alpha,\alpha} \in C_{\alpha,\alpha}$, vi)$(d(x) \circ d(y))_{\alpha,\alpha} \pm [x, y]_{\alpha,\alpha} \in C_{\alpha,\alpha}$, vii)$d([x, y]_{\alpha,\alpha}) \pm (x \circ y)_{\alpha,\alpha} \in C_{\alpha,\alpha}$, viii)$[d(x), d(y)]_{\alpha,\alpha} \pm (x \circ y)_{\alpha,\alpha} \in C_{\alpha,\alpha}$, ix)$d(x)\, d(y) \pm [x, y]_{\alpha,\alpha} \in C_{\alpha,\alpha}$, x)$d(x)\, d(y) \pm (x \circ y)_{\alpha,\alpha} \in C_{\alpha,\alpha}$, xi)$[d(x), y]_{\alpha,\alpha} \in C_{\alpha,\alpha}$, xii)$d\left([x, y]_{\alpha,\alpha}\right) \pm [d(x), y]_{\alpha,\alpha} \in C_{\alpha,\alpha}$,
xiii)$d\left((x \circ y)_{\alpha,\alpha}\right) \pm [d(x), y] \in C_{\alpha,\alpha}$, for all $x, y \in I$.

1 Introduction

Throughout R will represent an associative ring with center Z. A ring R is said to be prime if $xRy = (0)$ implies that either $x = 0$ or $y = 0$ and semiprime if $xRx = (0)$ implies that $x = 0$, where $x, y \in R$. A prime ring

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is obviously semiprime. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol $x \circ y$ stands for the commutator $xy + yx$. α a mapping from R into itself. For any $x, y \in R$, we write $[x, y]_{\alpha,\alpha}$ and $(x \circ y)_{\alpha,\alpha}$, for $x\alpha(y) - \alpha(y)x$ and $x\alpha(y) + \alpha(y)x$ respectively. We set $C_{\alpha,\alpha}$=$\{c \in R \mid c\alpha(x) = \alpha(x)c$ for all $x \in R\}$ and call (α, α)-center of R. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. A mapping $D(.,.) : R \times R \to R$ is said to be symmetric if $D(x, y) = D(y, x)$ for all $x, y \in R$. A mapping $d : R \to R$ is called the trace of $D(.,.)$ if $d(x) = D(x, x)$ for all $x \in R$. It is obvious that if $D(.,.)$ is bi-additive (i.e., additive in both arguments), then the trace d of $D(.,.)$ satisfies the identity $d(x + y) = d(x) + d(y) + 2D(x, y)$, for all $x, y \in R$. If $D(.,.)$ is bi-additive and satisfies the identities

$$D(xy, z) = D(x, z)y + xD(y, z)$$

and

$$D(x, yz) = D(x, y)z + yD(x, z),$$

for all $x, y, z \in R$. Then $D(.,.)$ is called a symmetric bi-derivation. A bi-additive mapping $D(.,.) : R \times R \to R$ is said to be bi-(α, α)-derivation if it satisfies the identities

$$D(xy, z) = D(x, z)\alpha(y) + \alpha(x)D(y, z)$$

and

$$D(x, yz) = D(x, y)\alpha(z) + \alpha(y)D(x, z),$$

for all $x, y, z \in R$. Of course a symmetric bi-$(1, 1)$-derivation where 1 is the identity map on R is symmetric bi-derivation.

The study of commuting mappings was initiaded by a well-known theorem due to Posner [9] which states that the existence of a nonzero commuting derivation on a prime ring R implies that R is commutative. A number of authors have extended the Posner's theorem in several ways. The notion of additive commuting mapping is closely connected with the notion of bi-derivation. Every additive commuting mapping $F : R \to R$ gives rise to a bi-derivation on R. Namely, linearizing $[F(x), x] = 0$, we get $[F(x), y] = [x, F(y)]$ and we note that the map $(x, y) \longmapsto [F(x), y]$ is a bi-derivation. The concept of bi-derivation was introduced by Maksa in [7]. It is shown in [8] that symmetric bi-derivations are related to general solution of some functional equations. Some results concerning symmetric bi-derivations in prime rings can found in [11] and [12]. Typical examples are mappings of the form $(x, y) \longmapsto \lambda[x, y]$ where $\lambda \in C$, the extended centroid of R. We shall call such maps inner bi-derivations. It was shown in [4] that every bi-derivation D of a noncommutative prime ring R is of the form $D(x, y) = \lambda[x, y]$ for some $\lambda \in C$.

F is called strong commutativity preserving (simply, SCP) on S if $[x, y] = [F(x), F(y)]$, for all $x, y \in S$. Derivations as well as SCP mappings have been extensively studied by researchers in the context of operator algebras, prime rings and semiprime rings too. For more information on SCP, we refere [3], [5], [1] and references therein.

On the other hand, in [6], Daif and Bell showed that if a semiprime ring R has a derivation d satisfiying the following condition, then I is a central ideal;

$$\text{there exists a nonzero ideal } I \text{ of } R \text{ such that}$$
$$\text{either } d([x,y]) = [x,y] \text{ for all } x,y \in I \text{ or } d([x,y]) = -[x,y] \text{ for all } x,y \in I.$$

This result was extended for semiprime rings in [2].

In this paper, we extend some well known results concerning of ideals in semiprime rings with bi-(α,α)-derivation. Throughout the present paper, we shall make use of the following basic identities without any specific mention:
i) $[x,yz] = y[x,z] + [x,y]z$
ii) $[xy,z] = [x,z]y + x[y,z]$
iii) $xyoz = (xoz)y + x[y,z] = x(yoz) - [x,z]y$
iv) $xoyz = y(xoz) + [x,y]z = (xoy)z + y[z,x]$
v) $[xy,z]_{\alpha,\alpha} = x\,[y,z]_{\alpha,\alpha} + [x,\alpha(z)]\,y = x[y,\alpha(z)] + [x,z]_{\alpha,\alpha}\,y$
vi) $[x,yz]_{\alpha,\alpha} = \alpha(y)\,[x,z]_{\alpha,\alpha} + [x,y]_{\alpha,\alpha}\,\alpha(z)$
vii) $(xz \circ y)_{\alpha,\alpha} = x(z \circ y)_{\alpha,\alpha} - [x,\alpha(y)]z$.

2 Results

Lemma 2.1. *[6, Lemma 2 (b)] If R is a semiprime ring, then the center of a nonzero ideal of R is contained in the center of R.*

Lemma 2.2. *[10, Theorem 2.1] Let R be a semiprime ring, I a nonzero two-sided ideal of R and $a \in I$ such that $axa = 0$ for all $x \in I$, then $a = 0$.*

Lemma 2.3. *Let R be a semiprime ring, I a nonzero ideal of R and α an automorphism. If $[I,I]_{\alpha,\alpha} = (0)$, then R contains a nonzero central ideal.*

Proof. By the hypothesis, we have

$$[x,y]_{\alpha,\alpha} = 0, \text{ for all } x,y \in I.$$

Replacing x by xz, $z \in I$ in this equation and using this equation, we obtain that

$$x[z,\alpha(y)] = 0, \text{ for all } x,y,z \in I,$$

and so

$$I[z,\alpha(y)] = 0, \text{ for all } y,z \in I.$$

By Lemma 2.2, we get

$$[z,\alpha(y)] = 0, \text{ for all } y,z \in I.$$

Thus, we obtain that $\alpha(y) \in Z$, for all $y \in I$ by Lemma 2.1. Since α is an automorphism, we get $I \subset Z$. We conclude that R contains a nonzero central ideal. This completes the proof. $\square$

Lemma 2.4. *Let R be a semiprime ring, I a nonzero ideal of R and α an automorphism. If $[I, I]_{\alpha,\alpha} \subset C_{\alpha,\alpha}$, then R contains a nonzero central ideal.*

Proof. By the hypothesis, we see that

$$[x, y]_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{ for all } x, y \in I.$$

Replacing x by $x\alpha(y)$ in the last expression and using this, we have

$$[x, y]_{\alpha,\alpha}\, \alpha(y) \in C_{\alpha,\alpha}, \text{ for all } x, y \in I.$$

That is

$$\Big[[x, y]_{\alpha,\alpha}\, \alpha(y), r\Big]_{\alpha,\alpha} = 0, \text{ for all } x, y \in I, r \in R.$$

Using the hypothesis in the last equation, we get

$$[x, y]_{\alpha,\alpha}\, [\alpha(y), \alpha(r)] = 0, \text{ for all } x, y \in I, r \in R.$$

and so

$$[x, y]_{\alpha,\alpha}\, [r, y]_{\alpha,\alpha} = 0, \text{ for all } x, y \in I, r \in R.$$

Taking r by rx in the last equation and using this, we see that

$$[x, y]_{\alpha,\alpha}\, \alpha(r)\, [x, y]_{\alpha,\alpha} = 0, \text{ for all } x, y \in I, r \in R.$$

Since α is an automorphism, we have

$$[x, y]_{\alpha,\alpha}\, R\, [x, y]_{\alpha,\alpha} = 0, \text{ for all } x, y \in I.$$

By the semiprimeness of R, we see that

$$[x, y]_{\alpha,\alpha} = 0, \text{ for all } x, y \in I.$$

By Lemma 2.3, we obtain that R contains a nonzero central ideal. This completes the proof. $\square$

Lemma 2.5. *Let R be a semiprime ring, I a nonzero ideal of R and α an automorphism. If $(I \circ I)_{\alpha,\alpha} \subset C_{\alpha,\alpha}$, then R contains a nonzero central ideal.*

Proof. By the hypothesis, we get

$$(x \circ y)_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{ for all } x, y \in I.$$

Replacing x by $x\alpha(y)$ in the last expression and using this, we obtain that

$$(x \circ y)_{\alpha,\alpha}\, \alpha(y) \in C_{\alpha,\alpha}, \text{ for all } x, y \in I.$$

This implies that

$$[(x \circ y)_{\alpha,\alpha}\,\alpha(y), r]_{\alpha,\alpha} = 0, \text{ for all } x, y \in I, r \in R.$$

Using the hypothesis, we get

$$(x \circ y)_{\alpha,\alpha}\,[\alpha(y), \alpha(r)] = 0, \text{ for all } x, y \in I, r \in R.$$

Since α is an automorphism, we have

$$(x \circ y)_{\alpha,\alpha}\,[\alpha(y), r] = 0, \text{ for all } x, y \in I, r \in R. \tag{1}$$

Replacing x by rx in the above expression, we get

$$r\,(x \circ y)_{\alpha,\alpha}\,[\alpha(y), r] - [r, \alpha(y)]x[\alpha(y), r] = 0.$$

Using equation (2.1), we find that

$$[r, \alpha(y)]x[\alpha(y), r] = 0.$$

Replacing r by z, $z \in I$ in this equation, we get

$$[z, \alpha(y)]I[z, \alpha(y)] = (0), \text{for all } y, z \in I.$$

By Lemma 2.2, we have $[z, \alpha(y)] = (0)$, for all $y, z \in I$. Since α is an automorphism, we see that $y \in Z$, for all $y \in I$ by Lemma 2.1. That is $I \subset Z$. We conclude that R contains a nonzero central ideal. This completes the proof. □

Theorem 2.6. *Let R be a $2-$torsion free semiprime ring, I a nonzero ideal of R, $D : R \times R \to R$ a symmetric bi-(α,α)-derivation, d be the trace of D and α an automorphism. If $d([x, y]_{\alpha,\alpha}) \pm [x, y]_{\alpha,\alpha} \in C_{\alpha,\alpha}$, for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof. By the hypothesis, we get

$$d([x, y]_{\alpha,\alpha}) \pm [x, y]_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{ for all } x, y \in I.$$

Replacing y by $y + z$, $z \in I$ in the above expression, we have

$$d([x, y]_{\alpha,\alpha}) + d([x, z]_{\alpha,\alpha}) + 2D([x, y]_{\alpha,\alpha}, [x, z]_{\alpha,\alpha}) \pm [x, y]_{\alpha,\alpha} \pm [x, z]_{\alpha,\alpha} \in C_{\alpha,\alpha}.$$

Using the hypothesis and R is $2-$torsion free, we obtain that

$$D([x, y]_{\alpha,\alpha}, [x, z]_{\alpha,\alpha}) \in C_{\alpha,\alpha}, \text{ for all } x, y, z \in I.$$

Taking z by y in last expression, we have

$$D([x, y]_{\alpha,\alpha}, [x, y]_{\alpha,\alpha}) \in C_{\alpha,\alpha},$$

and so,

$$d([x,y]_{\alpha,\alpha}) \in C_{\alpha,\alpha}, \text{for all } x,y \in I.$$

Using the hypothesis, we arrive at

$$[x,y]_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{ for all } x,y \in I.$$

By Lemma 2.4, we obtain that R contains a nonzero central ideal. This completes the proof. □

Theorem 2.7. *Let R be a $2-$torsion free semiprime ring, I a nonzero ideal of R, $D : R \times R \to R$ a symmetric bi-(α,α)-derivation, d be the trace of D and α an automorphism. If $[d(x), d(y)]_{\alpha,\alpha} \pm [x,y]_{\alpha,\alpha} \in C_{\alpha,\alpha}$, for all $x,y \in I$, then R contains a nonzero central ideal.*

Proof. By the hypothesis, we have

$$[d(x), d(y)]_{\alpha,\alpha} \pm [x,y]_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{for all } x,y \in I.$$

Writting y by $y+z$, $z \in I$ in above expression, we get

$$[d(x), d(y) + 2D(y,z) + d(z)]_{\alpha,\alpha} \pm [x, y+z]_{\alpha,\alpha} \in C_{\alpha,\alpha},$$

and so

$$[d(x), d(y)]_{\alpha,\alpha} + 2[d(x), D(y,z)]_{\alpha,\alpha} + [d(x), d(z)]_{\alpha,\alpha} \pm [x,y]_{\alpha,\alpha} \pm [x,z]_{\alpha,\alpha} \in C_{\alpha,\alpha}.$$

Using the hypothesis and R is $2-$torsion free, we obtain that

$$[d(x), D(y,z)]_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{ for all } x,y,z \in I.$$

Taking z by y in this expression, we get

$$[d(x), D(y,y)]_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{ for all } x,y \in I.$$

That is

$$[d(x), d(y)]_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{ for all } x,y \in I.$$

Using this expression in the hypothesis, we have

$$[x,y]_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{ for all } x,y \in I.$$

Hence, we conclude that R contains a nonzero central ideal by Lemma 2.4. This completes the proof. □

Theorem 2.8. *Let R be a $2-$torsion free semiprime ring, I a nonzero ideal of R, $D : R \times R \to R$ a symmetric bi-(α,α)-derivation and d be the trace of D and α an automorphism. If $d((x \circ y)_{\alpha,\alpha}) \pm (x \circ y)_{\alpha,\alpha} \in C_{\alpha,\alpha}$, for all $x,y \in I$, then R contains a nonzero central ideal.*

Proof. We get

$$d((x \circ y)_{\alpha,\alpha}) \pm (x \circ y)_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{ for all } x, y \in I.$$

Replacing y by $y+z$, $z \in I$ in this expression, we have

$$d((x \circ y)_{\alpha,\alpha}) + d((x \circ z)_{\alpha,\alpha}) + 2D((x \circ y)_{\alpha,\alpha}, (x \circ z)_{\alpha,\alpha}) \pm (x \circ y)_{\alpha,\alpha} \pm (x \circ z)_{\alpha,\alpha} \in C_{\alpha,\alpha}.$$

Applying the hypothesis, we obtain that

$$2D((x \circ y)_{\alpha,\alpha}, (x \circ z)_{\alpha,\alpha}) \in C_{\alpha,\alpha}.$$

Writting z by y, we have

$$d((x \circ y)_{\alpha,\alpha}) \in C_{\alpha,\alpha}, \text{ for all } x, y \in I.$$

Using this equation in our hypothesis, we see that

$$(x \circ y)_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{ for all } x, y \in I.$$

By Lemma 2.5, we conclude that R contains a nonzero central ideal. This completes the proof. □

Theorem 2.9. *Let R be a $2-$torsion free semiprime ring, I a nonzero ideal of R, α an automorphism, $D : R \times R \to R$ a symmetric bi-(α, α)-derivation, d be the trace of. If $(d(x) \circ d(y))_{\alpha,\alpha} \pm (x \circ y)_{\alpha,\alpha} \in C_{\alpha,\alpha}$ for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof. By the hypothesis, we get

$$(d(x) \circ d(y))_{\alpha,\alpha} \pm (x \circ y)_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{ for all } x, y \in I.$$

Taking y by $y+z$, $z \in I$ in the hypothesis, we see that

$$(d(x) \circ d(y))_{\alpha,\alpha} + (d(x) \circ d(z))_{\alpha,\alpha} + 2\,(d(x) \circ D(y,z))_{\alpha,\alpha} \pm (x \circ y)_{\alpha,\alpha} \pm (x \circ z)_{\alpha,\alpha} \in C_{\alpha,\alpha}.$$

Appliying the hypothesis and R is $2-$torsion free ring, we get

$$(d(x) \circ D(y,z))_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{ for all } x, y, z \in I.$$

Replacing z by y, we have

$$(d(x) \circ d(y))_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{ for all } x, y \in I.$$

From our hypothesis, we see that

$$(x \circ y)_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{ for all } x, y \in I.$$

By Lemma 2.5, we have R contains a nonzero central ideal. The proof is completed. □

Theorem 2.10. *Let R be a $2-$torsion free semiprime ring, I a nonzero ideal of R, α an automorphism, $D : R \times R \to R$ a symmetric bi-(α, α)-derivation and d be the trace of D. If $d((x \circ y)_{\alpha,\alpha}) \pm [x, y]_{\alpha,\alpha} \in C_{\alpha,\alpha}$, for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof. By the hypothesis, we get

$$d((x \circ y)_{\alpha,\alpha}) \pm [x, y]_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{for all } x, y \in I.$$

Taking y by $y + z, z \in I$, we have

$$d((x \circ y)_{\alpha,\alpha}) + d((x \circ z)_{\alpha,\alpha}) + 2D((x \circ y)_{\alpha,\alpha}, (x \circ z)_{\alpha,\alpha}) \pm [x, y]_{\alpha,\alpha} \pm [x, z]_{\alpha,\alpha} \in C_{\alpha,\alpha}.$$

Using the hypothesis, we find that

$$2D((x \circ y)_{\alpha,\alpha}, (x \circ z)_{\alpha,\alpha}) \in C_{\alpha,\alpha}, \text{for all } x, y, z \in I.$$

Since R is $2-$torsion free and replacing z by y in this expression, we have

$$D((x \circ y)_{\alpha,\alpha}, (x \circ y)_{\alpha,\alpha}) \in C_{\alpha,\alpha}, \text{ for all } x, y \in I,$$

and so,

$$d((x \circ y)_{\alpha,\alpha}) \in C_{\alpha,\alpha}, \text{for all } x, y \in I.$$

By our hypothesis, we see that

$$[x, y]_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{for all } x, y \in I.$$

We obtain that R contains a nonzero central ideal by Lemma 2.4. □

Theorem 2.11. *Let R be a $2-$torsion free semiprime ring, I a nonzero ideal of R, α an automorphism, $D : R \times R \to R$ a symmetric bi-(α, α)-derivation and d be the trace of D. If $(d(x) \circ d(y))_{\alpha,\alpha} \pm [x, y]_{\alpha,\alpha} \in C_{\alpha,\alpha}$, for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof. We get

$$(d(x) \circ d(y))_{\alpha,\alpha} \pm [x, y]_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{for all } x, y \in I.$$

Writting y by $y + z, z \in I$, we have

$$(d(x) \circ d(y))_{\alpha,\alpha} + (d(x) \circ d(z))_{\alpha,\alpha} + 2\,(d(x) \circ D(y, z))_{\alpha,\alpha} \pm [x, y]_{\alpha,\alpha} \pm [x, z]_{\alpha,\alpha} \in C_{\alpha,\alpha}.$$

Applying the hypothesis and R is $2-$torsion free, we obtain that

$$(d(x) \circ D(y, z))_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{for all } x, y, z \in I.$$

Taking z by y in the last expression, we have

$$(d(x) \circ d(y))_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{ for all } x, y \in I.$$

Using this equation in our hypothesis, we see that

$$[x,y]_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{for all } x,y \in I.$$

By Lemma 2.4, we obtain that R contains a nonzero central ideal. This completes the proof. □

Theorem 2.12. *Let R be a $2-$torsion free semiprime ring, I a nonzero ideal of R, α an automorphism, $D : R \times R \to R$ a symmetric bi-(α,α)-derivation and d be the trace of D. If $d([x,y]_{\alpha,\alpha}) \pm (x \circ y)_{\alpha,\alpha} \in C_{\alpha,\alpha}$, for all $x,y \in I$, then R contains a nonzero central ideal.*

Proof. We get

$$d([x,y]_{\alpha,\alpha}) \pm (x \circ y)_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{ for all } x,y \in I.$$

Replacing y by $y+z, z \in I$ in this expression, we have

$$d([x,y]_{\alpha,\alpha}) + d([x,y]_{\alpha,\alpha}) + 2D([x,y]_{\alpha,\alpha}, [x,z]_{\alpha,\alpha}) \pm (x \circ y)_{\alpha,\alpha} \pm (x \circ z)_{\alpha,\alpha} \in C_{\alpha,\alpha}.$$

By the hypothesis, we get

$$2D([x,y]_{\alpha,\alpha}, [x,z]_{\alpha,\alpha}) \in C_{\alpha,\alpha}, \text{ for all } x,y,z \in I.$$

Since R is $2-$torsion free and taking z by y in the above expression, we have

$$d([x,y]_{\alpha,\alpha}) \in C_{\alpha,\alpha}, \text{ for all } x,y \in I.$$

Our hypothesis reduces that

$$(x \circ y)_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{ for all } x,y \in I.$$

By Lemma 2.5, we conclude that R contains a nonzero central ideal. □

Theorem 2.13. *Let R be a $2-$torsion free semiprime ring, I a nonzero ideal of R, α an automorphism, $D : R \times R \to R$ a symmetric bi-(α,α)-derivation and d be the trace of D. If $[d(x), d(y)]_{\alpha,\alpha} \pm (x \circ y)_{\alpha,\alpha} \in C_{\alpha,\alpha}$, for all $x,y \in I$, then R contains a nonzero central ideal.*

Proof. Using our hypothesis, we have

$$[d(x), d(y)]_{\alpha,\alpha} \pm (x \circ y)_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{for all } x,y \in I.$$

Writting y by $y+z$ and R is $2-$torsion free, we get

$$[d(x), d(y)]_{\alpha,\alpha} + 2[d(x), D(y,z)]_{\alpha,\alpha} + [d(x), d(z)]_{\alpha,\alpha} \pm (x \circ y)_{\alpha,\alpha} \pm (x \circ z)_{\alpha,\alpha} \in C_{\alpha,\alpha}.$$

Using the hypothesis, we have

$$[d(x), D(y,z)]_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{for all } x,y,z \in I.$$

Taking z by y in this expression, we find that

$$[d(x), D(y,y)]_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{for all } x, y \in I,$$

and so

$$[d(x), d(y)]_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{for all } x, y \in I.$$

By our hypothesis, we have

$$(x \circ y)_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{for all } x, y \in I.$$

We get the required result by Lemma 2.5. □

Theorem 2.14. *Let R be a $2-$torsion free semiprime ring, I a nonzero ideal of R, α an automorphism, $D : R \times R \to R$ a symmetric bi-(α, α)-derivation and d be the trace of D. If $d(x)\, d(y) \pm [x, y]_{\alpha,\alpha} \in C_{\alpha,\alpha}$ for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof. We get

$$d(x)\, d(y) \pm [x, y]_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{ for all } x, y \in I.$$

Taking y by $y + z, z \in I$, we have

$$d(x)\,(d(y) + 2D(y,z) + d(z)) \pm [x, y]_{\alpha,\alpha} \pm [x, z]_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{for all } x, y, z \in I.$$

By the hypothesis, we obtain that

$$2d(x)D(y,z) \in C_{\alpha,\alpha}, \text{for all } x, y, z \in I.$$

Since R is $2-$torsion free, we get

$$d(x)D(y,z) \in C_{\alpha,\alpha}, \text{ for all } x, y, z \in I.$$

Replacing z by y in the last expression, we see that

$$d(x)d(y) \in C_{\alpha,\alpha}, \text{for all } x, y \in I.$$

Using the hypothesis, we have

$$[x, y]_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{for all } x, y \in I.$$

By Lemma 2.4, we obtain that R contains a nonzero central ideal. □

Theorem 2.15. *Let R be a $2-$torsion free semiprime ring, I a nonzero ideal of R, α an automorphism, $D : R \times R \to R$ a symmetric bi-(α, α)-derivation and d be the trace of D. If $d(x)\, d(y) \pm (x \circ y)_{\alpha,\alpha} \in C_{\alpha,\alpha}$ for all $x, y \in I$, then R contains a nonzero central ideal.*

Proof. We see that

$$d(x)\,d(y) \pm (x \circ y)_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{ for all } x, y \in I.$$

Taking y by $y+z$, $z \in I$, we have

$$d(x)\,(d(y) + 2D(y,z) + d(z)) \pm (x \circ y)_{\alpha,\alpha} \pm (x \circ z)_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{for all } x, y, z \in I.$$

By the hypothesis, we obtain that

$$2d(x)D(y,z) \in C_{\alpha,\alpha}, \text{for all } x, y, z \in I.$$

Since R is 2-torsion free, we have

$$d(x)D(y,z) \in C_{\alpha,\alpha}, \text{ for all } x, y, z \in I.$$

Writting z by y in the last equation, we get

$$d(x)d(y) \in C_{\alpha,\alpha}, \text{for all } x, y \in I.$$

Using this in our hypothesis, we find that

$$(x \circ y)_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{for all } x, y \in I.$$

We conclude that R contains a nonzero central ideal by Lemma 2.5. The proof is completed. □

Theorem 2.16. *Let R be a $2-$torsion free semiprime ring, I a nonzero ideal of R, α an automorphism, $D : R \times R \to R$ a symmetric bi-(α,α)-derivation, d be the trace of D and $\alpha(I) \subset I$. If $[d(x), y]_{\alpha,\alpha} \in C_{\alpha,\alpha}$, for all $x, y \in I$, then $[d(x), x]_{\alpha,\alpha} = 0$, for all $x \in I$.*

Proof. By the hypothesis, we get

$$[d(x), y]_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{ for all } x, y \in I.$$

Replacing y by yz in the hypothesis, we have

$$[d(x), y]_{\alpha,\alpha}\,\alpha(z) + \alpha(y)\,[d(x), z]_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{ for all } x, y, z \in I.$$

Commuting this term with $r, r \in R$, we get

$$\Big[[d(x), y]_{\alpha,\alpha}\,\alpha(z) + \alpha(y)\,[d(x), z]_{\alpha,\alpha}, r\Big]_{\alpha,\alpha} = 0.$$

Expanding this equation and using the hypothesis, we arrive at

$$[d(x), y]_{\alpha,\alpha}\,[\alpha(z), \alpha(r)] + [\alpha(y), \alpha(r)]\,[d(x), z]_{\alpha,\alpha} = 0, \text{ for all } x, y, z \in I, r \in R.$$

Replacing r by z in the last equation, we obtain that

$$[\alpha(y), \alpha(z)]\,[d(x), z]_{\alpha,\alpha} = 0, \text{ for all } x, y, z \in I.$$

Taking y by $ty, t \in R$ in above equation, we see that

$$[\alpha(t), \alpha(z)]\alpha(y)\,[d(x), z]_{\alpha,\alpha} = 0, \text{ for all } x, y, z \in I, r, t \in R.$$

Since α is an automorphism, we have

$$[t, \alpha(z)]\alpha(y)\,[d(x), z]_{\alpha,\alpha} = 0, \text{ for all } x, y, z \in I, r, t \in R.$$

That is,

$$[t, z]_{\alpha,\alpha}\,\alpha(y)\,[d(x), z]_{\alpha,\alpha} = 0, \text{ for all } x, y, z \in I, r, t \in R.$$

Replacing t by $d(x)$, we get

$$[d(x), z]_{\alpha,\alpha}\,\alpha(y)\,[d(x), z]_{\alpha,\alpha} = 0, \text{ for all } x, y, z \in I.$$

This implies that

$$[d(x), z]_{\alpha,\alpha}\,V\,[d(x), z]_{\alpha,\alpha} = 0, \text{ for all } x, y, z \in I,$$

where $\alpha(I) = V$ is a nonzero ideal. By Lemma 2.2, we have

$$[d(x), z]_{\alpha,\alpha} = 0, \text{ for all } x, z \in I.$$

In particular, we get $[d(x), x]_{\alpha,\alpha} = 0$, for all $x \in I$. The proof is completed. □

Theorem 2.17. *Let R be a $2-$torsion free semiprime ring, I a square-closed Lie ideal of R, α an automorphism, $D : R \times R \to R$ a symmetric bi-(α, α)-derivation, d be the trace of D and $\alpha(I) \subset I$. If $d\left([x, y]_{\alpha,\alpha}\right) \pm [d(x), y]_{\alpha,\alpha} \in C_{\alpha,\alpha}$, for all $x, y \in I$, then $[d(x), x]_{\alpha,\alpha} = 0$, for all $x \in I$.*

Proof. By the hypothesis, we have

$$d\left([x, y]_{\alpha,\alpha}\right) \pm [d(x), y]_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{ for all } x, y \in I.$$

Writting y by $y + z, z \in I$, we get

$$d\left([x, y]_{\alpha,\alpha}\right) + d([x, z]_{\alpha,\alpha}) + 2D([x, y]_{\alpha,\alpha}, [x, z]_{\alpha,\alpha}) \pm [d(x), y]_{\alpha,\alpha} \pm [d(x), z]_{\alpha,\alpha} \in C_{\alpha,\alpha}.$$

Using the hypothesis and R is $2-$torsion free, we see that

$$D([x, y]_{\alpha,\alpha}, [x, z]_{\alpha,\alpha}) \in C_{\alpha,\alpha}, \text{ for all } x, y \in I.$$

Replacing y by z in the last expression, we have

$$D([x,y]_{\alpha,\alpha}, [x,y]_{\alpha,\alpha}) \in C_{\alpha,\alpha}, \text{ for all } x, y \in I.$$

That is,

$$d([x,y]_{\alpha,\alpha}) \in C_{\alpha,\alpha}, \text{ for all } x, y \in I.$$

From our hypothesis, we have

$$[d(x), y]_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{ for all } x, y \in I.$$

By Theprem 2.16, we conclude that $[d(x), y]_{\alpha,\alpha} = 0$, for all $x, y \in I$. In particular, we get $[d(x), x]_{\alpha,\alpha} = 0$, for all $x \in I$. The proof is completed. □

Theorem 2.18. *Let R be a $2-$torsion free semiprime ring, I a nonzero ideal of R, α an automorphism, $D : R \times R \to R$ a symmetric bi-(α,α)-derivation, d be the trace of D and $\alpha(I) \subset I$. If $d\left((x \circ y)_{\alpha,\alpha}\right) \pm [d(x), y]_{\alpha,\alpha} \in C_{\alpha,\alpha}$, for all $x, y \in I$, then $[d(x), x]_{\alpha,\alpha} = 0$, for all $x \in I$.*

Proof. We assume that

$$d\left((x \circ y)_{\alpha,\alpha}\right) \pm [d(x), y]_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{ for all } x, y \in I.$$

Replacing y by $y + z, z \in I$, we get

$$d\left((x \circ y)_{\alpha,\alpha}\right) + d\left((x \circ z)_{\alpha,\alpha}\right) + 2D((x \circ y)_{\alpha,\alpha}, (x \circ z)_{\alpha,\alpha}) \pm [d(x), y]_{\alpha,\alpha} \pm [d(x), z]_{\alpha,\alpha} \in C_{\alpha,\alpha}.$$

Using the hypothesis and R is $2-$torsion free, we have

$$D((x \circ y)_{\alpha,\alpha}, (x \circ z)_{\alpha,\alpha}) \in C_{\alpha,\alpha}.$$

Writting y by z in the last expression, we get

$$D((x \circ y)_{\alpha,\alpha}, (x \circ y)_{\alpha,\alpha}) \in C_{\alpha,\alpha}, \text{ for all } x, y \in I,$$

and so,

$$d\left((x \circ y)_{\alpha,\alpha}\right) \in Z, \text{ for all } x, y \in I.$$

Using this equation in our hypothesis, we find that

$$[d(x), y]_{\alpha,\alpha} \in C_{\alpha,\alpha}, \text{ for all } x, y \in I.$$

We conclude that $[d(x), y]_{\alpha,\alpha} = 0$, for all $x, y \in I$ by Theorem 2.16. In particular, we get $[d(x), x]_{\alpha,\alpha} = 0$, for all $x \in I$. The proof is completed. □

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