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GROUP DIVISIBLE DESIGNS WITH TWO ASSOCIATE CLASSES AND $\lambda_2 = 5$

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Abstract

Necessary and sufficient conditions for the existence of Group divisible designs with two associate classes and $\lambda_2 = 5$ are here considered. We find that the necessary conditions, derived from graph theoretic conditions, are sufficient as well. We present some constructions to prove sufficiency.

1 Introduction

A *pairwise balanced design* is an ordered pair $(S, \mathcal{B})$, denoted PBD$(S, \mathcal{B})$, where S is a finite set of symbols and $\mathcal{B}$ is a collection of subsets of S called *blocks*, such that each pair of distinct elements of S occurs together in exactly one block of $\mathcal{B}$. Here $|S| = v$ is called the *order* of the PBD. Note that there is no condition on the size of the blocks in $\mathcal{B}$. If all blocks are of the same size k, then we have a *Steiner system* S(v, k). A PBD with index λ can be defined similarly: each pair of distinct elements occurs in λ blocks. If all blocks are same size, say k, then we get a balanced incomplete block design BIBD(v, b, r, k, λ). In other words, a BIBD(v, b, r, k, λ) is a set S of v elements together with a collection

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of b k-subsets of S, called blocks, where each point occurs in r blocks and each pair of distinct elements occurs in exactly λ blocks (see [2], [3], [4]).

Note that in a BIBD(v, b, r, k, λ) the parameters must satisfy the necessary conditions:

1. $vr = bk$ and
2. $\lambda(v-1) = r(k-1)$.

With these conditions a BIBD(v, b, r, k, λ) is usually written as BIBD(v, k, λ).

A *group divisible design* GDD$(v = v_1 + v_2 + \cdots + v_g, g, k, \lambda_1, \lambda_2)$ is a collection of k-subsets (called blocks) of a v-set of symbols, where the v-set is divided into g groups of size $v_1, v_2, \ldots, v_g$; each pair of symbols from the same group occurs in exactly λ_1 blocks; and each pair of symbols from different groups occurs in exactly λ_2 blocks (see [2], [3]). In this paper we consider the problem of determining necessary conditions for an existence of GDD$(v = m + n, 2, 3, \lambda_1, \lambda_2)$ and prove that the conditions are sufficient for some infinite families. Since we are dealing on GDDs with two groups and block size 3, we will use GDD$(m, n, \lambda_1, \lambda_2)$ for GDD$(v = m + n, 2, 3, \lambda_1, \lambda_2)$ from now on, and we refer to the blocks as triples. We denote $(X; Y, \mathcal{B})$ for a GDD$(m, n, \lambda_1, \lambda_2)$ if X and Y are m-set and n-set, respectively. Chaiyasena, et al [1] have written the first paper in this direction, followed by Pabhapote and Punnim [5]. In particular the first paper [1] completely solved the problem of determining all pairs of integers (n, λ) in which a GDD$(1, n, 1, \lambda)$ exists, while the second paper [5] found all triples of integers (m, n, λ) in which a GDD$(m, n, \lambda, 1)$ exists. We continue to investigate in this paper all triples of integers (m, n, λ) in which a GDD$(m, n, \lambda, 5)$ exists, where $\lambda \geq 5$. Surprisingly, this problem can be solved using just $\lambda-$fold triple system constructions, GDD$(1, v, 1, \lambda)$ in [1], and GDD$(m, n, \lambda, 1)$ in [5] as building blocks. There seems to be no need to consider GDD$(m, n, \lambda, 2)$, GDD$(m, n, \lambda, 3)$ nor GDD$(m, n, \lambda, 4)$.

Necessary conditions on the existence of a GDD$(m, n, \lambda_1, \lambda_2)$ can be obtained from a graph theoretic point of view as follows. Let λK_v denote the graph on v vertices in which each pair of vertices is joined by λ edges. Let G_1 and G_2 be graphs. The graph $G_1 \vee_\lambda G_2$ is formed from the union of G_1 and G_2 by joining each vertex in G_1 to each vertex in G_2 with λ edges. A *G-decomposition* of a graph H is a partition of the edges of H such that each element of the partition induces a copy of G. Thus the existence of a GDD$(m, n, \lambda_1, \lambda_2)$ is easily seen to be equivalent to the existence of a K_3-decomposition of $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$. The graph $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$ is of order $m + n$ and size $\lambda_1[\binom{m}{2} + \binom{n}{2}] + \lambda_2 mn$. It contains m vertices of degree $\lambda_1(m-1) + \lambda_2 n$ and n vertices of degree $\lambda_1(n-1) + \lambda_2 m$. Thus the existence of a K_3-decomposition of $\lambda_1 K_m \vee_{\lambda_2} \lambda_1 K_n$ implies

1. $3 \mid \lambda_1[\binom{m}{2} + \binom{n}{2}] + \lambda_2 mn$, and

2. $2 \mid \lambda_1(m-1) + \lambda_2 n$ and $2 \mid \lambda_1(n-1) + \lambda_2 m$.

2 Preliminaries

The following notation will be used for our constructions.

1. Let V be a v-set. We use $K(V)$ for the complete graph K_v on the vertex set V.

2. Let V be a v-set. Then there may be many different STS(v)s that can be constructed on the set V. Let STS(V) be defined as

$$\mathrm{STS}(V) = \{\mathcal{B} : (V, \mathcal{B}) \text{ is an } \mathrm{STS}(v)\}.$$

BIBD($V, 3, \lambda$) can be defined similarly, That is:

$$\mathrm{BIBD}(V,3,\lambda) = \{\mathcal{B} : (V, \mathcal{B}) \text{ is a } \mathrm{BIBD}(v,3,\lambda)\}.$$

Let X and Y be disjoint sets of cardinality m and n, respectively. We define GDD($X, Y, \lambda_1, \lambda_2$) as

$$\mathrm{GDD}(X,Y,\lambda_1,\lambda_2) = \{\mathcal{B} : (X; Y, \mathcal{B}) \text{ is a } \mathrm{GDD}(m,n,\lambda_1,\lambda_2)\}.$$

3. When we say that $\mathcal{B}$ is a *collection* of subsets (blocks) of a v-set V, $\mathcal{B}$ may contain repeated blocks. Thus "$\cup$" in our construction will be used for the union of multi-sets.

4. Finally, if we have a set X, the number of members or vertices of X shall be denoted by $|X|$.

The following results on existence of λ-fold triple systems are well known (see e.g. [4]).

Theorem 2.1. Let n be a positive integer. Then a BIBD($n, 3, \lambda$) exists if and only if λ and n are in one of the following cases:

(a) $\lambda \equiv 0 \pmod 6$ and for all positive integers $n \neq 2$,

(b) $\lambda \equiv 1$ *or* $5 \pmod 6$ and for all n with $n \equiv 1$ *or* $3 \pmod 6$,

(c) $\lambda \equiv 2$ *or* $4 \pmod 6$ and for all n with $n \equiv 0$ *or* $1 \pmod 3$, and

(d) $\lambda \equiv 3 \pmod 6$ and for all odd integers n.

3 GDD$(m, n, \lambda, 5)$

Let λ be a positive integer. We consider in this section the problem of determining all pairs of integers (m, n) in which a GDD$(m, n, \lambda, 5)$ exists. Recall that the existence of GDD$(m, n, \lambda, 5)$ implies $3 \mid \lambda[m(m-1)+n(n-1)]+mn$, $2 \mid \lambda(m-1)+n$ and $2 \mid \lambda(n-1)+m$. Let

$$S_5(\lambda) := \{(m, n) : a \ \text{GDD}(m, n, \lambda, 5) \ \text{exists}\}.$$

Lemma 3.1. Let t be a non-negative integer:

(a) If $(m, n) \in S_5(6t+1)$, then there exist non-negative integers h and k such that $\{m, n\} \in \{\{6k+1, 6h\}, \{6k, 6h+3\}, \{6k+3, 6h+4\}\}$.

(b) If $(m, n) \in S_5(6t+2)$, then there exist non-negative integers h and k such that $\{m, n\} \in \{\{6k, 6h\}, \{6k+2, 6h+4\}, \{6k, 6k+4\}, \{6k+2, 6h+2\}$.

(c) If $(m, n) \in S_5(6t+3)$, then there exist non-negative integers h and k such that $\{m, n\} \in \{\{6k, 6h+1\}, \{6k, 6h+3\}, \{6k+2, 6h+3), \{6k+4, 6h+3\}, \{6k, 6h+5\}\}$.

(d) If $(m, n) \in S_5(6t+4)$, then there exist non-negative integers h and k such that $\{m, n\} \in \{\{6k, 6h\}, \{6k, 6h+4\}\}$

(e) If $(m, n) \in S_5(6t+5)$, then there exist non-negative integers h and k such that $\{m, n\} \in \{\{6k, 6h+1\}, \{6k+1, 6h+2\}, \{6k+3, 6h+4), \{6k, 6h+3\}, \{6k+2, 6h+5\}, \{6k+4, 6h+5\}\}$.

(f) If $(m, n) \in S_5(6t+6)$, then there exist non-negative integers h and k such that $\{m, n\} \in \{\{6k, 6h\}, \{6k, 6h+2\}, \{6k, 6h+4\}\}$.

Proof. The proof follows from solving the corresponding systems of congruences. □

We now proceed with sufficiency for m and n not equal to 2. We note that for simplicity, we only prove sufficiency for say, GDD$(m, n, \lambda, 5)$, since the case of GDD$(n, m, \lambda, 5)$ can be dealt in an identical manner, simply by switching the sets involved. For the sake of economy of space, we will prove sufficiency for λ being the minimal value for the case involved. Once we have a GDD$(m, n, \lambda_1, 5)$, we can readily extend to any $\lambda_1 + 6t$ by the following technique.

Lemma 3.2. GDD$(m, n, \lambda_1, 5)$ can be extended to GDD$(m, n, \lambda_1+6t, 5)$, $t \geq 0$, provided neither m nor n is 2.

Proof. We let X be an m-set and Y be an n-set. We consider $(X; Y, \mathcal{B}_1)$ being a GDD$(m, n, \lambda_1, 5)$ as given. Let $\mathcal{B}_2 \in$ BIBD$(X, 3, 6)$ and $\mathcal{B}_3 \in$ BIBD$(Y, 3, 6)$. Both BIBDs exist by Theorem 2.1[(a)], since neither m nor n is 2. Now let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$. Then $(X; Y, \mathcal{B})$ forms a GDD$(m, n, \lambda_1 + 6t, 5)$ as required. □

Lemma 3.3. Let h and k be non-negative integers. Then $(6k, 6h+1), (6k, 6h+3), (6k+3, 6h+4) \in S_5(6t+1)$.

Proof Let (m, n) be such a pair from the list above. We want to construct a GDD$(m, n, 7, 5)$. Let X be an $m-$set and Y be an $n-$set. Then BIBD$(X \cup Y, 3, 5)$ is not empty since $|X \cup Y| = m+n \equiv 1$ or $\equiv 3 \pmod 6$. (Theorem 2.1[(b)]). Let $\mathcal{B}_1 \in$ BIBD$(X \cup Y, 3, 5)$. Furthermore, BIBD$(X, 3, 2)$ exists since $|X| = m \equiv 0 \pmod 3$ (Theorem 2.1[(c)]. So we let $\mathcal{B}_2 \in$ BIBD$(X, 3, 2)$. Also BIBD$(Y, 3, 2)$ exists as well since $|Y| = n \equiv 0$ or $\equiv 1 \pmod 3$ (Theorem 2.1[c)]. So let $\mathcal{B}_3 \in$ BIBD$(Y, 3, 2)$. We now let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$. Then $(X; Y, \mathcal{B})$ forms a GDD$(m, n, 7, 5)$ as desired. □

Lemma 3.4. Let h and k be non-negative integers. Then $(6k, 6h), (6k+2, 6h+4), (6k, 6h+4), (6k+2, 6h+2) \in S_5(6t+2)$.

Proof. Let (m, n) be such a pair from the list. We want to construct a GDD$(m, n, 8, 5)$. Let X be an $m-$set and Y be an $n-$set. Then there exists BIBD$(X \cup Y, 3, 4)$ since $|X \cup Y| = m + n \equiv 1$ or $\equiv 0 \pmod 3$. (Theorem 2.1[(c)]). Let $\mathcal{B}_1 \in$ BIBD$(X \cup Y, 3, 4)$. There exists GDD$(X, Y, 4, 1)$ by [5]. Let $\mathcal{B}_2 \in$ GDD$(X, Y, 4, 1)$. We now let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. Then $(X; Y, \mathcal{B})$ forms a GDD$(m, n, 8, 5)$ as desired. □

Lemma 3.5. Let h and k be non-negative integers. Then

(a) $(6k, 6h+1), (6k, 6h+3), (6k+3, 6h+4) \in S_5(6t+3)$,

(b) $(6k, 6h+5) \in S_5(6t+3)$.

(c) $(6k+3, 6h+2) \in S_5(6t+3)$.

Proof. (a) Let (m, n) be an ordered pair. We wish to construct GDD$(m, n, 9, 5)$. Let X be an $m-$ set and Y be an $n-$set . There exists BIBD$(X \cup Y, 3, 5)$ since $|X \cup Y| = m + n \equiv 1$ or $\equiv 3 \pmod 6$ (Theorem 2.1[(b)]). Hence let $\mathcal{B}_1 \in$ BIBD$(X \cup Y, 3, 5)$. Also there exists BIBD$(X, 3, 4)$ since $|X| = m \equiv 0 \pmod 3$ (Theorem 2.1[(c)]). Let $\mathcal{B}_2 \in$ BIBD$(X, 3, 4)$. Finally there exists BIBD$(Y, 3, 4)$ since $|X| = m \equiv 0$ or $\equiv 1 \pmod 3$ (Theorem 2.1[(c)]). Let

$\mathcal{B}_3 \in \text{BIBD}(Y, 3, 4)$. Now let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup B_3$. Then $(X; Y, \mathcal{B})$ forms a $\text{GDD}(m, n, 9, 5)$.

(b) Suppose we want $\text{GDD}(6k, 6h+5, 9, 5)$. Let X_k be a $6k$-set and Y_h be a $6h+5$-set containing the element a. Furthermore, let $Y_h' = Y_h - \{a\}$. There exists $\text{BIBD}(X_k \cup Y_h', 3, 4)$ since $|X_k \cup Y_h'| = 6k + 6h + 4 \equiv 1 \pmod 3$ (Theorem 2.1[(c)]). Hence let $\mathcal{B}_1 \in \text{BIBD}(X_k \cup Y_h', 3, 4)$. Also there exists $\text{GDD}(X_k, Y_h', 2, 1)$ by [5]. Let $\mathcal{B}_2 \in \text{GDD}(X_k, Y_h', 2, 1)$. We have the existence of $\text{GDD}(X_k, \{a\}, 1, 5)$ since $|X_k| = 6k$ ([1]). Let $\mathcal{B}_3 \in \text{GDD}(X_k, \{a\}, 1, 5)$. Now we have $\text{BIBD}(X_k, 3, 2)$ since $|X_k| \equiv 0 \pmod 3$ (Theorem 2.1[(c)]). Let $\mathcal{B}_4 \in \text{BIBD}(X_k, 3, 2)$. Finally, we have $\text{GDD}(\{a\}, Y_h', 1, 9)$ since $|Y_h'| = 6h+4 \equiv 4 \pmod 6$ ([1]). Let $\mathcal{B}_5 \in \text{GDD}(\{a\}, Y_h', 1, 9)$. Now we have $\text{BIBD}(Y_h', 3, 2)$ since $|Y_h'| \equiv 1 \pmod 3$ (Theorem 2.1[(c)]). Let $\mathcal{B}_6 \in \text{BIBD}(Y_h', 3, 2)$. Now let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4 \cup \mathcal{B}_5 \cup \mathcal{B}_6$. Then $(X_k; Y_h, \mathcal{B})$ forms $\text{GDD}(6k, 6h+5, 9, 5)$ as required.

(c) We want to construct $\text{GDD}(6k+3, 6h+2, 9, 5)$. Let X_k be a $6k+3$-set and Y_h be a $6h+2$-set containing the element a. Let $Y_h' = Y_h - \{a\}$. There exists $\text{BIBD}(X_k \cup Y_h, 3, 3)$ since $|X_k \cup Y_h| = 6k + 6h + 5$ is an odd number (Theorem 2.1[(d)]). (Note that Y_h, not Y_h' is used here.) Hence let $\mathcal{B}_1 \in \text{BIBD}(X_k \cup Y_h, 3, 3)$. We also have $\text{BIBD}(X_k \cup Y_h', 3, 2)$ since $|X_k \cup Y_h'| = 6k+4 \equiv 1 \pmod 3$ (Theorem 2.1[(c)]. Let $\mathcal{B}_2 \in \text{BIBD}(X_k \cup Y_h', 3, 2)$. We have the existence of $\text{GDD}(X_k, \{a\}, 1, 2)$ since $|X_k| = 6k+3 \equiv 3 \pmod 6$ ([1]). Let $\mathcal{B}_3 \in \text{GDD}(X_k, \{a\}, 1, 2)$. We also have $\text{GDD}(Y_h', \{a\}, 1, 6)$ since $|Y_h'| = 6h+1 \equiv 1 \pmod 6$ ([1]). Let $\mathcal{B}_4 \in \text{GDD}(Y_h', \{a\}, 1, 6)$. There exists $\text{BIBD}(X_k, 3, 3)$ since $|X_k| \equiv 1 \pmod 6$. Let $\mathcal{B}_5 \in \text{BIBD}(X_k, 3, 3)$. Finally, there exists $\text{BIBD}(Y_h', 3, 3)$ since $|Y_h'| \equiv 1 \pmod 6$. Let $\mathcal{B}_6 \in \text{BIBD}(Y_h', 3, 3)$. Now let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4 \cup \mathcal{B}_5 \cup \mathcal{B}_6$. Then $(X_k; Y_h, \mathcal{B})$ forms $\text{GDD}(6k+3, 6h+2, 9, 5)$ as required. □

Lemma 3.6. Let h and k be non-negative integers. Then $(6k, 6h), (6k, 6h+4) \in S_5(6t+4)$,

Proof. Let (m, n) be one of the ordered pairs delineated. We wish to construct $\text{GDD}(m, n, 10, 5)$. Let X be an $m-$ set and Y be an $n-$set. There exists $\text{BIBD}(X \cup Y, 3, 4)$ since $|X \cup Y| = m+n \equiv 0 \text{ or } \equiv 1 \pmod 3$.(Theorem 2.1[(c)]). Hence let $\mathcal{B}_1 \in \text{BIBD}(X \cup Y, 3, 4)$. There also exists $\text{GDD}(X, Y, 2, 1)$ by Theorem [5]. Let $\mathcal{B}_2 \in \text{GDD}(X, Y, 2, 1)$. We have $\text{BIBD}(Y, 3, 4)$ since $|Y| \equiv 0$ or $1 \pmod 3$ (Theorem 2.1[(c)]). Let $B_3 \in \text{BIBD}(\text{Y}, 3, 4)$. Finally we also have $\text{BIBD}(X, 3, 4)$ for the same reason. Let $B_4 \in \text{BIBD}(\text{X}, 3, 4)$. Now let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$. Then $(X_k; Y_h, \mathcal{B})$ forms $\text{GDD}(m, n, 10, 5)$ as required. □

Lemma 3.7. Let h and k be non-negative integers. Then $(6k, 6h+1), (6k+1, 6h+2), (6k+3, 6h+4), (6k, 6h+3), (6k+2, 6h+5), (6k+4, 6h+5) \in S_3(6t+5)$.

Proof. Let (m, n) be such an ordered pair. We want to build GDD$(m, n, 11, 5)$. To this end, let X be an $m-$set and Y be an $n-$set. Then BIBD$(X \cup Y, 3, 5)$ exists since $|X \cup Y| \equiv 0$ or $\equiv 1 \pmod 6$ (Theorem 2.1[(b)]). Let $\mathcal{B}_1 \in$ BIBD$(X \cup Y, 3, 5)$. We also have the existence of BIBD$(X, 3, 6)$ since $|X| \neq 2$ (Theorem 2.1[(a)]). So let $\mathcal{B}_2 \in$ BIBD$(X, 3, 6)$. Also BIBD$(Y, 3, 6)$ exists for the same reasons. So let $\mathcal{B}_3 \in$ BIBD$(Y, 3, 6)$. We now let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$. Then $(X, Y; \mathcal{B})$ forms a GDD$(m, n, 11, 5)$ as desired. □

Lemma 3.8. Let h and k be non-negative integers. Then

(a) $(6k, 6h), (6k, 6h + 4) \in S_5(6t + 6)$.

(b) $(6k, 6h + 2) \in S_5(6t + 6)$.

Proof. (a) Let (m, n) be such an ordered pair. We wish to construct GDD$(m, n, 6, 5)$. Let X be an $m-$ set and Y be an $n-$set. There exists BIBD$(X \cup Y, 3, 4)$ since $|X \cup Y| = m + n \equiv 0$ or $1 \pmod 3$(Theorem 2.1[(c)]). Hence let $\mathcal{B}_1 \in$ BIBD$(X \cup Y, 3, 4)$. There also exists GDD$(X, Y, 2, 1)$ by [5]. Let $\mathcal{B}_2 \in$ GDD$(X, Y, 2, 1)$. Now let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$. Then $(X_k; Y_h, \mathcal{B})$ forms GDD$(m, n, 6, 5)$ as required. □

(b) We construct GDD$(6k, 6h + 2, 6, 5)$. Let X_k be a $6k-$set and Y_h be a $6h + 2-$set containing a. Let $Y'_h = Y_h - \{a\}$. Then BIBD$(X_k \cup Y'_h, 3, 5)$ exists since $|X_k \cup Y'_h| = 6k + 6h + 1 \equiv 1 \pmod 6$ (Theorem 2.1[(b)]). Let $\mathcal{B}_1 \in$ BIBD$(X_k \cup Y'_h, 3, 5)$. Also there exists GDD$(X_k, \{a\}, 1, 5)$ since $|X_k| = 6k$ ([1]). We let $\mathcal{B}_2 \in$ GDD$((X_k, \{a\}, 1, 5)$. Finally, there exists GDD$(Y'_h, \{a\}, 1, 6)$ since $|Y'| = 6h + 1 \equiv 1 \pmod 6$ ([1]). We let $\mathcal{B}_3 \in$ GDD$((Y'_h, \{a\}, 1, 6)$. Now let $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$. Then $(X; Y, \mathcal{B})$ forms a GDD$(6k, 6h + 2, 6, 5)$ as desired.

4 Conclusion

We can now present our main result:

Theorem 4.1. Let m and n be positive integers with $m \neq 2$ or $n \neq 2$. There exists a GDD$(m, n, \lambda, 5)$ if and only if

1. $3 \mid \lambda[m(m-1) + n(n-1)] + mn$, and

2. $2 \mid \lambda(m-1) + n$ and $2 \mid \lambda(n-1) + m$.

Proof. The proof follows from Lemmas 3.1 - 3.8. □

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