# ON FULLY BOUNDED MODULES AND KRULL SYMMETRY

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#### Abstract

The concept of fully bounded modules introduced and as an special case of the Krull symmetry property, it is proved that if  $M_R$  is a fully bounded Noetherian projective module, which is a generator (NPG module for short) in the category of right R-modules (mod-R for short) and  $S = End(M_R)$ , then k-dim  $M_R = k$ -dim SM.

### 1 Introduction

Throughout this paper, all rings are associative with  $1 \neq 0$  and modules assumed to be unitary. An R-module M is called a generator if for every R-module N there is an epimorphism  $M^{(I)} \to N$  for some indexed set I. Obviously, R is a generator and if an epimorphic image of M is a generator, then so is M. This implies that  $R \oplus M$  is a generator for any R-module M, so every free R-module M is a generator. An R-module M is said to be self-generator if it generates all it's submodules. Note that an R-module M is self-generator if and only if for each submodule N of M, there exists  $\Delta \subseteq S = End(M_R)$  such that  $N = \sum_{f \in \Delta} f(M)$ . Also, an R-module M is called quasi-projective (or self-projective) if it is M-projective. The notation  $N \leq_e M$  means that

**Key words:** Keywords: bounded modules, fully bounded modules, Krull symmetry, Krull dimension, classical Krull dimension.

<sup>2010</sup> AMS Mathematics Classification: Primary: 16P60, 16P20; Secondary: 16P40.

N is an essential (or large) submodule of M, that is  $K \cap N \neq 0$  for every non-zero submodule K of M. Let R and S be two rings. the notion  $SM_R$ means that M is an (S-R)-bimodule, that is M is a left S-module and right R-module such that (rm)s = r(ms) for all  $r \in R, m \in M$  and  $s \in S$ . The bimodule  $SM_R$  will be called Noetherian if M is Noetherian both as a left S-module and as a right R-module. We shall say that a bimodule  $SM_R$  is Krull symmetric if M has Krull dimension both as a left S-module and as a right R-module and k-dim SM = k-dim  $M_R$ , see also [1]. T. H. Lenagan proved the zero-dimensional case for Noetherian bimodules, that is k-dim  $M_R = 0$  if and only if k-dim SM = 0, for any Noetherian bimodule  $SM_R$ , see [2, Theorem, 7.10]. Recall that a ring R is right (left) bounded if every essential right (left) ideal of R contains an (2-sided) ideal which is essential as a right (left) ideal. If both condition hold, R is called a bounded ring. Any commutative ring and any semisimple ring is bounded. Note that every nonzero ideal of a prime ring R is right and left essential, so a prime ring R is bounded if and only if every one sided ideal of R contains a nonzero (2-sided) ideal. A ring Ris right (left) fully bounded if  $\frac{R}{P}$  is right (left) bounded for any prime ideal P of R. If R is both left and right fully bounded, it called a fully bounded ring. A right (left) FBN ring is any right (left) fully bounded right (left) Noetherian ring. An FBN ring is any right and left FBN ring. Jategaonkar has shown that if S is a left FBN ring and R is a right FBN ring, then any Noetherian bimodule  $_SM_R$  is Krull symmetric, see [2, Theorem 13.15]. In particular, k-dim R = k-dim R for any FBN ring R. From these facts, naturally raised the Krull symmetry problems that is still open in general. The krull symmetry problems is as follows:

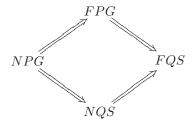
- 1. Does k-dim  $R_R = k$ -dim  $R_R$ , for any Noetherian ring R?
- 2. Does k-dim SM = k-dim  $M_R$  for any Noetherian bimodule  $SM_R$ ?

After that, several authors tried to solve this problems and each one has investigated this issue in a special way, see for example [6]. The main result of this paper, Theorem 2.16, states another special case of Krull symmetry of modules. Now, let us give a brief outline of this paper. First of all, we introduce abreviated symbols in this paper as follows:

- 1. FQS module means a finitely generated quasi-projective module, which is self-generator.
- 2. NQS module means a Noetherian quasi-projective module, which is self-generator.
- 3. *FPG* module means a finitly-generated projective module, which is generator and finally,

 NPG module means a Noetherian projective module, which is a generator in mod-R.

Note that



In [3], we investigate FQS modules. We show that if M is an FQS module, then M is Goldie (critical, dual critical, etc.) if and only if the endomorphism ring  $S = End(M_R)$  has all the latter properties for right hand, respectively. We also prove that an FQS module M has the classical Krull dimension, denoted by cl.k-dim M if and only if it satisfies ACC on prime submodules. The reader is referred to [3] for more details.

In this paper we will discuss some other properties of the FQS modules. First, we defined a bounded and a fully bounded module. Then we proved that if M is an FQS module and  $S = End(M_R)$ , then M is bounded (resp. fully bounded) if and only if S is right bounded (resp. fully bounded) ring. Also we show that if M is an NQS-module, which is FBN, then k-dim  $M_R = k$ -dim S = cl.k-dim S = cl

### 2 main results

We recall that a fully invariant submodule of M is any sub-bimodule of  $SM_R$ , where  $S = End(M_R)$ . Note that, X is a fully invariant submodule of M, if  $f(X) \subseteq X$  for any  $f \in S = End(M_R)$ . According to [5, Definition 1.1], a prime submodule of an R-module M is a fully invariant proper submodule P such that  $IX \subseteq P$  implies  $X \subseteq P$  or  $IM \subseteq P$ , for any ideal I of  $S = End(M_R)$  and any fully invariant submodule X of M. An R-module M is called prime if 0 is a prime submodule of M. Any maximal of the set of all fully invariant submodules of M is prime, see [5, Proposition 1.6]. The set of all prime submoules of M will be denoted by Spec(M). Next, we introduce the concept of bounded and fully bounded modules and investigate some related properties, similar to the definitions in ring theory. We note that by a fully

invariant essential submodule of  $M_R$ , we mean a fully invariant submodule which is essential as an R-submodule.

**Definition 2.1.** An R-module M is bounded if every essential submodule of M, contains a fully invariant essential submodule.

Note that, if M is an FQS prime module, then every nonzero fully invariant submodule of M is an essential R-submodule of M, see [3, Lemma 3.11]. So a prime FQS module M is bounded if and only if every essential submodule of M contains a nonzero fully invariant submodule.

We recall that  $I_X = \{ f \in S : f(M) \subseteq X \}$  and  $AM = \sum_{f \in A} f(M)$ , for any submodule  $X \subseteq M$  and  $A \subseteq S = End(M_R)$ .

Proposition 2.2. Let M be an FQS module and  $S = End(M_R)$ . M is bounded if and only if S is a right bounded ring.

Proof. Let M be bounded and A be an essential right ideal of S, then AM is an essential submodule of M, by [3, Lemma 3.4 and Theorem 3.5], so there exists a fully invariant essential submodule U such that  $U \subseteq AM$ . Then  $I_U$  is an essential ideal of S and  $I_U \subseteq A$ , by the above mentioned facts and therefore S is right bounded. Conversely, let S be right bounded and E be an essential submodule of M. Then  $I_E$  is an essential right ideal of S, so it contains an ideal X, which is right essential. Again, by the mentioned facts, XM is a fully invariant essential submodule of M and  $XM \subseteq I_EM = E$ . Therefore M is bounded.

**Definition 2.3.** An R-module M is fully bounded if  $\frac{M}{P}$  is bounded, for every  $P \in spec(M)$ .

Note that M is a fully bounded R-module if for any prime submodule P and any submodule A of M which  $P \subsetneq A$ , there exists a fully invariant essential submodule X of M such that  $P \subsetneq X \subseteq A$ .

Proposition 2.4. Let M be an FQS module and  $S = End(M_R)$ . M is fully bounded if and only if S if a right fully bounded ring.

Proof. First, let M be fully bounded. In order to show that S is a right fully bounded ring, let  $P \in spec(S)$  and A be a right ideal of S such that  $P \subsetneq A$ . According to [3, Lemma 3.9],  $PM \subsetneq AM$  are submodules of M and  $PM \in spec(M)$ . By definition, there exists a fully invariat essential submodule X of M such that  $PM \subsetneq X \subseteq AM$ . Then  $I_X$  is an ideal of S, which is essential as a right ideal and  $P \subsetneq I_X \subseteq A$ , by [3, Lemmas 3.4 and 3.5]. Therefore S is right fully bounded. Conversely, let S be a right fully bounded ring. In order to show that M is fully bounded, by the comment after Definition 2.3, let  $P \in spec(M)$  and A be a submodule of M such that  $P \subsetneq A$ . Then  $I_P$  is a prime ideal of S properly contained in the right ideal  $I_A$  and so there exists an ideal X of S, which is essential as a right ideal and  $I_P \subsetneq X \subseteq I_A$ . In this

case, XM is a fully invariant essential submodule of M and  $P \subsetneq XM \subseteq A$ . Therefore M is fully bounded and we are done.

Let us recall two results to prove the next theorem.

Theorem 2.5. [3, Theorem 4.3] Let M be an FQS module,  $S = End_R(M)$  and X be a submodule of M. Then we have:

- 1.  $G dim M_R = G dim S_S$ .
- 2. k-dim  $M_R = k$ -dim  $S_S$ , if either side exists.
- 3. k-dim X = k-dim  $I_X$ .
- 4. n-dim  $M_R = n$ -dim  $S_S$ , if either side exists.
- 5. n-dim X = n-dim  $I_X$ .
- 6. k-dim  $\frac{M}{X} = k$ -dim  $\frac{S}{I_X}$ , if X is fully invariant.
- 7. n-dim  $\frac{M}{X} = n$ -dim  $\frac{S}{I_X}$ , if X is fully invariant.

Theorem 2.6. [3, Theorem 4.21] Let M be an FQS module and  $S = End_R(M)$ . Then M has classical Krull dimension if and only if S has classical Krull dimension and in this case cl.k-dim M = cl.k-dim S.

We recall that, in [4] Krause showed that if R is a right fully bounded ring with right Krull dimension then cl.k-dim  $R \le k$ -dim R and equality holds, where R is a right FBN ring. The next result is an analogue of this fact for modules.

Theorem 2.7. Let M be a fully bounded NQS-module. Then:

$$k$$
-dim  $M_R = k$ -dim  $S_S = cl.k$ -dim  $S = cl.k$ -dim  $M_R$ .

*Proof.* Note that M is an FQS module and so k-dim  $M_R = k$ -dim  $S_S$  and cl.k-dim  $M_R = cl.k$ -dim S, by Theorems 2.5, 2.6. Since M is fully bounded, S is a right fully bounded ring, by Proposition 2.4. Also, M is Noetherian and so S is right Noetherian, by Theorem 2.5. Thus S is a right FBN ring and so k-dim  $S_S = cl.k$ -dim S, by the above comment and therefore

$$k$$
-dim  $M_R = k$ -dim  $S_S = cl.k$ -dim  $S = cl.k$ -dim  $M_R$ .

Let M be a right R-module,  $S = End(M_R)$  and T = End(sM) be the endomorphism ring of sM. In the literature, T is called the biendomorphism ring of  $M_R$ , abbreviated  $Biend(M_R)$ . Recall that the canon  $\psi: R \to T$ , by  $\psi(r) = \bar{r}$ , where  $\bar{r}(m) = mr$ , is a ring homomorphism. The module  $M_R$  is called faithful, when the canonical map  $\psi$  is injective or equivalently,  $ann_R(M) = \{r \in R : Mr = 0\} = 0$ . In other case that  $\psi$  is surjective, M is called to be balanced. We cite the following fact.

Theorem 2.8. [7, 18.8] Let M be a right R-module and  $S = End(M_R)$ , then  $M_R$  is a generator if and only if

- 1. sM is FP(i.e., finitely generated and projective), and
- 2.  $R \cong T = Biend(M_R)$

The following result is now immediate.

Corollary 2.9. Let  $M_R$  be a generator, then  $M_R$  is both faithful and balanced. Now, we have the following.

Theorem 2.10. Let  $M_R$  be a module.

- 1. If  $M_R$  is FPG, then so is  $_SM$ .
- 2. If  $M_R$  is faithful and balanced, then the converse of (1) holds, i.e., if SM is an FPG-module, so is  $M_R$ .

*Proof.* (1) Since  $M_R$  is a generator, thus  $_SM$  is FP and  $R \cong T$ , by Theorem 2.8. In this case,  $M_T$  is FP and  $S = End(M_R) \cong End(M_T) = Biend(_SM)$ . Again, we may invoke the left-hand version of Theorem 2.8 to show that  $_SM$  is a generator. Therefore  $_SM$  is an FPG-module.

(2) If  $_SM$  is FPG, then so is  $M_T$ , by item (1). If in addition,  $M_R$  is faithful and balanced, then  $R\cong T$ . Therefore  $M_R$  is an FPG-module and we are done.  $\sqcap$ 

According to [5, Theorem 2.4], a fully invariant proper submodule P of an R-module M is prime if  $\phi X \subseteq P$  implies that  $X \subseteq P$  or  $\phi M \subseteq P$  for any  $\phi \in S = End(M_R)$  and any fully invariant submodule X of M. Similarly, a prime submodule of SM is a fully invariant submodule P of SM such that  $X\psi \subseteq P$  implies that  $X \subseteq P$  or  $M\psi \subseteq P$ , for any  $\psi \in T = End(SM) = Biend(M_R)$  and fully invariant submodule X of SM.

**Remark 2.11.** Let  $M_R$  be a faithful and balanced module, then  $R \cong T = Biend(M_R)$  and it is easy to see that X is a fully invariant submodule of  $M_R$  if and only if X is a fully invariant submodule of SM for any  $X \subseteq M$ . In other words, for both modules  $M_R$  and SM, the sets of fully invariant submodules are coincide. Thus, according to the above comment, P is a prime submodule of SM, if and only if, P is a fully invariant submodule of  $M_R$  such that  $X \subseteq P$  implies that  $X \subseteq P$  or  $M \subseteq P$ , for every  $P \in R$  and fully invariant submodule  $P \cap M_R$ .

Now, with respect to the previous remark, we will have the following result. We note that, by  $Spec(M_R)$  and Spec(SM) we means the set of all prime submodules of  $M_R$  and SM.

Theorem 2.12. Let  $M_R$  be an FPG module,  $S = End(M_R)$  and  $P \subseteq M$ . Then  $P \in Spec(M_R)$  if and only if  $P \in Spec(SM)$ .

Proof. First, let  $P \in Spec(M_R)$ . Since  $M_R$  is a generator, it is faithful and balanced and so P is a fully invariant submodule of  ${}_SM$  too, by the previous remark. In order to show that  $P \in Spec({}_SM)$ , let X be a fully invariant submodule of  $M_R$  and  $r \in R$  such that  $Xr \subseteq P$ . Define  $\phi : M \to M$ , by  $\phi(m) = mr$ , then clearly  $\phi \in S$  and  $Xr = \phi X \subseteq P$ . Since  $P \in Spec(M_R)$ , we see that  $X \subseteq P$  or  $\phi M = Mr \subseteq P$ . Now we use the previous remark to conclude that  $P \in Spec(SM)$ . Conversely, let  $P \in Spec(SM)$ . Note that SM is an SM module, by Theorem 2.10. The previous reasoning implies that  $P \in Spec(M_T)$ , where  $T = Biend(M_R)$ . But  $M_R$  is faithful and balanced and so there is the ring isomorphism  $R \cong T$ . Consequently,  $P \in Spec(M_R)$  and this copmlete the proof.

We cite the following definition from [3, Definition 4.18].

**Definition 2.13.** Let M be an R-module. We denote the set of all prime submodules of M, by Spec(M). Let X(M) = Spec(M) and  $X_0(M)$  denote the set of all maximal fully invariant submodules of M. For an ordinal number  $\alpha > 0$ ,  $X_{\alpha}(M)$  denote the set of all prime submodules P of M such that each prime submodule Q properly containing P, belongs to  $X_{\beta}$  for some  $\beta < \alpha$ . Hence we have  $X_0(M) \subseteq X_1(M) \subseteq X_2(M) \subseteq \ldots$  The smallest ordinal  $\alpha$  for which  $X_{\alpha}(M) = X(M)$  is called the classical Krull dimension of M and is denoted by cl.k-dim M.

For left hand modules, provided a similar definition. Now we have the next important result.

Theorem 2.14. Let  $M_R$  be an FPG module and  $S = End(M_R)$ . Then cl.k-dim  $M_R = cl.k$ -dim SM, if either side exists.

*Proof.* By Theorem 2.12, immediately it results that for every ordinal  $\alpha$ ,  $X_{\alpha}(M_R) = X_{\alpha}(SM)$  and so we are done.

Now we return to our main purpose to find a sort of modules  $M_R$  for which k-dim  $M_R = k$ -dim SM. We cite the next known fact.

Lemma 2.15. [2, Theorem 13.15] Let S be a left FBN ring, R a right FBN ring and  $_SM_R$  a bimodule, which is finitely generated on both sides. Then  $k\text{-}\dim M_R = k\text{-}\dim _SM$ .

We will now state our main result:

Theorem 2.16. Let R be a left FBN ring,  $M_R$  a fully bounded NPG module and  $S = End(M_R)$ . Then k-dim  $M_R = k$ -dim SM.

*Proof.* Obviously M is both NQS and FPG, so by Theorems 2.7 and 2.14, k-dim  $M_R = cl.k$ -dim  $M_R$ 

the left-hand version of Theorem 2.7 to see that k-dim  $_SM=cl.k$ -dim  $_SM$ . Consequently, k-dim  $M_R=k$ -dim  $_SM$ .

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