East-West J. of Mathematics: Vol. 22, No 2 (2020) pp. 153-164
https://doi.org/10.36853/ewjm0377

SOME TYPES OF PARTIAL GENERALIZED HYPERSUBTITUTIONS OF MANY-SORTED ALGEBRAS

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Abstract

One of important study in Universal algebra is to classify algebras into varieties and classify varieties into hypervarieties. The concept of a hypersubstitution, which is a tool used to study hyperidentities, was introduced by K. Denecke, D. Lau, R. Pöschel and D. Schweigert [3]. In 2000, S. Leeratanavalee and K. Denecke [6] extended the above concept to the concept of a generalized hypersubstitution. In Universal algebra, we do not study only algebras which have one base set but many base sets. In 1970, G. Birkhoff and John D. Lipson [1] extended the concept of base structure of algebras from one-sorted to many-sorted, that is called heterogeneous algebras or many-sorted algebras. In this present paper, we show that the set of partial generalized hypersubstitutions $\Sigma^{|I|,n}(i)$-Hyp_G forms a monoid.

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Key words: many-sorted algebra, i-sorted partial Σ-generalized hypersubstitution, i-sorted Σ-algebras, Σ-terms.
2010 AMS Mathematics Classification: 08A99; 03C05.

1 Introduction

In computer programming, there are two major types of data. The first one is a basic type such as integer, float, character and string which can be used to solve some simple problems. However, it proves to be difficult to solve more complex problems using only this type of data, so the abstract data type (ADT) has been invented. We can describe the structure of ADT as an algebra. In Universal algebra, we have not studied only structure of algebras, but we classify algebras using identities into collections called *varieties* and classify varieties into a high level of varieties called *hypervarieties*.

For the usual definition of algebra, when we mention about an algebra, we always imagine an algebra which has only one base set. It is very interesting to study an algebra which has more than one base set and all of operations can be defined on different base sets. In some situations, for instance, colors, as we know all colors can be created by mixing the primary colors together. If we let the mixing of two colors and the mixing ratio be the operations and the collection of all colors and the amount of each color added be the base sets, then we can explain this situation using many-sorted algebra. The concept of many-sorted algebras was introduced in 1970 by G. Birkhoff and John D. Lipson [1]. A vector space $\mathcal{V}$ over field $\mathbb{F}$ is one of examples of many-sorted algebra.

Let I be a nonempty set, $I^* := \bigcup_{n\geq 1} I^n$ and $\Sigma \subseteq I^* \times I$ with $\Sigma_n := \Sigma \cap I^{n+1}$. Let $A := (A_i)_{i\in I}$ be an I-sorted set, an I-indexed family of sets, where A_i is the set of elements of sort i of A, for $i \in I$. A pair $\underline{A} := (A, ((f_\gamma^A)_k)_{k\in K_\gamma,\gamma\in\Sigma})$ is called an I-sorted Σ-algebra where $f_\gamma^A : A_{k_1} \times \ldots \times A_{k_n} \to A_i$ is a mapping, is called an I-sorted n-ary operation on A, where $\gamma := (k_1, \ldots, k_n, i) \in I^{n+1}$ and K_γ is the set of indices with respect to γ. For $\gamma \in I^*$, let $\gamma(j)$ denote the j-th component of γ.

Example 1.1. A vector space over field F:
The structure $\underline{A} := (\{V, F\}, \{+^A_{(1,1,1)}, \cdot^A_{(2,1,1)}\})$ is an I-sorted Σ-algebra with $I = \{1, 2\}$, $A = \{V, F\}$ and $\Sigma = \{(1, 1, 1), (2, 1, 1)\}$, that is there are two binary operations consist of $+^A_{(1,1,1)}$ (addition) and $\cdot^A_{(2,1,1)}$ (scalar multiplication), i.e.,

$$+^A_{(1,1,1)} : V \times V \to V \quad \text{and} \quad \cdot^A_{(2,1,1)} : F \times V \to V.$$

For $i \in I$, we set $\Lambda_n(i) := \{\alpha \in I^{n+1} |\ \alpha(n + 1) = i\}$, $\Lambda(i) := \bigcup_{n=1}^{\infty} \Lambda_n(i)$ and $\Lambda := \bigcup_{i\in I} \Lambda(i)$.

Let $\Sigma_m(i) := \{\gamma \in \Sigma_m |\ \gamma(m + 1) = i\}$ and $\Sigma(i) := \bigcup_{n=1}^{\infty} \Sigma_m(i)$.

The concept of terms for many-sorted algebras was introduced by K. Denecke and S. Lekkoksung [4] in 2008.

Definition 1.2. Let $n \in \mathbb{N}^+$ and I be an indexed set. Let $X^{(n)} := (X_i^{(n)})_{i\in I}$ be an I-sorted set of n variables, is called an n-element I-sorted alphabet, with $X_i^{(n)} = \{x_{i1}, x_{i2}, ..., x_{in}\}, i \in I$ and let $X = (X_i)_{i\in I}$ be an I-sorted set of variables, is called an I-sorted alphabet, with $X_i = \{x_{i1}, x_{i2}, x_{i3}, ...\}, i \in I$. Let $((f_\gamma)_k)_{k\in K_\gamma,\gamma\in\Sigma}$ be a Σ-sorted set of operation symbols. Then for each $i \in I$, a set $W_n(i)$ which is called the set of all n-ary Σ-terms of sort i, is inductively defined as follows:

1. $W_0^n(i) := X_i^{(n)}$,
2. $W_{l+1}^n(i) := W_l^n(i) \cup \{f_\gamma(t_{k_1}, ..., t_{k_n}) \mid \gamma = (k_1, ..., k_n, i) \in \Sigma,\ t_{k_j} \in W_l^n(k_j)\},\ l \in \mathbb{N}$. Here we inductively assume that the set $W_l^n(i)$ are already defined for all sorts $i \in I$.

Then $W_n(i) := \bigcup_{l=0}^{\infty} W_l^n(i)$ and $W(i) := \bigcup_{n\in\mathbb{N}} W_n(i)$. $W(i)$ is called an I-sorted set of all Σ-terms of sort i. The set $W_\Sigma(X) := (W(i))_{i\in I}$ is called an I-sorted set of all Σ-terms and its elements are called I-sorted Σ-terms.

To study hypervariety, we first need to study hypersubstitutions. In order to do that, we need to define a binary operation on a set of hypersubstitutions which satisfies an associative law. This also holds true in the case of many-sorted algebra. For each $i \in I$, an arbitary mapping

$$\sigma_i : \{f_\gamma \mid \gamma \in \Sigma(i)\} \to W(i)$$

is called a Σ-generalized hypersubstitution of sort i. The set of all Σ-generalized hypersubstitutions of sort i is denoted by $\Sigma(i)$-Hyp_G. To define a binary operation on $\Sigma(i)$-Hyp_G, we need the concept of the superposition operation.

Definition 1.3. The superposition operation

$$S_\beta : W(i) \times W(k_1) \times ... \times W(k_n) \to W(i),$$

for $\beta = (k_1, ..., k_n, i) \in \Lambda$, is defined inductively by the following steps:

1. If $t = x_{ij} \in X_i$, then
   (a) $S_\beta(x_{ij}, t_1, ..., t_n) = x_{ij}$ if $i \neq k_j, \forall j$ and,
   (b) $S_\beta(x_{ij}, t_1, ..., t_n) = t_j$ if $i = k_j, 1 \leq j \leq n$ and,
   (c) $S_\beta(x_{ij}, t_1, ..., t_n) = x_{ij}$ if $j > n$.

2. If $t = f_\gamma(s_1, ..., s_m) \in W(i)$, for $\gamma = (i_1, ..., i_m, i) \in \Sigma$ and $s_q \in W(i_q), 1 \leq q \leq m$, and assume that $S_{\beta_q}(s_q, t_1, ..., t_n)$ with $\beta_q = (k_1, ..., k_n, i_q) \in \Lambda(i_q)$ are already defined, then

$$S_\beta(f_\gamma(s_1, ..., s_m), t_1, ..., t_n) := f_\gamma(S_{\beta_1}(s_1, t_1, ..., t_n), ..., S_{\beta_m}(s_m, t_1, ..., t_n)),$$

for $t_j \in W(k_j),\ 1 \leq j \leq n$.

For any Σ-generalized hypersubstitution σ_i of sort i can be extended to a mapping $\hat{\sigma}_i : W(i) \to W(i)$ is definded by

1. $\hat{\sigma}[x_{ij}] := x_{ij}$, for $x_{ij} \in X_i$,

2. $\hat{\sigma}[f_\gamma(t_1, ..., t_n)] := S_\gamma(\sigma_i(f_\gamma), \hat{\sigma}_{k_1}[t_1], ..., \hat{\sigma}_{k_n}[t_n])$ where $\gamma = (k_1, ..., k_n, i)$ and $t_j \in W(k_j), 1 \leq j \leq n$, assume that $\hat{\sigma}_{k_j}[t_j]$ are already defined.

Since the extension of a Σ-generalized hypersubstitution of sort i is unique, we can define a binary operation $\circ_G^i$ on $\Sigma(i)$-Hyp_G by

$$(\sigma_1)_i \circ_G^i (\sigma_2)_i := (\hat{\sigma_1})_i \circ (\sigma_2)_i,$$

for $(\sigma_1)_i, (\sigma_2)_i \in \Sigma(i)$-$Hyp_G$ and $\circ$ is the usual composition of mapping. Let $(\sigma_{id})_i \in \Sigma(i)$-$Hyp_G$ which maps each operation symbol f_γ to the Σ-term $f_\gamma(x_{k_1 1}, ..., x_{k_n n})$, for $\gamma = (k_1, ..., k_n, i) \in \Sigma(i)$, i.e.,

$$(\sigma_{id})_i(f_\gamma) := f_\gamma(x_{k_1 1}, ..., x_{k_n n}).$$

Example 1.4. Let $\Sigma = \{(2, 2, 1), (2, 1, 1, 1)\}$, i.e., there are two operations f_γ, f_β with $\gamma = (2, 2, 1)$,
$\beta = (2, 1, 1, 1)$. Let $\sigma_1, \sigma_2, \sigma_3 \in \Sigma(i)$-$Hyp_G$ such that $\sigma_1(f_\gamma) = x_{13}, \sigma_1(f_\beta) = x_{13}$, $\sigma_2(f_\gamma) = f_\beta(x_{21}, x_{12}, x_{15})$, $\sigma_2(f_\beta) = f_\gamma(x_{22}, x_{21})$ and $\sigma_3(f_\beta) = f_\beta(x_{23}, f_\gamma(x_{25}, x_{22}), x_{15})$. We have

$$\begin{aligned}
(\sigma_1 \circ_G^i (\sigma_2 \circ_G^i \sigma_3))(f_\beta) &= (\hat{\sigma}_1 \circ (\hat{\sigma}_2 \circ \sigma_3))(f_\beta) = \hat{\sigma}_1[\hat{\sigma}_2[\sigma_3(f_\beta)]] \\
&= \hat{\sigma}_1[\hat{\sigma}_2[f_\beta(x_{23}, f_\gamma(x_{25}, x_{22}), x_{15})]] \\
&= \hat{\sigma}_1[S_\beta(\sigma_2(f_\beta), x_{23}, \hat{\sigma}_2[f_\gamma(x_{25}, x_{22})], x_{15})] \\
&= \hat{\sigma}_1[S_\beta(f_\gamma(x_{22}, x_{21}), x_{23}, f_\beta(x_{25}, x_{12}, x_{15}), x_{15})] \\
&= \hat{\sigma}_1[f_\gamma(x_{22}, x_{23})] \\
&= S_\gamma(\sigma_1(f_\gamma), x_{22}, x_{23}) \\
&= S_\gamma(x_{13}, x_{22}, x_{23}) = x_{13},
\end{aligned}$$

$$\begin{aligned}((\sigma_1 \circ_G^i \sigma_2) \circ_G^i \sigma_3)(f_\beta) &= (\sigma_1 \circ_G^i \sigma_2)\hat{}[\sigma_3(f_\beta)] \\ &= (\sigma_1 \circ_G^i \sigma_2)\hat{}[f_\beta(x_{23}, f_\gamma(x_{25}, x_{22}), x_{15})] \\ &= S_\beta((\sigma_1 \circ_G^i \sigma_2)(f_\beta), x_{23}, (\sigma_1 \circ_G^i \sigma_2)\hat{}[f_\gamma(x_{25}, x_{22})], x_{15}) \\ &= S_\beta(\hat{\sigma}_1[\sigma_2(f_\beta)], x_{23}, x_{15}, x_{15}) \\ &= S_\beta(\hat{\sigma}_1[f_\gamma(x_{22}, x_{21})], x_{23}, x_{15}, x_{15}) \\ &= S_\beta(S_\gamma(\sigma_1(f_\gamma), x_{22}, x_{21}), x_{23}, x_{15}, x_{15}) \\ &= S_\beta(S_\gamma(x_{13}, x_{22}, x_{21}), x_{23}, x_{15}, x_{15}) \\ &= S_\beta(x_{13}, x_{23}, x_{15}, x_{15}) = x_{15}.\end{aligned}$$

That is $(\sigma_1 \circ_G^i \sigma_2) \circ_G^i \sigma_3 \neq \sigma_1 \circ_G^i (\sigma_2 \circ_G^i \sigma_3)$.

we figured out that $(\Sigma(i)\text{-}Hyp_G, \circ_G^i, (\sigma_{id})_i)$ is a non associative (with identity). So, we need to put some conditions for each Σ-generalized hypersubstitution of sort i, $i \in I$. In this paper, we consider the structure of many-sorted algebra which all of operation symbols of sort i have the same arity n ($n \geq 2$) and have the same structure, i.e., for each $i \in I$, $\Sigma(i) = \{\gamma\}$ and each $k \in K_\gamma$, $(f_\gamma)_k$ is n-ary. We denote a set of type of operation symbols by $\Sigma^{|I|,n}(i)$.

In 2006, S. Busaman and K. Denecke [2] established the definition of a partial hypersubstitution. Motivated by these concepts, we are interested to study partial generalized hypersubstitutions in many-sorted algebras.

2 Main Results

For $i \in I$, a partial generalized hypersubstitution on $\{f_\gamma |\ \gamma \in \Sigma^{|I|,n}(i)\}$ is a partial function

$$\sigma_i : \{f_\gamma |\ \gamma \in \Sigma^{|I|,n}(i)\} \multimap W(i),$$

that is $dom\sigma_i \subseteq \{f_\gamma |\ \gamma \in \Sigma^{|I|,n}(i)\}$ and $f_\gamma \in dom\sigma_i$ if $\sigma_i(f_\gamma)$ is defined. Denote $\Sigma^{|I|,n}(i)\text{-}PHyp_G$ the set of all partial generalized hypersubstitutions of sort i. If $dom\sigma_i = \{f_\gamma |\ \gamma \in \Sigma^{|I|,n}(i)\}$, we have σ_i is a generalized hypersubstitution and let $\Sigma^{|I|,n}(i)\text{-}Hyp_G$ be the set of all generalized hypersubstitutions of sort i.

Next, we give the definition of a partial superposition operation and prove some of it properties.

Definition 2.1. For $\beta = (k_1, \ldots, k_n, i) \in \Lambda$, the partial superposition operation

$$S_\beta^{'} : W(i) \times W(k_1) \times \ldots \times W(k_n) \multimap W(i)$$

is a partial function of the superposition operation S_β which is defined if all of $n+1$ input terms are defined.

Lemma 2.2. Let $m, n \in \mathbb{N}^+$ with $m \leq n$. Then

$$S_\beta^{'}(s, S_{\beta_1}^{'}(l_1, t_1, \ldots, t_m), \ldots, S_{\beta_n}^{'}(l_n, t_1, \ldots, t_m)) = S_\gamma^{'}(S_\beta^{'}(s, l_1, \ldots, l_n), t_1, \ldots, t_m)$$

where $\beta = (i_1, \ldots, i_n, i), \gamma = (i_1, \ldots, i_m, i)$ and $\beta_j = (i_1, \ldots, i_m, i_j)$.
Proof. We have

$$\begin{aligned} S'_\beta(s, S'_{\beta_1}(l_1, t_1, \ldots, t_m), \ldots, S'_{\beta_n}(l_n, t_1, \ldots, t_m)) \text{ is defined } \Leftrightarrow \\ s, S'_{\beta_j}(l_j, t_1, \ldots, t_m), \forall j \in \{1, \ldots, n\} \text{ are defined.} \Leftrightarrow \\ s, l_j, t_q \text{ are defined}, \forall j \in \{1, \ldots, n\}, q \in \{1, \ldots, m\}. \Leftrightarrow \\ S'_\gamma(S'_\beta(s, l_1, \ldots, l_n), t_1, \ldots, t_m) \text{ is defined.} \end{aligned}$$

Next, we show that $S'_\beta(s, S'_{\beta_1}(l_1, t_1, \ldots, t_m), \ldots, S'_{\beta_n}(l_n, t_1, \ldots, t_m)) = S'_\gamma(S'_\beta(s, l_1, \ldots, l_n), t_1, \ldots, t_m)$. We prove by induction on the complexity of the Σ-term $s \in W(i)$.

(i) $s = x_{ij} \in X(i)$, we consider into three cases.

Case 1: $i \neq k_j$.

$$\begin{aligned} & S'_\beta(s, S'_{\beta_1}(l_1, t_1, \ldots, t_m), \ldots, S'_{\beta_n}(l_n, t_1, \ldots, t_m)) = \\ & \quad = S'_\beta(x_{ij}, S'_{\beta_1}(l_1, t_1, \ldots, t_m), \ldots, S'_{\beta_n}(l_n, t_1, \ldots, t_m)) \\ & \quad = x_{ij} \\ & \quad = S'_\gamma(x_{ij}, t_1, \ldots, t_m) \\ & \quad = S'_\gamma(S'_\beta(x_{ij}, l_1, \ldots, l_n), t_1, \ldots, t_m). \end{aligned}$$

Case 2 : $i = i_j, 1 \leq j \leq n$.

$$\begin{aligned} & S'_\beta(s, S'_{\beta_1}(l_1, t_1, \ldots, t_m), \ldots, S'_{\beta_n}(l_n, t_1, \ldots, t_m)) = \\ & \quad = S'_\beta(x_{ij}, S'_{\beta_1}(l_1, t_1, \ldots, t_m), \ldots, S'_{\beta_n}(l_n, t_1, \ldots, t_m)) \\ & \quad = S'_{\beta_j}(l_j, t_1, \ldots, t_m) \\ & \quad = S'_{\beta_j}(S'_\beta(x_{ij}, l_1, \ldots, l_n), t_1, \ldots, t_m) \\ & \quad = S'_\gamma(S'_\beta(x_{ij}, l_1, \ldots, l_n), t_1, \ldots, t_m). \end{aligned}$$

Case 3 : $j > n$.

$$\begin{aligned} & S'_\beta(s, S'_{\beta_1}(l_1, t_1, \ldots, t_m), \ldots, S'_{\beta_n}(l_n, t_1, \ldots, t_m)) = \\ & \quad = S'_\beta(x_{ij}, S'_{\beta_1}(l_1, t_1, \ldots, t_m), \ldots, S'_{\beta_n}(l_n, t_1, \ldots, t_m)) \\ & \quad = x_{ij} \\ & \quad = S'_\gamma(x_{ij}, t_1, \ldots, t_m) \\ & \quad = S'_\gamma(S'_\beta(x_{ij}, l_1, \ldots, l_n), t_1, \ldots, t_m). \end{aligned}$$

(ii) $s = f_\alpha(s_1,\ldots,s_h) \in W(i)$ with $\alpha = (p_1,\ldots,p_h,i) \in \Sigma^{|I|,h}(i)$ and $s_r \in W(p_r), 1 \leq r \leq h$.

We assume that $S'_{\alpha_r}(s_r, S'_{\beta_1}(l_1,t_1,\ldots,t_m),\ldots,S'_{\beta_n}(l_n,t_1,\ldots,t_m)) = S'_{\gamma_r}(S'_{\alpha_r}(s_r,l_1,\ldots,l_n),t_1,\ldots,t_m)$ where $\alpha_r = (k_1,\ldots,k_n,p_r) \in \Lambda(p_r)$ and $\gamma_r = (i_1,\ldots,i_m,p_r) \in \Lambda(p_r), 1 \leq r \leq h$. Then

$$\begin{aligned}
S'_\beta(s, S'_{\beta_1}&(l_1,t_1,\ldots,t_m),\ldots,S'_{\beta_n}(l_n,t_1,\ldots,t_m)) = \\
&= S'_\beta(f_\alpha(s_1,\ldots,s_h), S'_{\beta_1}(l_1,t_1,\ldots,t_m),\ldots,S'_{\beta_n}(l_n,t_1,\ldots,t_m)) \\
&= f_\alpha(S'_{\alpha_1}(s_1, S'_{\beta_1}(l_1,t_1,\ldots,t_m),\ldots,S'_{\beta_n}(l_n,t_1,\ldots,t_m)),\ldots, \\
&\qquad S'_{\alpha_h}(s_h, S'_{\beta_1}(l_1,t_1,\ldots,t_m),\ldots,S'_{\beta_n}(l_n,t_1,\ldots,t_m))) \\
&= f_\alpha(S'_{\gamma_1}(S'_{\alpha_1}(s_1,l_1,\ldots,l_n),t_1,\ldots,t_m),\ldots, \\
&\qquad S'_{\gamma_h}(S'_{\alpha_h}(s_h,l_1,\ldots,l_n),t_1,\ldots,t_m)) \\
&= S'_\gamma(f_\alpha(S'_{\alpha_1}(s_1,l_1,\ldots,l_n),\ldots,S'_{\alpha_h}(s_h,l_1,\ldots,l_n)),t_1,\ldots,t_m) \\
&= S'_\gamma(S'_\beta(f_\alpha(s_1,\ldots,s_h),l_1,\ldots,l_n),t_1,\ldots,t_m).
\end{aligned}$$

Then we complete the proof of this lemma. □

For $\sigma_i \in \Sigma^{|I|,n}(i)$-$PHyp_G$, it can be extended to partial mapping $\hat{\sigma}_i : W(i) \multimap\to W(i)$ defined by

1. $\hat{\sigma}_i[x_{ij}] = x_{ij}$, for $x_{ij} \in W(i)$,

2. $\hat{\sigma}_i[f_\gamma(t_1,\ldots,t_n)] = S'_\gamma(\sigma_i(f_\gamma), \hat{\sigma}_1[t_1],\ldots,\hat{\sigma}_n[t_n])$, where $\gamma = (k_1,\ldots,k_n,i) \in \Sigma^{|I|,n}(i)$ and $t_j \in W(k_j)$ such that $t_j \in dom\hat{\sigma}_{k_j}$, $\hat{\sigma}_{k_j}[t_j]$ are already defined and $f_\gamma \in dom\sigma_i$.

Next, we define a binary operation $\circ^i_p$ on $\Sigma^{|I|,n}(i)$-$PHyp_G$ by for $(\sigma_1)_i, (\sigma_2)_i \Sigma^{|I|,n}(i)$-$PHyp_G$,

$$(\sigma_1)_i \circ^i_p (\sigma_2)_i := (\hat{\sigma}_1)_i \circ (\sigma_2)_i$$

and $dom((\sigma_1)_i \circ^i_p (\sigma_2)_i) = \{f_\gamma |\ f_\gamma \in dom(\sigma_2)_i \text{ and } (\sigma_2)_i(f_\gamma) \in dom(\hat{\sigma}_1)_i\}$.

Example 2.3. Let $I = \{1,2\}$, $\Sigma^{|I|,2}(1) = \{(2,1,1)\}, K_{(2,1,1)} = \{1,2\}$ and $\Sigma^{|I|,2}(2) = \{(2,2,2)\}$.
Denote $\gamma = (2,1,1), \beta = (2,1,1)$ and $\alpha = (2,2,2)$. Let $\sigma_1, \sigma_2 \in \Sigma^{|I|,2}(1)$-$PHyp_G$ be defined by

$\sigma_1(f_\gamma) = f_\beta(f_\alpha(x_{24},x_{21}),x_{11}), \sigma_1(f_\beta) = x_{11}$ and $\sigma_2(f_\gamma)$ is undefined, $\sigma_2(f_\beta) = f_\gamma(x_{23},x_{14})$.

Then $(\sigma_1 \circ_p^i \sigma_2)(f_\gamma) = (\hat{\sigma}_1 \circ \sigma_2)(f_\gamma) = \hat{\sigma}_1[\sigma_2(f_\gamma)]$ is undefined,

$$\begin{aligned}(\sigma_1 \circ_p^i \sigma_2)(f_\beta) &= (\hat{\sigma}_1 \circ \sigma_2)(f_\beta) = \hat{\sigma}_1[\sigma_2(f_\beta)] = \hat{\sigma}_1[f_\gamma(x_{23}, x_{14})] \\ &= S^{'}_{(2,1,1)}(\sigma_1(f_\gamma), x_{23}, x_{14}) \\ &= S^{'}_{(2,1,1)}(\, f_\beta(f_\alpha(x_{24}, x_{21}), x_{11}), x_{23}, x_{14}) \\ &= f_\beta(f_\alpha(x_{24}, x_{23}), x_{11}).\end{aligned}$$

Example 2.4. Let $I = \{1, 2\}$ and $i = 1$. Let $\Sigma^{|I|,2}(i) = \{(2, 1, 1)\}$ and $K_\gamma = \{1, 2\}$, i.e., there are two binary operation symbols $(f_\gamma)_1$ and $(f_\gamma)_2$ where $\gamma = (2, 1, 1)$. Define $\sigma \in \Sigma^{|I|,2}(i)$-$PHyp_G$ by $\sigma((f_\gamma)_1) = x_{12}$, $\sigma((f_\gamma)_2)$ is undefined. Let $t = x_{11}, t_1 = (f_\gamma)_1(x_{21}, x_{15})$ and $t_2 = (f_\gamma)_2(x_{23}, (f_\gamma)_1(x_{22}, x_{12}))$. Then

$$\begin{aligned}\hat{\sigma}[S^{'}_{(1,1,1)}(x_{11}, (f_\gamma)_1(x_{21}, x_{15}),(f_\gamma)_2(x_{23}, (f_\gamma)_1(x_{22}, x_{12})))] &= \hat{\sigma}[(f_\gamma)_1(x_{21}, x_{15})] \\ &= S^{'}_{(1,1,1)}(\sigma((f_\gamma)_1), x_{21}, x_{15}) \\ &= S^{'}_{(1,1,1)}(x_{12}, x_{21}, x_{15}) = x_{15},\end{aligned}$$

and $S^{'}_{(1,1,1)}(\hat{\sigma}[x_{12}], \hat{\sigma}[(f_\gamma)_1(x_{21}, x_{15})], \hat{\sigma}[(f_\gamma)_2(x_{23}, (f_\gamma)_1(x_{22}, x_{12}))])$ is undefined, since $\hat{\sigma}[(f_\gamma)_2(x_{23}, (f_\gamma)_1(x_{22}, x_{12}))]$ is undefined. Hence, $\hat{\sigma}[S^{'}_{(1,1,1)}(t, t_1, t_2)] \neq S^{'}_{(1,1,1)}(\hat{\sigma}[t], \hat{\sigma}[t_1], \hat{\sigma}[t_2])$.

Lemma 2.5. Let $\sigma_i \in \Sigma^{|I|,n}(i)$-$PHyp_G$. If $S^{'}_\alpha(\hat{\sigma}_i[t], \hat{\sigma}_{k_1}[t_1], \ldots, \hat{\sigma}_{k_n}[t_n])$ is defined, then

$$\hat{\sigma}_i[S^{'}_\alpha(t, t_1, \ldots, t_n)] = S^{'}_\alpha(\hat{\sigma}_i[t], \hat{\sigma}_{k_1}[t_1], \ldots, \hat{\sigma}_{k_n}[t_n])$$

where $\alpha = (k_1, \ldots, k_n, i) \in \Lambda$.
Proof. We prove by induction on the complexity of Σ-term t of sort i.
If $t = x_{ij} \in X(i)$,

$$\begin{aligned}&S^{'}_\alpha(\hat{\sigma}_i[t], \hat{\sigma}_{k_1}[t_1], \ldots, \hat{\sigma}_{k_n}[t_n]) \text{ is defined } \Rightarrow \hat{\sigma}_i[t], \hat{\sigma}_{k_j}[t_j] \text{ exist.} \\ &\quad \Rightarrow \hat{\sigma}_i[t], t_j \in dom\hat{\sigma}_{k_j} \text{ that is } t_j \text{ exists}, \forall j \in \{1, \ldots, n\}.\end{aligned}$$

Case 1 : $i \neq k_j$. Then

$$\begin{aligned}\hat{\sigma}_i[S^{'}_\alpha(t, t_1, \ldots, t_n)] &= \hat{\sigma}_i[S^{'}_\alpha(x_{ij}, t_1, \ldots, t_n)] \\ &= \hat{\sigma}_i[x_{ij}] = x_{ij} \\ &= S^{'}_\alpha(x_{ij}, \hat{\sigma}_{k_1}[t_1], \ldots, \hat{\sigma}_{k_n}[t_n]) \\ &= S^{'}_\alpha(\hat{\sigma}_i[x_{ij}], \hat{\sigma}_{k_1}[t_1], \ldots, \hat{\sigma}_{k_n}[t_n]).\end{aligned}$$

Case 2 : $i = k_j, 1 \leq j \leq n$. Then

$$\begin{aligned}
\hat{\sigma}_i[S'_\alpha(t, t_1, \ldots, t_n)] &= \hat{\sigma}_i[S'_\alpha(x_{ij}, t_1, \ldots, t_n)] \\
&= \hat{\sigma}_i[t_j] \\
&= \hat{\sigma}_{k_j}[t_j] \\
&= S'_\alpha(x_{ij}, \hat{\sigma}_{k_1}[t_1], \ldots, \hat{\sigma}_{k_n}[t_n]) \\
&= S'_\alpha(\hat{\sigma}_i[x_{ij}], \hat{\sigma}_{k_1}[t_1], \ldots, \hat{\sigma}_{k_n}[t_n]).
\end{aligned}$$

Case 3 : $j > n$. Then

$$\begin{aligned}
\hat{\sigma}_i[S'_\alpha(t, t_1, \ldots, t_n)] &= \hat{\sigma}_i[S'_\alpha(x_{ij}, t_1, \ldots, t_n)] \\
&= \hat{\sigma}_i[x_{ij}] = x_{ij} \\
&= S'_\alpha(x_{ij}, \hat{\sigma}_{k_1}[t_1], \ldots, \hat{\sigma}_{k_n}[t_n]) \\
&= S'_\alpha(\hat{\sigma}_i[x_{ij}], \hat{\sigma}_{k_1}[t_1], \ldots, \hat{\sigma}_{k_n}[t_n]).
\end{aligned}$$

If $t = f_\gamma(s_1, \ldots, s_n) \in W(i)$ with $\gamma = (i_1, \ldots, i_n, i) \in \Sigma^{|I|,n}(i)$. Assume that $S'_{\alpha_j}(\hat{\sigma}_{i_j}[s_j], \hat{\sigma}_{k_1}[t_1], \ldots, \hat{\sigma}_{k_n}[t_n])$ is defined and $\hat{\sigma}_{i_j}[S'_{\alpha_j}(s_j, t_1, \ldots, t_n)] = S'_{\alpha_j}(\hat{\sigma}_{i_j}[s_j], \hat{\sigma}_{k_1}[t_1], \ldots, \hat{\sigma}_{k_n}[t_n]), \alpha_j = (k_1, \ldots, k_n, i_j), \forall j$. Then

$$\begin{aligned}
S'_\alpha(\hat{\sigma}_i[t], \hat{\sigma}_{k_1}[t_1], \ldots, \hat{\sigma}_{k_n}[t_n]) \text{ is defined } &\Rightarrow\ \hat{\sigma}_i[f_\gamma(s_1, \ldots, s_n)], \hat{\sigma}_{k_j}[t_j] \text{ exist.} \\
&\Rightarrow\ S'_\gamma(\sigma_i(f_\gamma), \hat{\sigma}_{i_1}[s_1], \ldots, \hat{\sigma}_{i_n}[s_n]), \hat{\sigma}_{k_j}[t_j] \text{ exist.} \\
&\Rightarrow\ f_\gamma \in dom\sigma_i \text{ and } \hat{\sigma}_{i_j}[s_j], \hat{\sigma}_{k_j}[t_j] \text{ exist.}
\end{aligned}$$

And we have,

$$\begin{aligned}
\hat{\sigma}_i[S'_\alpha(t, t_1, \ldots, t_n)] &= \hat{\sigma}_i[S'_\alpha(f_\gamma(s_1, \ldots, s_n), t_1, \ldots, t_n)] \\
&= \hat{\sigma}_i[f_\gamma(S'_{\alpha_1}(s_1, t_1, \ldots, t_n), \ldots, S'_{\alpha_n}(s_n, t_1, \ldots, t_n))] \\
&= S'_\gamma(\sigma_i(f_\gamma), \hat{\sigma}_{i_1}[S'_{\alpha_1}(s_1, t_1, \ldots, t_n)], \ldots, \hat{\sigma}_{i_n}[S'_{\alpha_n}(s_n, t_1, \ldots, t_n)]) \\
&= S'_\gamma(\sigma_i(f_\gamma), S'_{\alpha_1}(\hat{\sigma}_{i_1}[s_1], \hat{\sigma}_{k_1}[t_1], \ldots, \hat{\sigma}_{k_n}[t_n]), \ldots, \\
&\quad S'_{\alpha_n}(\hat{\sigma}_{i_n}[s_n], \hat{\sigma}_{k_1}[t_1], \ldots, \hat{\sigma}_{k_n}[t_n])) \\
&= S'_\alpha(S'_\gamma(\sigma_i(f_\gamma), \hat{\sigma}_{i_1}[s_1], \ldots, \hat{\sigma}_{i_n}[s_n]), \hat{\sigma}_{k_1}[t_1], \ldots, \hat{\sigma}_{k_n}[t_n]) \\
&= S'_\alpha(\hat{\sigma}_i[f_\gamma(s_1, \ldots, s_n)], \hat{\sigma}_{k_1}[t_1], \ldots, \hat{\sigma}_{k_n}[t_n]) \\
&= S'_\alpha(\hat{\sigma}_i[t], \hat{\sigma}_{k_1}[t_1], \ldots, \hat{\sigma}_{k_n}[t_n]).
\end{aligned}$$

So $\hat{\sigma}_i[S'_\alpha(t, t_1, \ldots, t_n)] = S'_\alpha(\hat{\sigma}_i[t], \hat{\sigma}_{k_1}[t_1], \ldots, \hat{\sigma}_{k_n}[t_n])$. □

Lemma 2.6. For $(\sigma_1)_i, (\sigma_2)_i \in \Sigma^{|I|,n}(i)$-$PHyp_G$, $((\sigma_1)_i \circ_p^i (\sigma_2)_i)\,\hat{}= (\hat{\sigma}_1)_i \circ (\hat{\sigma}_2)_i$.

Proof. We prove by induction on the complexity of Σ-term t of sort i.
If $t = x_{ij} \in X(i)$.

Since $((\sigma_1)_i \circ_p^i (\sigma_2)_i)\hat{}, (\hat{\sigma}_1)_i, (\hat{\sigma}_2)_i$ are defined on variables, $x_{ij} \in dom((\sigma_1)_i \circ_p^i (\sigma_2)_i)\hat{}, dom(\hat{\sigma}_1)_i, dom(\hat{\sigma}_2)_i$.

So, $x_{ij} \in dom((\sigma_1)_i \circ_p^i (\sigma_2)_i)\hat{}, dom((\hat{\sigma}_1)_i \circ (\hat{\sigma}_2)_i)$ and $((\sigma_1)_i \circ_p^i (\sigma_2)_i)\hat{}[x_{ij}] = x_{ij} = ((\hat{\sigma}_1)_i \circ (\hat{\sigma}_2)_i)[x_{ij}]$.
If $t = f_\gamma(t_1, \ldots, t_n) \in W(i)$ with $\gamma = (i_1, \ldots, i_n, i) \in \Sigma^{|I|,n}(i)$.
Assume that $t_j \in dom((\sigma_1)_{i_j} \circ_p^{i_j} (\sigma_2)_{i_j})\hat{}, dom((\hat{\sigma}_1)_{i_j} \circ (\hat{\sigma}_2)_{i_j})$ and $((\sigma_1)_{i_j} \circ_p^{i_j} (\sigma_2)_{i_j})\hat{}[t_j] = ((\hat{\sigma}_1)_{i_j} \circ (\hat{\sigma}_2)_{i_j})[t_j]$.
First, we show that $t \in dom((\sigma_1)_i \circ_p^i (\sigma_2)_i)\hat{} \Leftrightarrow t \in dom((\hat{\sigma}_1)_i \circ (\hat{\sigma}_2)_i)$.

$$\begin{aligned}
t = f_\gamma(t_1, \ldots, t_n) \in dom((\sigma_1)_i \circ_p^i (\sigma_2)_i)\hat{} \Leftrightarrow \\
&\Leftrightarrow f_\gamma \in dom((\sigma_1)_i \circ_p^i (\sigma_2)_i) \text{ and } t_j \in dom((\sigma_1)_{i_j} \circ_p^{i_j} (\sigma_2)_{i_j})\hat{} \\
&\Leftrightarrow f_\gamma \in dom(\sigma_2)_i, (\sigma_2)_i(f_\gamma) \in dom(\hat{\sigma}_1)_i \\
&\quad \text{and } t_j \in dom((\hat{\sigma}_1)_{i_j} \circ (\hat{\sigma}_2)_{i_j}) \\
&\Leftrightarrow f_\gamma \in dom(\sigma_2)_i, t_j \in dom(\hat{\sigma}_2)_{i_j} \text{ and} \\
&\quad (\sigma_2)_i(f_\gamma) \in dom(\hat{\sigma}_1)_i, (\hat{\sigma}_2)_{i_j}[t_j] \in dom(\hat{\sigma}_1)_{i_j} \\
&\Leftrightarrow f_\gamma(t_1, \ldots, t_n) \in dom(\hat{\sigma}_2)_i \text{ and } (\hat{\sigma}_2)_i[f_\gamma(t_1, \ldots, t_n)] \in dom(\hat{\sigma}_1)_i \\
&\Leftrightarrow t = f_\gamma(t_1, \ldots, t_n) \in dom((\hat{\sigma}_1)_i \circ (\hat{\sigma}_2)_i).
\end{aligned}$$

And we have,

$$\begin{aligned}
&((\sigma_1)_i \circ_p^i (\sigma_2)_i)\hat{}[t] = ((\sigma_1)_i \circ_p^i (\sigma_2)_i)\hat{}[f_\gamma(t_1, \ldots, t_n)] \\
&= S_\gamma^{'}(\, ((\sigma_1)_i \circ_p^i (\sigma_2)_i)(f_\gamma), ((\sigma_1)_{i_1} \circ_p^{i_1} (\sigma_2)_{i_1})\hat{}[t_1], \ldots, ((\sigma_1)_{i_n} \circ_p^{i_n} (\sigma_2)_{i_n})\hat{}[t_n]) \\
&= S_\gamma^{'}(\, ((\hat{\sigma}_1)_i \circ (\sigma_2)_i)(f_\gamma), ((\hat{\sigma}_1)_{i_1} \circ (\hat{\sigma}_2)_{i_1})[t_1], \ldots, ((\hat{\sigma}_1)_{i_n} \circ (\hat{\sigma}_2)_{i_n})[t_n]) \\
&= S_\gamma^{'}(\, (\hat{\sigma}_1)_i[(\sigma_2)_i(f_\gamma)], (\hat{\sigma}_1)_{i_1}[(\hat{\sigma}_2)_{i_1}[t_1]], \ldots, (\hat{\sigma}_1)_{i_n}[(\hat{\sigma}_2)_{i_n}[t_n]]) \\
&= (\hat{\sigma}_1)_i[S_\gamma^{'}((\sigma_2)_i(f_\gamma), (\hat{\sigma}_2)_{i_1}[t_1], \ldots, (\hat{\sigma}_2)_{i_n}[t_n])] \\
&= (\hat{\sigma}_1)_i[(\hat{\sigma}_2)_i[f_\gamma(t_1, \ldots, t_n)]] \\
&= (\hat{\sigma_1})_i \circ (\hat{\sigma_2})_i[f_\gamma(t_1, \ldots, t_n)] \\
&= (\hat{\sigma_1})_i \circ (\hat{\sigma_2})_i[t].
\end{aligned}$$

Therefore $((\sigma_1)_i \circ_p^i (\sigma_2)_i)\hat{} = (\hat{\sigma}_1)_i \circ (\hat{\sigma}_2)_i$. ☐

Lemma 2.7. For $(\sigma_1)_i, (\sigma_2)_i, (\sigma_3)_i \in \Sigma^{|I|,n}(i)$-$PHyp_G$,

$$((\sigma_1)_i \circ_p^i (\sigma_2)_i) \circ_p^i (\sigma_3)_i = (\sigma_1)_i \circ_p^i ((\sigma_2)_i \circ_p^i (\sigma_3)_i).$$

Proof. We first prove that $dom(((\sigma_1)_i \circ_p^i (\sigma_2)_i) \circ_p^i (\sigma_3)_i) = dom((\sigma_1)_i \circ_p^i ((\sigma_2)_i \circ_p^i$

$(\sigma_3)_i))$.

$$
\begin{aligned}
f_\gamma \in dom(((\sigma_1)_i \circ_p^i (\sigma_2)_i) \circ_p^i (\sigma_3)_i) \Leftrightarrow & \\
&\Leftrightarrow f_\gamma \in dom(((\sigma_1)_i \circ_p^i (\sigma_2)_i)\hat{} \circ (\sigma_3)_i) \\
&\Leftrightarrow f_\gamma \in dom(\sigma_3)_i \text{ and } (\sigma_3)_i(f_\gamma) \in dom((\sigma_1)_i \circ_p^i (\sigma_2)_i)\hat{} \\
&\Leftrightarrow f_\gamma \in dom(\sigma_3)_i \text{ and } (\sigma_3)_i(f_\gamma) \in dom((\hat{\sigma}_1)_i \circ (\hat{\sigma}_2)_i) \\
&\Leftrightarrow f_\gamma \in dom(\sigma_3)_i \text{ and} \\
&\quad (\sigma_3)_i(f_\gamma) \in dom(\hat{\sigma}_2)_i, (\hat{\sigma}_2)_i[(\sigma_3)_i(f_\gamma)] \in dom(\hat{\sigma}_1)_i \\
&\Leftrightarrow f_\gamma \in dom((\hat{\sigma}_2)_i \circ (\sigma_3)_i) \text{ and } (\hat{\sigma}_2)_i[(\sigma_3)_i(f_\gamma)] \in dom(\hat{\sigma}_1)_i \\
&\Leftrightarrow f_\gamma \in dom((\hat{\sigma}_1)_i \circ ((\hat{\sigma}_2)_i \circ (\sigma_3)_i)) \\
&\Leftrightarrow f_\gamma \in dom((\sigma_1)_i \circ_p^i ((\sigma_2)_i \circ_p^i (\sigma_3)_i)).
\end{aligned}
$$

Next, we prove that $((\sigma_1)_i \circ_p^i (\sigma_2)_i) \circ_p^i (\sigma_3)_i = (\sigma_1)_i \circ_p^i ((\sigma_2)_i \circ_p^i (\sigma_3)_i)$.

$$
\begin{aligned}
((\sigma_1)_i \circ_p^i (\sigma_2)_i) \circ_p^i (\sigma_3)_i &= ((\sigma_1)_i \circ_p^i (\sigma_2)_i)\hat{} \circ (\sigma_3)_i \\
&= ((\hat{\sigma_1})_i \circ (\hat{\sigma_2})_i) \circ (\sigma_3)_i \\
&= (\hat{\sigma_1})_i \circ ((\hat{\sigma_2})_i \circ (\sigma_3)_i) \\
&= (\hat{\sigma_1})_i \circ ((\sigma_2)_i \circ_p^i (\sigma_3)_i) \\
&= (\sigma_1)_i \circ_p^i ((\sigma_2)_i \circ_p^i (\sigma_3)_i).
\end{aligned}
$$

Hence $\Sigma^{|I|,n}(i)$-$PHyp_G$ satisfies an associative law. □

Let $(\sigma_{id})_i \in \Sigma^{|I|,n}(i)$-$PHyp_G$ which maps each f_γ to the Σ-term $f_\gamma(x_{k_1 1}, \ldots, x_{k_n n}), \forall\gamma = (k_1, \ldots, k_n, i) \in \Sigma^{|I|,n}(i)$. For $\sigma_i \in \Sigma^{|I|,n}(i)$-$PHyp_G$,

$$dom((\sigma_{id})_i \circ_p^i \sigma_i) = dom\sigma_i = dom(\sigma_i \circ_p^i (\sigma_{id})_i)$$

and we can prove that $(\sigma_{id})_i \circ_p^i \sigma_i = \sigma_i = \sigma_i \circ_p^i (\sigma_{id})_i$, or see [4].

Theorem 2.8. *$(\Sigma^{|I|,n}(i)$-$PHyp_G, \circ_p^i)$ is a monoid.*

Proof. By Lemma 2.7, we can conclude that $(\Sigma^{|I|,n}(i)$-$PHyp_G, \circ_p^i)$ forms a monoid. □

Corollary 2.9. $(\Sigma^{|I|,n}(i)$-$Hyp_G, \circ_p^i)$ is a monoid.

Proof. This follows from the previous theorem which is stated that $dom\sigma_i = \{f_\gamma | \gamma \in \Sigma^{|I|,n}(i)\}$. □

Acknowledgement This research was supported by Chiang Mai University, Chiang Mai 50200, Thailand.

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