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ON BEHAVIOR OF THE SIXTH LANNES-ZARATI HOMOMORPHISM

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Abstract

In this paper, we determine the image of the indecomposable elements in $\mathrm{Ext}_A^{6,*}(\mathbb{F}_2, \mathbb{F}_2)$ for $6 \le t \le 120$ through the sixth Lannes-Zarati homomorphism $\varphi_6 := \varphi_6^{\mathbb{F}_2}$.

1 Introduction and statement of results

Let D be the destabilization functor from the category $\mathcal{M}$ of left modules over the mod 2 Steenrod algebra A to the category $\mathcal{U}$ of unstable modules, which is the left adjoint to the forgetful functor $\mathcal{U} \to \mathcal{M}$. Hence, it is right exact and, therefore, it admits the left derived functor $\mathrm{D}_s : \mathcal{M} \to \mathcal{U}$ for each $s \ge 0$. By definition of D (see Section 2), for any $M \in \mathcal{M}$, there exists a natural homomorphism $\mathrm{D}(M) \to \mathbb{F}_2 \otimes_A M$, and then, this homomorphism in turns induces natural maps $i_s^M : \mathrm{D}_s(M) \to \mathrm{Tor}_s^A(\mathbb{F}_2, M)$ between corresponding derived functors. In addition, as the result of Lannes and Zarati [16], for any $M \in \mathcal{U}$ and for each $s \ge 0$, there is an isomorphism $\alpha_s(\Sigma M) : \mathrm{D}_s(\Sigma^{1-s}M) \to \Sigma \mathrm{R}_s M$, where R_s is the Singer construction, which is an exact functor from $\mathcal{U}$ to itself (see Singer [18], [19], Lannes-Zarati [16], see also Hai [9], and citations

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therein for a detail description). Therefore, for any unstable A-module M, there exists a natural homomorphism, for each $s \geq 0$,

$$(\bar{\varphi}_s^M)^\# : \mathrm{R}_s M \to \mathrm{Tor}_s^A(\mathbb{F}_2, \Sigma^{-s} M).$$

Since the Steenrod algebra A has acted trivially on the target, $(\bar{\varphi}_s^M)^\#$ factors through $\mathbb{F}_2 \otimes_A \mathrm{R}_s M$. Hence, there exists a natural homomorphism

$$(\varphi_s^M)^\# : (\mathbb{F}_2 \otimes_A \mathrm{R}_s M)^t \to \mathrm{Tor}_{s,t}^A(\mathbb{F}_2, \Sigma^{-s} M) \simeq \mathrm{Tor}_{s,s+t}^A(\mathbb{F}_2, M). \tag{1.1}$$

Taking (linear) dual, we have a homomorphism (the so-called Lannes-Zarati homomorphism), for each $s \geq 0$,

$$\varphi_s^M : \mathrm{Ext}_{\mathrm{A}}^{s,s+t}(M, \mathbb{F}_2) \to \mathrm{Ann}(\mathrm{R}_s(M)^\#)_t.$$

Here, for any A-module N, we denote $N^\#$ the (linear) dual of N and $\mathrm{Ann}(N^\#)$ the subspace of $N^\#$ spanned by all elements annihilated by all Steenrod operations of positive degree. The Lannes-Zarati homomorphism is also considered as an associated graded of the Hurewicz map

$$H : \pi_*^S(S^0) \to H_*(Q_0 S^0),$$

on the base-point component $Q_0 S^0$ of the infinite loop space $QS^0 = \varinjlim \Omega^n \Sigma^n S^0$ (see Lannes and Zarati [14], [15] for the sketch of proof). Therefore, the study of the Lannes-Zarati homomorphism is related to the study of the image of the Hurewicz map and then Curtis's conjecture on the spherical classes [8] (see [7] for discussion).

The Lannes-Zarati homomorphism was first constructed by Lannes-Zarati in [16]. Therein, they showed that $\varphi_1^{\mathbb{F}_2}$ is an isomorphism, $\varphi_2^{\mathbb{F}_2}$ is an epimorphism. Later, Hung et. al also proved $\varphi_s^{\mathbb{F}_2}$ is trivial in any positive stems for $3 \leq s \leq 5$ (see [11] for the case $s = 3$, [10] for the case $s = 4$ and [12] for the case $s = 5$). The results of Hung et. al essentially based on the information of "hit" problem for the Dickson algebra. Since the "hit" problem for the Dickson algebra of six variables is still unsolved, it is difficult to apply this method for $\varphi_6^{\mathbb{F}_2}$.

In this paper, we use the method of Chon-Nhu [6, 7] to determine the image of $\varphi_6^{\mathbb{F}_2}$. Thereby, we obtain the following result.

Theorem 1.1. *The homomorphism* $\varphi_6^{\mathbb{F}_2} : \mathrm{Ext}_A^{6,6+t}(\mathbb{F}_2, \mathbb{F}_2) \to \mathrm{Ann}((\mathrm{R}_6\mathbb{F}_2)^\#)_t$ *is trivial on indecomposable elements in* $\mathrm{Ext}_A^{6,t}(\mathbb{F}_2, \mathbb{F}_2)$ *for* $6 \leq t \leq 120$.

The advantage of this method is to avoid using the knowledge of the "hit" problem for the Dickson algebra.

2 Preliminaries

Denote $\mathcal{M}$ as the category of graded left A-modules and degree zero A-linear maps. An A-module $M \in \mathcal{M}$ is called unstable if $Sq^i x = 0$ for $i > \deg x$ and for all $x \in M$.

Given an A-module M and an integer s, let $\Sigma^s M$ denote the s-th iterated suspension of M. We define $(\Sigma^s M)^n = M^{n-s}$, then an element in degree n of $\Sigma^s M$ is usually written in the form $\Sigma^s m$, where $m \in M^{n-s}$.

Let $\mathcal{U}$ is the full subcategory of $\mathcal{M}$ of all unstable modules. The destabilization functor $\mathrm{D}\colon \mathcal{M} \to \mathcal{U}$ is the left adjoint to the inclusion $\mathcal{U} \to \mathcal{M}$. It can be described more explicitly as follows:

$$\mathrm{D}(M) := M/EM,$$

where $EM := Span_{\mathbb{F}_2}\{Sq^i x : 2i > \deg(x), x \in M\}$ is an A-submodule of M, that is a consequence of the Adem relations. In particular, EM is the subspace of elements in a negative degree if M is a graded vector space which is considered as an A-module with trivial action. Then $\mathrm{D}(M)$ is an A-submodule of M consisting of all elements in non-negative degrees. It is simple to observe the following construction.

For any A-module M, then there is an A-homomorphism $\mathrm{D}(M) \to \mathrm{D}(\mathbb{F}_2 \otimes_A M)$, which is induced by the projection $M \to \mathbb{F}_2 \otimes_A M$ and the canonical embedding $\mathrm{D}(\mathbb{F}_2 \otimes_A M) \hookrightarrow \mathbb{F}_2 \otimes_A M$. Thus, there exists a natural A-homomorphism $\mathrm{D}(M) \to \mathbb{F}_2 \otimes_A M$ which is the composition

$$\mathrm{D}(M) \to \mathrm{D}(\mathbb{F}_2 \otimes_A M) \hookrightarrow \mathbb{F}_2 \otimes_A M.$$

Therefore, maps between corresponding derived functors are induced by this exact sequence

$$i_s^M \colon \mathrm{D}_s(M) \to \mathrm{Tor}_s^A(\mathbb{F}_2, M).$$

The possibility of understanding the homology of the Steenrod algebra via knowledge of derived functors of the destabilization functor is raised by the natural map i_s^M. However, computing D_s is generally very difficult, except in one important situation in which Lannes and Zarati [16], [21] discovered that it can be described in terms of the Singer functors R_s.

We recall the definition of the Lannes-Zarati homomorphism. For any A-module M, let the short exact sequence

$$0 \to P_1 \otimes M \to \hat{P} \otimes M \to \Sigma^{-1} M \to 0,$$

where, $P_1 = \mathbb{F}_2[x_1]$ be the polynomial algebra over $\mathbb{F}_2$ generated by x_1 with $|x_1| = 1$ and $\hat{P}$ is the A-module extension of P_1 by formally adding the generator x_1^{-1} in degree -1. The action of A on $\hat{P}$ is given by $Sq^n(x_1^{-1}) = x_1^{n-1}$.

Moreover, we have the following theorem.

Theorem 2.1 (Lannes and Zarati [16]). *For any unstable A-module M, the homomorphism $\alpha_s(\Sigma M) : \mathrm{D}_s(\Sigma^{1-s}M) \to \Sigma \mathrm{R}_s M$ is an isomorphism of unstable A-modules.*

For any unstable A-module M and for $s \geq 0$, there exists a homomorphism $(\bar{\varphi}_s^M)^{\#}$ such that the following diagram commutes (see Chon-Nhu [7] for a detail construction):

$$\begin{array}{ccccc} \mathrm{D}_s(\Sigma^{1-s}M) & \xrightarrow{\alpha_s(\Sigma M)} & \Sigma \mathrm{R}_s M & \hookrightarrow & \Sigma P_s \otimes M \\ \downarrow{\scriptstyle i_s^{\Sigma^{1-s}M}} & \swarrow{\scriptstyle (\bar{\varphi}_s^M)^{\#}} & & & \\ \mathrm{Tor}_s^A(\mathbb{F}_2, \Sigma^{1-s}M). & & & & \end{array} \qquad (2.1)$$

where, $P_s = \mathbb{F}_2[x_1, x_2, \cdots, x_s]$ be the polynomial algebra over $\mathbb{F}_2$ generated by the indicated variables, each of degree 1.

$(\bar{\varphi}_s^M)^{\#}$ factors through $\mathbb{F}_2 \otimes_A \Sigma \mathrm{R}_s M$ because of acting trivially on the target of the Steenrod algebra A. Therefore, after desuspending, we obtain the dual of the Lannes-Zarati homomorphism

$$(\varphi_s^M)^{\#} : (\mathbb{F}_2 \otimes_A \mathrm{R}_s M)^t \to \mathrm{Tor}_{s,t}^A(\mathbb{F}_2, \Sigma^{-s}M) \simeq \mathrm{Tor}_{s,s+t}^A(\mathbb{F}_2, M).$$

The linear dual

$$\varphi_s^M : \mathrm{Ext}_A^{s,s+t}(M, \mathbb{F}_2) \to (\mathbb{F}_2 \otimes_A \mathrm{R}_s M)_t^{\#} = \mathrm{Ann}((\mathrm{R}_s M)^{\#})_t,$$

is the so-called Lannes-Zarati homomorphism.

In [1] (see also [2]), for computing the cohomology of the Steenrod algebra, Bousfield et. al defined a differential algebra and so-called the Lambda algebra. The dual of Lambda algebra as differential $\mathbb{F}_2$-module is isomorphic to Γ^+ (see in [13]). Because the sign is not compatible, extending an isomorphism between chain complexes $\Gamma^+ M$ and $\Lambda^{\#} \otimes M$ is difficult. Therefore, as naturally, we used the opposite algebra of the Lambda algebra, also denoted Λ, which corresponds to the original Lambda algebra under the anti-isomorphism of differential $\mathbb{F}_2$-modules. In the literature, it is also called the Lambda algebra.

The Lambda algebra, Λ, which is defined as the differential, graded, associative algebra with unit over $\mathbb{F}_2$, is generated by $\lambda_i, i \geq 0$, of degree i, satisfying the Adem relations

$$\lambda_i \lambda_j = \sum_t \binom{t-j-1}{2t-i} \lambda_{i+j-t} \lambda_t, \qquad (2.2)$$

for all $i, j \geq 0$. Here $\binom{n}{k}$ is interpreted as the coefficient of x^k in expansion of $(x+1)^n$ so that it is defined for all integer n and all non-negative integer k (see Chon-Ha [5]).

For $i_j, j = 1, \cdots, s$ is non-negative integers, a monomial $\lambda_I = \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_s}$ in Λ is called the monomial of the length s. And λ_I is also called an admissible monomial if $i_1 \leq 2i_2, \cdots, i_{s-1} \leq 2i_s$, and define the excess of λ_I or I to be

$$exc(\lambda_I) = exc(I) = i_1 - \sum_{j=2}^{s} i_j.$$

The Dyer-Lashof algebra $\mathcal{R}$ is an important quotient algebra of the lambda algebra over the ideal generated by the monomials of negative excess. Let the canonical projection $\pi : \Lambda \to \mathcal{R}$, put $Q_I = Q^{i_1}Q^{i_2}\cdots Q^{i_s}$ be the image of λ_I under π. Let $\mathcal{R}_s$ be the subspace of $\mathcal{R}$ spanned by the monomials of length s.

From the results of Chon-Nhu [6], we have

Proposition 2.2 (Chon-Nhu [6, Proposition 6.2]). *The projection*

$$\widetilde{\varphi}_s^{\mathbb{F}_2} : \Lambda_s \otimes \mathbb{F}_2^{\#} \to (\mathrm{R}_s\mathbb{F}_2)^{\#},$$

given by

$$\lambda_I \otimes \ell \mapsto [Q^I \otimes \ell]$$

is a chain-level representation of the mod 2 Lannes-Zarati homomorphism $\varphi_s^{\mathbb{F}_2}$.

Proposition 2.3 (Chon-Nhu [6, Proposition 6.3]). *The following diagram is commutative*

$$\begin{array}{ccc}
\mathrm{Ext}_A^{s,s+t}(\mathbb{F}_2,\mathbb{F}_2) & \xrightarrow{Sq^0} & \mathrm{Ext}_A^{s,2(s+t)}(\mathbb{F}_2,\mathbb{F}_2) \\
\downarrow{\scriptstyle \varphi_s^{\mathbb{F}_2}} & & \downarrow{\scriptstyle \varphi_s^{\mathbb{F}_2}} \\
(\mathbb{F}_2 \otimes_A \mathrm{R}_s\mathbb{F}_2)_t^{\#} & \xrightarrow[Sq^0]{} & (\mathbb{F}_2 \otimes_A \mathrm{R}_s\mathbb{F}_2)_{2t+s}^{\#}.
\end{array}$$

3 The proof of Theorem 1.1

In this section, we use the chain-level representation map of the $\varphi_s^{\mathbb{F}_2}$ constructed in the previous section to investigate the behavior of the sixth Lannes-Zarati homomorphism $\varphi_6^{\mathbb{F}_2}$.

Lemma 3.1. *If* $\lambda_I \in \Lambda_s$ *and* $\lambda_J \in \Lambda_\ell$ *such that* $\tilde{\varphi}_s^{\mathbb{F}_2}(\lambda_I) = 0$ *or* $\tilde{\varphi}_\ell^{\mathbb{F}_2}(\lambda_J) = 0$ *then* $\tilde{\varphi}_{s+\ell}^{\mathbb{F}_2}(\lambda_I\lambda_J) = 0$.

Now, we need to prove Theorem 1.1

Proof. From Chen's result [4], indecomposable elements in $\mathrm{Ext}_A^{6,t}(\mathbb{F}_2,\mathbb{F}_2)$ for $6 \le t \le 120$, is listed as follows

$$(1)\ r = \left\{\begin{array}{l} e_0\lambda_1\lambda_{12} + (\lambda_3^2\lambda_{11}\lambda_2 + \lambda_7^2\lambda_2\lambda_3)\lambda_1\lambda_{10} + f_0(\lambda_2\lambda_{10} \\ +\lambda_3\lambda_9)\lambda_7^2\lambda_4\lambda_2\lambda_1\lambda_9 + (\lambda_3^2\lambda_9\lambda_5 + \lambda_3^2\lambda_{11}\lambda_4\lambda_2 \\ +\lambda_3\lambda_9\lambda_3^2\lambda_5 + \lambda_3\lambda_9\lambda_5\lambda_3^2 + \lambda_7^2\lambda_4\lambda_2\lambda_3)\lambda_7 \\ +(\lambda_3\lambda_{11}\lambda_9 + \lambda_{23}\lambda_0^2 + \lambda_7\lambda_5\lambda_{11} + \lambda_7\lambda_9\lambda_7)\lambda_0^2\lambda_7 \\ +\lambda_7^2\lambda_2\lambda_7\lambda_0\lambda_7 + f_0\lambda_6^2 + (\lambda_3^2\lambda_{11}\lambda_2 + \lambda_7^2\lambda_2\lambda_3)\lambda_5\lambda_6 \\ +\lambda_7^2(\lambda_4\lambda_2\lambda_5^2 + \lambda_0\lambda_{10}\lambda_3^2) + (\lambda_3^2\lambda_9 + \lambda_9\lambda_3^2)\lambda_9\lambda_3^2 \\ +\lambda_7^2\lambda_4\lambda_6\lambda_3^2 \end{array}\right\} \in \mathrm{Ext}_A^{6,36}(\mathbb{F}_2,\mathbb{F}_2);$$

$$(2)\ q = \left\{\begin{array}{l} \lambda_{15}(\lambda_3^2\lambda_2\lambda_1\lambda_8 + \lambda_{11}\lambda_0^3\lambda_6) + \lambda_7^3\lambda_6\lambda_3\lambda_2 + g_1\lambda_3\lambda_9 \\ +(\lambda_{15}\lambda_3\lambda_0\lambda_6 + \lambda_7\lambda_9\lambda_5\lambda_3 + \lambda_{11}\lambda_7\lambda_4\lambda_2)\lambda_3\lambda_5 \\ +\lambda_{15}(\lambda_1\lambda_2\lambda_1\lambda_8 + \lambda_1^2\lambda_4\lambda_6 + \lambda_1\lambda_4\lambda_2\lambda_5 + \lambda_3\lambda_2\lambda_3\lambda_4 \\ +\lambda_1^3\lambda_9 + \lambda_3\lambda_4\lambda_2\lambda_3 + \lambda_1\lambda_2\lambda_6\lambda_3)\lambda_5 + \lambda_{15}\lambda_3^2\lambda_2\lambda_5\lambda_4 \\ +\lambda_{15}(\lambda_5\lambda_3^3 + \lambda_1\lambda_2\lambda_5\lambda_6 + \lambda_1^2\lambda_8\lambda_4 + \lambda_1\lambda_5\lambda_4^2 \\ +\lambda_1\lambda_4\lambda_6\lambda_8 + \lambda_5\lambda_3\lambda_4\lambda_2)\lambda_3 + \lambda_{15}\lambda_3\lambda_7\lambda_4\lambda_1\lambda_2 \end{array}\right\} \in \mathrm{Ext}_A^{6,38}(\mathbb{F}_2,\mathbb{F}_2);$$

$$(3)\ t = \left\{\begin{array}{l} n_0\lambda_5 + (\lambda_7\lambda_{15}\lambda_3\lambda_0\lambda_8 + \lambda_7^2\lambda_5\lambda_9\lambda_5 + \\ \lambda_7\lambda_{15}\lambda_3\lambda_2\lambda_6 + \lambda_{15}\lambda_3\lambda_7\lambda_5\lambda_3)\lambda_3 \end{array}\right\} \in \mathrm{Ext}_A^{6,42}(\mathbb{F}_2,\mathbb{F}_2);$$

$$(4)\ y = \left\{\begin{array}{l} \lambda_{15}^2\lambda_0^3\lambda_8 + [\lambda_{23}(\lambda_1^2\lambda_4\lambda_2 + \lambda_1\lambda_4\lambda_2\lambda_1 + \lambda_1^3\lambda_5 \\ +\lambda_4\lambda_1^2\lambda_2) + \lambda_7\lambda_{15}\lambda_3\lambda_0\lambda_6 + \lambda_7^2\lambda_5\lambda_{10}\lambda_2 \\ +\lambda_{15}\lambda_3\lambda_7\lambda_4\lambda_2 + \lambda_7^2\lambda_5\lambda_3\lambda_9]\lambda_7 + \lambda_{15}^2[\lambda_0\lambda_2\lambda_1\lambda_5 \\ +(\lambda_0^2\lambda_4 + \lambda_0\lambda_4\lambda_0 + \lambda_4\lambda_0^2)\lambda_4 + \lambda_0^2\lambda_2\lambda_6 \\ +(\lambda_0^2\lambda_5 + \lambda_0\lambda_5\lambda_0 + \lambda_5\lambda_0^2)\lambda_3 \\ +(\lambda_0\lambda_4 + \lambda_4\lambda_0)\lambda_2^2] + \lambda_{15}^2\lambda_4\lambda_2\lambda_1 \end{array}\right\} \in \mathrm{Ext}_A^{6,44}(\mathbb{F}_2,\mathbb{F}_2);$$

$$(5)\ C = \left\{\begin{array}{l} c_1\lambda_7\lambda_5 + [(\lambda_{15}^2\lambda_5\lambda_7 + \lambda_{15}\lambda_{11}\lambda_7\lambda_9 + \lambda_7^2\lambda_{23})\lambda_5 + \\ \lambda_{15}^2\lambda_{11}\lambda_0\lambda_6 + \lambda_7\lambda_{23}\lambda_{15}\lambda_1^2 + \lambda_{15}^2\lambda_9\lambda_5\lambda_3 \end{array}\right\} \in \mathrm{Ext}_A^{6,56}(\mathbb{F}_2,\mathbb{F}_2);$$

(6) $G = \{D_1(0)\lambda_2\} \in \mathrm{Ext}_A^{6,60}(\mathbb{F}_2,\mathbb{F}_2)$;

(7) $D_2 = \{\lambda_{47}\lambda_{11}\lambda_0^4\} \in \mathrm{Ext}_A^{6,64}(\mathbb{F}_2,\mathbb{F}_2)$;

(8) $\mathcal{A} = \{D_1(0)\lambda_9 + \lambda_{47}d_0\lambda_0 + \lambda_{15}^2\lambda_{11}^2\lambda_6\lambda_3\} \in \mathrm{Ext}_A^{6,67}(\mathbb{F}_2,\mathbb{F}_2)$;

(9) $\mathcal{A}' = \left\{ \begin{array}{l} c_2[\lambda_0\lambda_2\lambda_{18} + \lambda_2\lambda_3\lambda_{15} + \lambda_0\lambda_6\lambda_{14} + \lambda_2\lambda_5\lambda_{13} \\ +\lambda_6\lambda_1\lambda_{13} + \lambda_0\lambda_8\lambda_{12} + \lambda_0^2\lambda_{20} + \lambda_8\lambda_0\lambda_{12} \\ +\lambda_6\lambda_3\lambda_{11} + \lambda_0\lambda_{10}^2 + \lambda_8\lambda_2\lambda_{10} + (\lambda_9\lambda_2 + \lambda_{10}\lambda_1)\lambda_9 \\ +(\lambda_6\lambda_7 + \lambda_8\lambda_5)\lambda_7 + \lambda_{10}\lambda_5^2] + \lambda_{15}^2\lambda_{11}\lambda_2\lambda_1\lambda_{17} \\ +(\lambda_{15}\lambda_{11}\lambda_7\lambda_9\lambda_8 + \lambda_{15}^2\lambda_{13}\lambda_7\lambda_0)\lambda_{11} + \lambda_{31}f_0\lambda_{12} \\ +\lambda_{15}^3\lambda_4\lambda_2\lambda_{10} + D_1(0)\lambda_9 + \lambda_{15}^2(\lambda_{13}\lambda_7\lambda_4\lambda_7 \\ +\lambda_{15}\lambda_0\lambda_{10}\lambda_6) + [\lambda_{31}(\lambda_3^2\lambda_9 + \lambda_9\lambda_3^2)\lambda_9 + \lambda_{31}\lambda_3 \\ (\lambda_9\lambda_5 + \lambda_3\lambda_{11})\lambda_7 + \lambda_{15}^2(\lambda_{11}\lambda_8 + \lambda_{15}\lambda_4)\lambda_6]\lambda_6 \\ +[\lambda_{15}^2(\lambda_{15}\lambda_2\lambda_9 + \lambda_{15}\lambda_{10}\lambda_1) + \lambda_{31}(\lambda_3^2\lambda_{11}\lambda_8 \\ +\lambda_3\lambda_9\lambda_5\lambda_8 + \lambda_3^3\lambda_6 + \lambda_{11}\lambda_1^2\lambda_{12} + (\lambda_3^2\lambda_9 + \lambda_9\lambda_3^2)\lambda_{10} \\ +\lambda_7(\lambda_1\lambda_9 + \lambda_9\lambda_1)\lambda_8 + \lambda_7(\lambda_5\lambda_7\lambda_6 + \lambda_9\lambda_5\lambda_4))]\lambda_5 \\ +[\lambda_{15}^2(\lambda_{15}\lambda_2\lambda_{11} + \lambda_{15}\lambda_5\lambda_8 + \lambda_{15}\lambda_8\lambda_5 + \lambda_{11}^2\lambda_6) \\ +\lambda_{31}(\lambda_{11}\lambda_1^2\lambda_{14} + \lambda_3^2\lambda_8\lambda_{13} + \lambda_3^2\lambda_9\lambda_{12} + \lambda_3\lambda_{11}\lambda_2\lambda_{11} \\ +\lambda_3^2\lambda_{11}\lambda_{10} + \lambda_3\lambda_9\lambda_5\lambda_{10} + \lambda_{11}\lambda_7\lambda_0\lambda_9 \\ +\lambda_7\lambda_5\lambda_7\lambda_8 + \lambda_3\lambda_{11}\lambda_7\lambda_6 + \lambda_{11}\lambda_7\lambda_4\lambda_5)]\lambda_3 \end{array} \right\}$

$\in \mathrm{Ext}_A^{6,67}(\mathbb{F}_2, \mathbb{F}_2)$;

(10) $\mathcal{A}'' = \{D_1(0)\lambda_{12} + \lambda_{15}^2\lambda_{11}\lambda_7\lambda_8^2 + \lambda_{47}e_0\lambda_0\} \in \mathrm{Ext}_A^{6,70}(\mathbb{F}_2, \mathbb{F}_2)$;

(11) $r_1 = \left\{ \begin{array}{l} f_1\lambda_7\lambda_{19} + g_2\lambda_{11}^2 + (\lambda_{31}\lambda_3\lambda_{11}\lambda_7 + \lambda_{23}\lambda_{15}\lambda_9\lambda_5)\lambda_7^2 \\ +\lambda_{15}^2\lambda_9^2\lambda_7\lambda_{11} + \lambda_{31}[\lambda_7^2\lambda_0\lambda_{14} + (\lambda_3^2\lambda_9 + \lambda_9\lambda_3^2)\lambda_{13} \\ +(\lambda_3^2\lambda_{11} + \lambda_3\lambda_9\lambda_5)\lambda_{11}]\lambda_7 \end{array} \right\}$

$\in \mathrm{Ext}_A^{6,72}(\mathbb{F}_2, \mathbb{F}_2)$;

(12) $x_{6,77} = \left\{ \begin{array}{l} \lambda_{47}\lambda_3^2\lambda_2\lambda_1\lambda_{15} + c_2(\lambda_7\lambda_{12} + \lambda_{19}\lambda_0)\lambda_{11} + D_1(0) \\ \lambda_{19} + \lambda_{15}^2(\lambda_{27}\lambda_0\lambda_7 + \lambda_{11}\lambda_{19}\lambda_4 + \lambda_{11}\lambda_{23}\lambda_0)\lambda_7 \end{array} \right\}$

$\in \mathrm{Ext}_A^{6,77}(\mathbb{F}_2, \mathbb{F}_2)$;

(13) $x_{6,82} = \left\{ \begin{array}{l} H_1(0)\lambda_{14} + \lambda_{15}^2\lambda_{11}^3\lambda_{13} + (\lambda_{15}^2\lambda_{11}\lambda_{15}\lambda_9 \\ +\lambda_{15}^2\lambda_{13}\lambda_7 + \lambda_{31}\lambda_7\lambda_{23}\lambda_4\lambda_0 + D_3(0)\lambda_4 \\ +\lambda_{15}\lambda_{47}\lambda_0^2\lambda_3 + \lambda_{47}\lambda_3^2\lambda_6^2 + \lambda_{47}\lambda_3^2\lambda_2\lambda_{10} \\ +\lambda_{31}\lambda_{23}\lambda_1^2\lambda_9 + \lambda_{47}\lambda_{11}\lambda_0^2\lambda_9 + \lambda_{31}\lambda_{23}\lambda_1\lambda_5^2 \\ +\lambda_{31}\lambda_{23}\lambda_5\lambda_3^2 + \lambda_{47}\lambda_{11}\lambda_1^2\lambda_5 + \lambda_{47}\lambda_3\lambda_7\lambda_5\lambda_3)\lambda_{11} \\ +(\lambda_{47}\lambda_3^2\lambda_{10}\lambda_6 + \lambda_{47}\lambda_7\lambda_1\lambda_8\lambda_6 + \lambda_{47}\lambda_3\lambda_7\lambda_6^2 \\ +\lambda_{47}\lambda_7\lambda_5\lambda_4\lambda_6 + \lambda_{31}\lambda_{23}\lambda_1\lambda_9\lambda_5 + \lambda_{31}\lambda_{23}\lambda_9\lambda_1\lambda_5 \\ +\lambda_{15}\lambda_{47}\lambda_0\lambda_4\lambda_3 + \lambda_{15}\lambda_{47}\lambda_4\lambda_0\lambda_3 + \lambda_{15}\lambda_{47}\lambda_1^2\lambda_5 \\ +\lambda_{15}^2\lambda_{11}\lambda_{21}\lambda_7)\lambda_7 \end{array} \right\}$

$\in \mathrm{Ext}_A^{6,82}(\mathbb{F}_2, \mathbb{F}_2)$;

(14) $t_1 = Sq^0 t \in \mathrm{Ext}_A^{6,84}(\mathbb{F}_2, \mathbb{F}_2)$;

(15) $x_{6,90} = \left\{ \begin{array}{l} d_2\lambda_{15}\lambda_2 + [d_2\lambda_{16} + \lambda_{31}(\lambda_7\lambda_{23}\lambda_8 \\ +\lambda_{23}\lambda_{15}\lambda_0)\lambda_{15} + D_3(0)\lambda_{23}]\lambda_1 \end{array} \right\} \in \mathrm{Ext}_A^{6,90}(\mathbb{F}_2, \mathbb{F}_2)$;

(16) $C_1 = Sq^0 C \in \mathrm{Ext}_A^{6,112}(\mathbb{F}_2, \mathbb{F}_2)$;

(17) $x_{6,114} = \left\{ \begin{array}{c} [\lambda_{31}(\lambda_{23}\lambda_{15}\lambda_{19}\lambda_{13} + \lambda_{31}\lambda_9\lambda_{11}\lambda_9) \\ +c_3(\lambda_5\lambda_{11} + \lambda_9\lambda_7)]\lambda_7 + f_2\lambda_{15}\lambda_9 \\ +c_3(\lambda_{15}\lambda_0\lambda_8 + \lambda_{15}\lambda_2\lambda_6 + \lambda_{15}\lambda_4^2) \\ +\lambda_{31}^2\lambda_{19}^2\lambda_3\lambda_5 \end{array} \right\} \in \mathrm{Ext}_A^{6,114}(\mathbb{F}_2, \mathbb{F}_2)$;

(18) $G_1 = Sq^0 G \in \mathrm{Ext}_A^{6,120}(\mathbb{F}_2, \mathbb{F}_2)$.

In [11], Hung-Peterson proved that $\varphi_s^{\mathbb{F}_2}$ vanishes on decomposable elements for $s > 2$. Therefore, it is sufficient to prove that $\varphi_6^{\mathbb{F}_2}$ is vanishing on indecomposable elements of $\mathrm{Ext}_A^{6,*}(\mathbb{F}_2, \mathbb{F}_2)$. In order to show this claim, we prove that images of cycles which represented indecomposable elements of $\mathrm{Ext}_A^{6,t}(\mathbb{F}_2, \mathbb{F}_2)$ under the homomorphism $\tilde{\varphi}_6^{\mathbb{F}_2} : \Lambda_6 \otimes \mathbb{F}_2 \to (\mathrm{R}_6\mathbb{F}_2)^\#$ are trivial. For convenience, we write $\mathrm{Ext}_A^{s,t} = \mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$.

Since the canonical projection $\pi : \Lambda_s \to \mathcal{R}_s$ is an A-algebra homomorphism. If λ_I contains a factor of negative excess, then $\tilde{\varphi}_s^{\mathbb{F}_2}(\lambda_I) = 0$. Moreover, the actions of Sq^0 on $\mathrm{Ext}_A^{s,t}$ and on $\mathrm{R}_s\mathbb{F}_2$ commute with each other through $\varphi_s^{\mathbb{F}_2}$.

In the Lambda algebra, we have (see Wang [20], Lin-Mahowald [17], and Chen [3])

- $\varphi_3^{\mathbb{F}_2}(c_i) = 0$ with $c_i = \{(Sq^0)^i(\lambda_3^2\lambda_2)\} \in \mathrm{Ext}_A^{3,11.2^i}, i \geq 0$.

  In fact, for $c_0 = \lambda_3^2\lambda_2$, we have $\tilde{\varphi}_3^{\mathbb{F}_2}(c_0) = 0$ since $e(c_0) = -2 < 0$. This implies $\varphi_3^{\mathbb{F}_2}(c_0) = 0$. Then,

  $$\varphi_3^{\mathbb{F}_2}(c_i) = \varphi_3^{\mathbb{F}_2}((Sq^0)^i(c_0)) = (Sq^0)^i(\varphi_3^{\mathbb{F}_2}(c_0)) = 0.$$

- $\varphi_4^{\mathbb{F}_2}(d_i) = 0$ with $d_i = \{(Sq^0)^i(\lambda_3^2\lambda_2\lambda_6 + \lambda_3^2\lambda_4^2 + \lambda_3\lambda_5\lambda_4\lambda_2)\} \in \mathrm{Ext}_A^{4,18.2^i}, i \geq 0$. By direct inspection, we have $\varphi_4^{\mathbb{F}_2}(d_0) = 0$, so

  $$\varphi_4^{\mathbb{F}_2}(d_i) = \varphi_4^{\mathbb{F}_2}((Sq^0)^i(d_0)) = (Sq^0)^i(\varphi_4^{\mathbb{F}_2}(d_0)) = 0.$$

- $\varphi_4^{\mathbb{F}_2}(e_i) = 0$ with

  $$e_i = \{(Sq^0)^i(\lambda_3^3\lambda_8 + (\lambda_3\lambda_5^2 + \lambda_3^2\lambda_7)\lambda_4 + (\lambda_3^2\lambda_9 + \lambda_9\lambda_3^2)\lambda_2)\} \in \mathrm{Ext}_A^{4,22.2^i}, i \geq 0.$$

  Since $\tilde{\varphi}_4^{\mathbb{F}_2}(\lambda_3^3\lambda_8) = 0, \tilde{\varphi}_4^{\mathbb{F}_2}(\lambda_3\lambda_5^2\lambda_4) = 0, \tilde{\varphi}_4^{\mathbb{F}_2}(\lambda_3^2\lambda_7\lambda_4) = 0$ and $\tilde{\varphi}_4^{\mathbb{F}_2}(\lambda_3^2\lambda_9\lambda_2) = 0$, we have

  $$\tilde{\varphi}_4^{\mathbb{F}_2}(e_0) = \tilde{\varphi}_4^{\mathbb{F}_2}(\lambda_9\lambda_3\lambda_3\lambda_2).$$

  Applying the Adem relation, we have $\lambda_9\lambda_3 = \lambda_7\lambda_5$. Then

  $$\tilde{\varphi}_4^{\mathbb{F}_2}(e_0) = \tilde{\varphi}_4^{\mathbb{F}_2}(\lambda_9\lambda_3\lambda_3\lambda_2) = \tilde{\varphi}_4^{\mathbb{F}_2}(\lambda_7\lambda_5\lambda_3\lambda_2) = 0.$$

  Hence, $\varphi_4^{\mathbb{F}_2}(e_0) = 0$ and then $\varphi_4^{\mathbb{F}_2}(e_i) = \varphi_4^{\mathbb{F}_2}((Sq^0)^i(e_0)) = (Sq^0)^i(\varphi_4^{\mathbb{F}_2}(e_0)) = 0$.

- $\varphi_4^{\mathbb{F}_2}(f_i) = 0$ with

  $$f_i = \{(Sq^0)^i(\lambda_7^2\lambda_0\lambda_4 + (\lambda_3^2\lambda_9 + \lambda_7\lambda_5\lambda_3)\lambda_3 + \lambda_7^2\lambda_2^2)\} \in \mathrm{Ext}_A^{4,22.2^i}, i \geq 0.$$

  By direct inspection, we have

  $\tilde{\varphi}_4^{\mathbb{F}_2}(\lambda_7^2\lambda_0\lambda_4) = 0, \tilde{\varphi}_4^{\mathbb{F}_2}(\lambda_3^2\lambda_9\lambda_3) = 0, \tilde{\varphi}_4^{\mathbb{F}_2}(\lambda_7\lambda_5\lambda_3^2) = 0$, and $\tilde{\varphi}_4^{\mathbb{F}_2}(\lambda_7^2\lambda_2^2) = 0$,

  then $\tilde{\varphi}_4^{\mathbb{F}_2}(f_0) = 0$, and so

  $$\varphi_4^{\mathbb{F}_2}(f_i) = \varphi_4^{\mathbb{F}_2}((Sq^0)^i(f_0)) = (Sq^0)^i(\varphi_4^{\mathbb{F}_2}(f_0)) = 0.$$

  Similarly, by direct inspection, we also have

- $\varphi_4^{\mathbb{F}_2}(g_{i+1}) = 0$ with $g_{i+1} = \big\{(Sq^0)^i(\lambda_7^2\lambda_0\lambda_6 + (\lambda_3^2\lambda_9 + \lambda_7\lambda_5\lambda_3)\lambda_5$

  $$+(\lambda_3\lambda_9\lambda_5 + \lambda_3^2\lambda_{11})\lambda_3)\big\} \in \mathrm{Ext}_A^{4,24.2^i},\ i \geq 0.$$

- $\varphi_4^{\mathbb{F}_2}(D_3(i)) = 0$ with $D_3(i) = \{(Sq^0)^i(\lambda_{31}\lambda_7\lambda_{23}\lambda_0)\} \in \mathrm{Ext}_A^{4,65.2^i}, i \geq 0$.
  Applying the Adem relation, we have

  $$\tilde{\varphi}_4^{\mathbb{F}_2}(D_3(0)) = \tilde{\varphi}_4^{\mathbb{F}_2}(\lambda_{31}\lambda_7\lambda_{23}\lambda_0) = \tilde{\varphi}_4^{\mathbb{F}_2}(\lambda_{15}\lambda_{23}\lambda_{23}\lambda_0) = 0.$$

  Then

  $$\varphi_4^{\mathbb{F}_2}(D_3(i)) = \varphi_4^{\mathbb{F}_2}((Sq^0)^i(D_3(0))) = (Sq^0)^i(\varphi_4^{\mathbb{F}_2}(D_3(0))) = 0.$$

  It is easy to check these details,

- $\varphi_5^{\mathbb{F}_2}(n_i) = 0$ with $n_i = \{(Sq^0)^i(\lambda_7^2\lambda_5\lambda_3\lambda_9 + \lambda_7\lambda_{15}\lambda_3\lambda_0\lambda_6 + \lambda_7\lambda_{15}\lambda_1\lambda_5\lambda_3)\}$ $\in \mathrm{Ext}_A^{5,36.2^i}, i \geq 0$.

- $\varphi_5^{\mathbb{F}_2}(D_1(i)) = 0$ with $D_1(i) = \{(Sq^0)^i(\lambda_{15}^2\lambda_{11}\lambda_7\lambda_4)\} \in \mathrm{Ext}_A^{5,57.2^i}, i \geq 0$.

- $\varphi_5^{\mathbb{F}_2}(H_1(i)) = 0$ with $H_1(i) = \big\{(Sq^0)^i(\lambda_{15}^2\lambda_{11}\lambda_7\lambda_{14} + \lambda_{15}^2\lambda_{11}^2\lambda_{10}$

  $$+\lambda_{15}\lambda_{31}\lambda_7\lambda_1\lambda_8 + \lambda_{15}\lambda_{31}\lambda_3\lambda_7\lambda_6 + \lambda_{15}\lambda_{31}\lambda_7\lambda_5\lambda_4)\} \in \mathrm{Ext}_A^{5,67.2^i}, i \geq 0.$$

Using Adem relations, we have

$$\lambda_{23}\lambda_1 = \lambda_{11}\lambda_{13} + \lambda_7\lambda_{17} + \lambda_3\lambda_{21}; \quad (1')$$
$$\lambda_{23}\lambda_4 = \lambda_{15}\lambda_{12} + \lambda_{11}\lambda_{16} + \lambda_9\lambda_{18}; \quad (2')$$
$$\lambda_{47}\lambda_{11} = \lambda_{31}\lambda_{27} + \lambda_{23}\lambda_{35}; \quad (3')$$
$$\lambda_{31}\lambda_3 = \lambda_{15}\lambda_{19} + \lambda_7\lambda_{27}; \quad (4')$$

$$\begin{aligned}
\lambda_{31}\lambda_9 &= \lambda_{23}\lambda_{17} + \lambda_{19}\lambda_{21}; && (5')\\
\lambda_{31}\lambda_{11} &= \lambda_{23}\lambda_{19}; && (6')\\
\lambda_{31}\lambda_7 &= \lambda_{15}\lambda_{23}; && (7')\\
\lambda_{47}\lambda_3 &= \lambda_{15}\lambda_{35} + \lambda_7\lambda_{43} && (8')\\
\lambda_{47}\lambda_7 &= \lambda_{15}\lambda_{39}; && (9')\\
\lambda_{27}\lambda_0 &= \lambda_{13}\lambda_{14} + \lambda_{11}\lambda_{16} + \lambda_5\lambda_{22} + \lambda_3\lambda_{24} + \lambda_1\lambda_{26}; && (10')\\
\lambda_{23}\lambda_0 &= \lambda_{11}\lambda_{12} + \lambda_9\lambda_{14} + \lambda_7\lambda_{16} + \lambda_3\lambda_{20} + \lambda_1\lambda_{22}; && (11')\\
\lambda_{11}\lambda_1 &= \lambda_3\lambda_9. && (12')
\end{aligned}$$

Now, we prove that $\tilde{\varphi}_6^{\mathbb{F}_2}$ sends the above eighteen indecomposable elements (from (1) to (18)) to zero.

- By replacing (11') in (1), combined with results $\varphi_4^{\mathbb{F}_2}(e_0) = 0, \varphi_4^{\mathbb{F}_2}(f_0) = 0$, we have $\tilde{\varphi}_6^{\mathbb{F}_2}(r) = 0$. Then $\varphi_6^{\mathbb{F}_2}(r) = 0$.
- By direct inspection and $\varphi_4^{\mathbb{F}_2}(g_1) = 0$, we imply $\tilde{\varphi}_6^{\mathbb{F}_2}(q) = 0$. Then $\varphi_6^{\mathbb{F}_2}(q) = 0$.
- From result $\varphi_5^{\mathbb{F}_2}(n_0) = 0$ and the excess of other terms is negative. Therefore, $\tilde{\varphi}_6^{\mathbb{F}_2}(t) = 0$, then $\varphi_6^{\mathbb{F}_2}(t) = 0$.
- By replacing (1') and (2') in (4), then excess of all terms of y is negative. Therefore, $\tilde{\varphi}_6^{\mathbb{F}_2}(y) = 0$, imply $\varphi_6^{\mathbb{F}_2}(q) = 0$.
- From the result, $\varphi_4^{\mathbb{F}_2}(e_1) = 0$ and by direct inspection, under $\varphi_6^{\mathbb{F}_2}$, the image of element C is trivial.
- From the result, $\varphi_5^{\mathbb{F}_2}(D_1(0)) = 0$ and by direct inspection, under $\varphi_6^{\mathbb{F}_2}$, the image of element G is trivial.
- From (3') and (10'), we have

  $$\begin{aligned}
  \tilde{\varphi}_6^{\mathbb{F}_2}(D_2) &= \tilde{\varphi}_6^{\mathbb{F}_2}(\lambda_{47}\lambda_{11}\lambda_0^4) = \tilde{\varphi}_6^{\mathbb{F}_2}(\lambda_{31}\lambda_{27}\lambda_0^4 + \lambda_{23}\lambda_{35}\lambda_0^4)\\
  &= \tilde{\varphi}_6^{\mathbb{F}_2}(\lambda_{31}(\lambda_{13}\lambda_{14} + \lambda_{11}\lambda_{16} + \lambda_5\lambda_{22} + \lambda_3\lambda_{24}\\
  &\qquad + \lambda_1\lambda_{26})\lambda_0^3 + \lambda_{23}\lambda_{35}\lambda_0^4)\\
  &= 0.
  \end{aligned}$$

  Then $\varphi_6^{\mathbb{F}_2}(D_2) = 0$.
- From results $\varphi_5^{\mathbb{F}_2}(D_1(0)) = 0$ and $\varphi_4^{\mathbb{F}_2}(d_0) = 0$, we have $\tilde{\varphi}_6^{\mathbb{F}_2}(\mathcal{A}) = 0$. Therefore, $\varphi_6^{\mathbb{F}_2}(\mathcal{A}) = 0$.
- Using (4'), (5'), (6'), (7') and the results

  $$\varphi_3^{\mathbb{F}_2}(c_2) = 0, \varphi_4^{\mathbb{F}_2}(f_0) = 0, \varphi_5^{\mathbb{F}_2}(D_1(0)) = 0,$$

  we have $\tilde{\varphi}_6^{\mathbb{F}_2}(\mathcal{A}') = 0$. Then, $\varphi_6^{\mathbb{F}_2}(\mathcal{A}') = 0$.

- Similarly, taking (8'), (9'),(12') and $D_1(0)$ replace on $\mathcal{A}''$, we have

$$\begin{aligned}\mathcal{A}'' &= \{D_1(0)\lambda_{12} + \lambda_{15}^2\lambda_{11}\lambda_7\lambda_8^2 + \lambda_{47}e_0\lambda_0\}\\ &= \{\lambda_{15}^2\lambda_{11}\lambda_7\lambda_4\lambda_{12} + \lambda_{15}^2\lambda_{11}\lambda_7\lambda_8^2 + \lambda_{47}(\lambda_3^3\lambda_8 + (\lambda_3\lambda_5^2 + \lambda_3^2\lambda_7)\lambda_4\\ &+ (\lambda_3^2\lambda_9 + \lambda_9\lambda_3^2)\lambda_2)\lambda_0\}\\ &= \{\lambda_{15}^2\lambda_{11}\lambda_7\lambda_4\lambda_{12} + \lambda_{15}^2\lambda_{11}\lambda_7\lambda_8^2 + \lambda_{47}(\lambda_3^3\lambda_8 + (\lambda_3\lambda_5^2 + \lambda_3^2\lambda_7)\lambda_4\\ &+ (\lambda_3\lambda_{11}\lambda_1 + \lambda_7\lambda_5\lambda_3)\lambda_2)\lambda_0\}\\ &= \{\lambda_{15}^2\lambda_{11}\lambda_7\lambda_4\lambda_{12} + \lambda_{15}^2\lambda_{11}\lambda_7\lambda_8^2 + (\lambda_{15}\lambda_{35} + \lambda_7\lambda_{43})\lambda_3^2\lambda_8\\ &+ (\lambda_{15}\lambda_{35} + \lambda_7\lambda_{43})\lambda_5^2 + (\lambda_{15}\lambda_{35} + \lambda_7\lambda_{43})\lambda_3\lambda_7)\lambda_4\\ &+ (\lambda_{15}\lambda_{35} + \lambda_7\lambda_{43})\lambda_{11}\lambda_1 + \lambda_{15}\lambda_{39}\lambda_5\lambda_3)\lambda_2)\lambda_0\}\end{aligned}$$

  Obviously, the excess of all terms of $\mathcal{A}''$ is negative. Therefore, $\tilde{\varphi}_6^{\mathbb{F}_2}(\mathcal{A}'') = 0$, then $\varphi_6^{\mathbb{F}_2}(\mathcal{A}'') = 0$.

- Since $\varphi_4^{\mathbb{F}_2}(f_1) = 0, \varphi_5^{\mathbb{F}_2}(g_2) = 0$ and the excess of the other terms of r_1 is negative. Therefore, $\tilde{\varphi}_6^{\mathbb{F}_2}(r_1) = 0$, then $\varphi_6^{\mathbb{F}_2}(r_1) = 0$.
- Depend on the results $\varphi_3^{\mathbb{F}_2}(c_2) = 0, \varphi_5^{\mathbb{F}_2}(D_1(0)) = 0$ and (8'). It is easy to prove that under $\tilde{\varphi}_6^{\mathbb{F}_2}$, image of $x_{6,77}$ is trivial, then $\varphi_6^{\mathbb{F}_2}(x_{6,77}) = 0$.
- Since $\varphi_5^{\mathbb{F}_2}(H_1(0)) = 0, \varphi_5^{\mathbb{F}_2}(D_3(0)) = 0$ and relations (3'),(8') and (9'), we have $\varphi_6^{\mathbb{F}_2}(x_{6,82}) = 0$.
- The actions of Sq^0 on $\mathrm{Ext}_A^{s,t}$ and on $\mathrm{R}_s\mathbb{F}_2$ commute with each other through $\varphi_s^{\mathbb{F}_2}$. Therefore,

$$\varphi_6^{\mathbb{F}_2}(t_1) = \varphi_6^{\mathbb{F}_2}(Sq^0 t) = Sq^0(\varphi_6^{\mathbb{F}_2}(t)) = 0.$$

- From results $\varphi_4^{\mathbb{F}_2}(D_3(0)) = 0$ and $\varphi_4^{\mathbb{F}_2}(d_2) = 0$, we have $\tilde{\varphi}_6^{\mathbb{F}_2}(x_{6,90}) = 0$. Therefore, $\varphi_6^{\mathbb{F}_2}(x_{6,90}) = 0$.
- Similarly, $\varphi_6^{\mathbb{F}_2}(C_1) = \varphi_6^{\mathbb{F}_2}(Sq^0 C) = Sq^0(\varphi_6^{\mathbb{F}_2}(C)) = 0$.
- From results $\varphi_3^{\mathbb{F}_2}(c_3) = 0$ and $\varphi_4^{\mathbb{F}_2}(f_2) = 0$, we have $\tilde{\varphi}_6^{\mathbb{F}_2}(x_{6,114}) = 0$. Therefore, $\varphi_6^{\mathbb{F}_2}(x_{6,114}) = 0$.
- Similarly, $\varphi_6^{\mathbb{F}_2}(G_1) = \varphi_6^{\mathbb{F}_2}(Sq^0 G) = Sq^0(\varphi_6^{\mathbb{F}_2}(G)) = 0$.

The proof is complete. □

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