East-West J. of Mathematics: Vol. 21, No 2 (2019) pp. 158-170
https://doi.org/10.36853/ewjm0360

ON SOME PROPERTIES OF CUBIC IDEALS OF SEMIRINGS

Debabrata Mandal

Department of Mathematics
Raja Peary Mohan College, Uttarpara, Hooghly-712258
e-mail: dmandaljumath@gmail.com

Abstract

In this paper, we have defined cubic ideals, cubic bi-ideals and cubic quasi-ideals of a semiring and obtain some of the basic properties of these ideals using several operation on them. We also obtain some inter-relation between these ideals. Among all the results, we also obtain some characterizations of regular semiring.

1 Introduction

The notion of a semiring as first introduced by H. S. Vandiver [9] in 1934 with two associative binary operations where one distributes over the other. But semirings had appeared in earlier studies on the theory of ideals of rings. In structure, semirings lie between semigroups and rings. The results which hold in rings but not in semigroups may hold in semirings since semiring is a generalization of ring. The study of rings shows that multiplicative structure of ring is an independent of additive structure whereas in semiring multiplicative structure of semiring is not an independent of additive structure of semiring.

The additive and the multiplicative structure of a semiring play an important role in determining the structure of a semiring. The theory of rings and theory of semigroups have considerable impact on the development of theory of semirings. Also, semirings has some applications to the theory of automata,

Key words: cubic ideal, cubic bi-ideal, cubic quasi-ideal, cartesian product, normal cubic ideal, intersection, regular, semiring.
AMS Classification 2010: 08A72,16Y60.

formal languages, optimization theory and other branches of applied mathematics. Ideals of semiring play a central role in the structure theory and useful for many purposes.

The theory of fuzzy sets, proposed by Zadeh [10], has provided a useful mathematical tool for describing the behavior of the systems that are too complex or illdefined to admit precise mathematical analysis by classical methods and tools. The study of fuzzy algebraic structure has started by Rosenfeld [8]. Since then many researchers, for example [3, 6], developed this ideas. As an extension of this, the concept of cubic subgroups and cubic sets were initiated by Jun et al [4, 5]. Khan et al [7] applied this in case of cubic h-ideals of hemirings. Chinnadurai [1, 2] used this notion to study cubic bi-ideals and cubic lateral ideals in near-ring and ternary near-ring, respectively.

The main aim of this paper is to introduce and study cubic ideal of semiring. At first we define some basic operation such as intersection, cartesian product, composition etc on them and use these to obtain some of its basic properties. After that we define cubic bi-ideals and cubic quasi-ideals and obtain some characterization of regular semiring.

2 Preliminaries

We recall the following preliminaries for subsequent use.

Definition 2.1. A hemiring [respectively semiring] is a nonempty set S on which operations addition and multiplication have been defined such that the following conditions are satisfied:
(i) $(S, +)$ is a commutative monoid with identity 0;
(ii) $(S, .)$ is a semigroup (resp. monoid with identity 1_S);
(iii) Multiplication distributes over addition from either side;
(iv) $0s = 0 = s0$ for all $s \in S$.
(v) $1_S \neq 0$.

A subset A of a semiring S is called a left(resp. right) ideal of S if A is closed under addition and $SA \subseteq A$ (resp.$AS \subseteq A$). A subset A of a semiring S is called an ideal if it is both left and right ideal of S.

A subset A of a semiring S is called a quasi-ideal of S if A is closed under addition and $SA \cap AS \subseteq A$.

A subset A of a semiring S is called a bi-ideal if A is closed under addition and $ASA \subseteq A$.

Definition 2.2. A fuzzy subset of a non-empty set X is defined as a function μ : X $\rightarrow$ [0,1].

Definition 2.3. Let X be a non-empty set. A cubic set A in X is a structure $A = \{< x, \widetilde{\mu}, f >: x \in X\}$ which briefly denoted as $A =< \widetilde{\mu}, f >$ where

$\widetilde{\mu} = [\mu^-, \mu^+]$ is an interval valued fuzzy set (briefly, IVF) in X and f is a fuzzy set in X.

Definition 2.4. For any non-empty set G of a set X, the characteristic cubic set of G is defined to be the structure $\chi_G(x) =< x, \widetilde{\zeta}_{\chi_G}(x), \eta_{\chi_G}(x) : x \in X >$ where

$$\begin{aligned} \widetilde{\zeta}_{\chi_G}(x) &= [1,1] \approx \widetilde{1} \; if \; x \in G \\ &= [0,0] \approx \widetilde{0} \; otherwise. \end{aligned}$$

and

$$\begin{aligned} \eta_{\chi_G}(x) &= 0 \; if \; x \in G \\ &= 1 \; otherwise. \end{aligned}$$

Throughout this paper unless otherwise mentioned S denotes the semiring and for any two set P and Q, we use the following notation:

$$\cap\{P, Q\} = P \cap Q \text{ and } \cup\{P, Q\} = P \cup Q.$$

3 Cubic ideals

In this section, the notions of cubic ideals in semiring are introduced and some of their basic properties are investigated.

Definition 3.1. Let $< \widetilde{\mu}, f >$ be a non empty cubic subset of a semiring S. Then $< \widetilde{\mu}, f >$ is called a cubic left ideal [cubic right ideal] of S if

(i) $\widetilde{\mu}(x + y) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$, $f(x + y) \leq \max\{f(x), f(y)\}$ and

(ii) $\widetilde{\mu}(xy) \supseteq \widetilde{\mu}(y)$, $f(xy) \leq f(y)$ [respectively $\widetilde{\mu}(xy) \supseteq \widetilde{\mu}(x), f(xy) \leq f(x)$].

for all $x, y \in S$.

A cubic ideal of a semiring S is a non empty cubic subset of S which is a cubic left ideal as well as a cubic right ideal of S. Note that if $< \widetilde{\mu}, f >$ is a cubic left or right ideal of a semiring S, then $\widetilde{\mu}(0) \supseteq \widetilde{\mu}(x)$ and $f(0) \leq f(x)$ for all $x \in S$.

A cubic right ideal is defined similarly.

By a cubic ideal $< \widetilde{\mu}, f >$, we mean that $< \widetilde{\mu}, f >$ is both cubic left and cubic right ideal.

Example 3.2. *Let $S = Z$,the set of integers. Then S forms a semiring with usual addition and multiplication of integers. Define $< \widetilde{\mu}, f >$ be a cubic subset of S as follows*

$$\widetilde{\mu}(x) = \begin{cases} [0,1] & if \;\; x = 0 \\ (0.1, 0.7] & if \; x \; is \; even \\ (0.2, 0.5) & if \; x \; is \; odd \end{cases} \quad and \; f(x) = \begin{cases} 0 & if \;\; x = 0 \\ 0.4 & if \; x \; is \; even \\ 0.9 & if \; x \; is \; odd \end{cases}$$

The cubic subset $< \widetilde{\mu}, f >$ of S is a cubic ideal S.

Throughout this section, we prove results only for cubic left ideals. Similar results can be obtained for cubic right ideals and cubic ideals.

Theorem 3.3. *A cubic set $C =< \widetilde{\mu}, f >$ of S is a cubic left ideal of S if and only if any level subset $C_t =< \widetilde{\mu}_t, f_t >:= \{x \in S : \widetilde{\mu}(x) \supseteq [t,t]$ and $f(x) \leq t, t \in [0,1]\}$ is a left ideal of S, provided it is non-empty.*

Proof. Let $< \widetilde{\mu}, f >$ be a cubic left ideal of S and assume that $< \widetilde{\mu}_t, f_t > \neq \phi$ for $t \in$ [0,1]. Let $x, z \in S$ and $a, b \in < \widetilde{\mu}_t, f_t >$. Then $\widetilde{\mu}(a+b) \supseteq \cap\{\widetilde{\mu}(a), \widetilde{\mu}(b)\} \supseteq [t,t]$ and $f(a+b) \leq \max\{f(a), f(b)\} \leq t$; implies that $a+b \in < \widetilde{\mu}_t, f_t >$. Also $\widetilde{\mu}(xa) \supseteq \widetilde{\mu}(a) \supseteq [t,t]$ and $f(xa) \leq f(a) \leq t$ which implies $xa \in < \widetilde{\mu}_t, f_t >$. So, $< \widetilde{\mu}_t, f_t >$ is a left ideal of S.

Conversely, suppose $< \widetilde{\mu}_t, f_t >$ is a left ideal of S. If possible, suppose $< \widetilde{\mu}, f >$ is not a cubic left ideal of S. Then there exist $a, b \in S$ such that $\widetilde{\mu}(a+b) \subset \cap\{\widetilde{\mu}(a), \widetilde{\mu}(b)\}$ or $f(a+b) > \max\{f(a), f(b)\}$. Taking $t_0 = \frac{1}{2}[f(a+b)+\max\{f(a), f(b)\}]$, we see that $t_0 \in$ [0,1] and $f(a+b) > t_0 > \max\{f(a), f(b)\}$ whence $a, b \in f_{t_0}$ but $a+b \notin f_{t_0}$ - which is a contradiction.

Therefore $\widetilde{\mu}(a+b) \supseteq \cap\{\widetilde{\mu}(a), \widetilde{\mu}(b)\}$ and $f(a+b) \leq \max\{f(a), f(b)\}$ for all $a, b \in S$. The other property can be proved similarly. Consequently, $< \widetilde{\mu}, f >$ is a cubic left ideal of S. □

Theorem 3.4. *Let A be a non-empty subset of a semiring S. Then A is a left ideal of S if and only if the characteristic function $\chi_A =< \widetilde{\mu}_{\chi_A}, f_{\chi_A} >$ is a cubic left ideal of S.*

Proof. Assume that A is a left ideal of S and $x, y \in S$. Suppose $\widetilde{\mu}_{\chi_A}(x+y) \subset \cap\{\widetilde{\mu}_{\chi_A}(x), \widetilde{\mu}_{\chi_A}(y)\}$ and $f_{\chi_A}(x+y) > \max\{f_{\chi_A}(x), f_{\chi_A}(y)\}$. It follows that $\widetilde{\mu}_{\chi_A}(x+y) = \widetilde{0}, \cap\{\widetilde{\mu}_{\chi_A}(x), \widetilde{\mu}_{\chi_A}(y)\} = \widetilde{1}$ and $f_{\chi_A}(x+y) = 1, \max\{f_{\chi_A}(x), f_{\chi_A}(y)\} = 0$. This implies that $x, y \in A$ but $x+y \notin A$ $-$ a contradiction. So, $\widetilde{\mu}_{\chi_A}(x+y) \supseteq \cap\{\widetilde{\mu}_{\chi_A}(x), \widetilde{\mu}_{\chi_A}(y)\}$ and $f_{\chi_A}(x+y) \leq \max\{f_{\chi_A}(x), f_{\chi_A}(y)\}$.

Similarly we can show that $\widetilde{\mu}_{\chi_A}(xy) \supseteq \widetilde{\mu}_{\chi_A}(y)$, $f_{\chi_A}(xy) \leq f_{\chi_A}(y)$. Therefore $\chi_A =< \widetilde{\mu}_{\chi_A}, f_{\chi_A} >$ is a cubic left ideal of S.

Conversely, assume that $\chi_A =< \widetilde{\mu}_{\chi_A}, f_{\chi_A} >$ is a cubic left ideal of S for any subset A of S. Let $x, y \in A$. Then $\widetilde{\mu}_{\chi_A}(x) = \widetilde{\mu}_{\chi_A}(y) = \widetilde{1}$ and $f_{\chi_A}(x) = f_{\chi_A}(y) = 0$. Now $\widetilde{\mu}_{\chi_A}(x+y) \supseteq \cap\{\widetilde{\mu}_{\chi_A}(x), \widetilde{\mu}_{\chi_A}(y)\} = \widetilde{1}$, $f_{\chi_A}(x+y) \leq \max\{f_{\chi_A}(x), f_{\chi_A}(y)\} = 0$ and $\widetilde{\mu}_{\chi_A}(xy) \supseteq \widetilde{\mu}_{\chi_A}(y) = \widetilde{1}$, $f_{\chi_A}(xy) \leq f_{\chi_A}(y) = 0$. This implies $x+y, xy \in A$. Hence A is a left ideal of S. □

Definition 3.5. Let $A =< \widetilde{\mu}, f >$ and $B =< \widetilde{\theta}, g >$ be two cubic sets of a semiring S. Define intersection of A and B by

$$A \cap B =< \widetilde{\mu}, f > \cap < \widetilde{\theta}, g > =< \widetilde{\mu} \cap \widetilde{\theta}, f \cup g > .$$

Proposition 3.6. *Intersection of a non-empty collection of cubic left ideals is a cubic left ideal of S.*

Proof. Let $A_i = \{< \widetilde{\mu}_i, f_i >: i \in I\}$ be a non-empty family of ideals of S. Let $x, y \in S$. Then

$$\begin{aligned}(\underset{i\in I}{\cap}\widetilde{\mu}_i)(x+y) &= \underset{i\in I}{\cap}\{\mu_i(x+y)\} \supseteq \underset{i\in I}{\cap}\{\cap\{\widetilde{\mu}_i(x), \widetilde{\mu}_i(y)\}\} \\ &= \cap\{\underset{i\in I}{\cap}\widetilde{\mu}_i(x), \underset{i\in I}{\cap}\widetilde{\mu}_i(y)\} = \cap\{(\underset{i\in I}{\cap}\widetilde{\mu}_i)(x), (\underset{i\in I}{\cap}\widetilde{\mu}_i)(y)\}.\end{aligned}$$

$$\begin{aligned}(\underset{i\in I}{\cup}f_i)(x+y) &= \sup_{i\in I}\{f(x+y)\} \le \sup_{i\in I}\{\max\{f_i(x), f_i(y)\}\} \\ &= \max\{\sup_{i\in I} f_i(x), \sup_{i\in I} f_i(y)\} = \max\{(\underset{i\in I}{\cup}f_i)(x), (\underset{i\in I}{\cup}f_i)(y)\}.\end{aligned}$$

Again

$$(\underset{i\in I}{\cap}\widetilde{\mu}_i)(xy) = \underset{i\in I}{\cap}\{\widetilde{\mu}_i(xy)\} \supseteq \underset{i\in I}{\cap}\{\widetilde{\mu}_i(y)\} = (\underset{i\in I}{\cap}\widetilde{\mu}_i)(y).$$

$$(\underset{i\in I}{\cup}f_i)(xy) = \sup_{i\in I}\{f_i(xy)\} \le \sup_{i\in I}\{f_i(y)\} = (\underset{i\in I}{\cup}f_i)(y).$$

Hence $\underset{i\in I}{\cap}A_i = \{< \underset{i\in I}{\cap}\widetilde{\mu}_i, \underset{i\in I}{\cup}f_i >: i \in I\}$ is a cubic left ideal of S. □

Proposition 3.7. *Let $f : R \to S$ be a morphism of Γ-semirings and $A =< \widetilde{\phi}, g >$ be a cubic left ideal of S, then $f^{-1}(A)$ is a cubic left ideal of R where $f^{-1}(A)(x) =< f^{-1}(\widetilde{\phi})(x), f^{-1}(g)(x) >=< \widetilde{\phi}(f(x)), g(f(x)) >$*

Proof. Let $f : R \to S$ be a morphism of semirings and $A =< \widetilde{\phi}, g >$ be a cubic left ideal of S. Assume $r, s \in R$. Then

$$\begin{aligned}f^{-1}(\widetilde{\phi})(r+s) &= \widetilde{\phi}(f(r+s)) = \widetilde{\phi}(f(r)+f(s)) \\ &\supseteq \cap\{\widetilde{\phi}(f(r)), \widetilde{\phi}(f(s))\} = \cap\{(f^{-1}(\widetilde{\phi}))(r), (f^{-1}(\widetilde{\phi}))(s)\}\end{aligned}$$

$$\begin{aligned}f^{-1}(g)(r+s) &= g(f(r+s)) = g(f(r)+f(s)) \\ &\le \max\{g(f(r)), g(f(s))\} = \max\{(f^{-1}(g))(r), (f^{-1}(g))(s)\}\end{aligned}$$

Again $(f^{-1}(\widetilde{\phi}))(rs) = \widetilde{\phi}(f(rs)) = \widetilde{\phi}(f(r)f(s)) \supseteq \widetilde{\phi}(f(s)) = (f^{-1}(\widetilde{\phi}))(s)$.
$(f^{-1}(g))(rs) = g(f(rs)) = g(f(r)f(s)) \le g(f(s)) = (f^{-1}(g))(s)$.
Thus $< f^{-1}(\widetilde{\phi})(x), f^{-1}(g)(x) >$ is a cubic left ideal of R. □

Definition 3.8. A cubic left ideal $< \widetilde{\mu}, f >$ of a semiring S, is said to be normal cubic left ideal if $\widetilde{\mu}(0) = \widetilde{1},\ f(0) = 0$.

Proposition 3.9. *Given a cubic left ideal $< \widetilde{\mu}, f >$ of a semiring S, let $< \widetilde{\mu}_+, f_+ >$ be a cubic set in S obtained by $\widetilde{\mu}_+(x) = \widetilde{\mu}(x) + \widetilde{1} - \widetilde{\mu}(0),\ f_+(x) = f(x) - f(0)$ for all $x \in S$. Then $< \widetilde{\mu}_+, f_+ >$ is a normal cubic left ideal of S.*

Proof. For all $x, y \in S$, we have $\widetilde{\mu}_+(0) = \widetilde{\mu}(0) + \widetilde{1} - \widetilde{\mu}(0) = \widetilde{1}$, $f_+(0) = f(0) - f(0) = 0$ Now,

$$\begin{aligned}\widetilde{\mu}_+(x+y) &= \widetilde{\mu}(x+y) + \widetilde{1} - \widetilde{\mu}(0) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\} + \widetilde{1} - \widetilde{\mu}(0) \\ &= \cap\{\{\widetilde{\mu}(x) + \widetilde{1} - \widetilde{\mu}(0)\}, \{\widetilde{\mu}(y) + \widetilde{1} - \widetilde{\mu}(0)\}\} = \cap\{\widetilde{\mu}_+(x), \widetilde{\mu}_+(y)\}\end{aligned}$$

$$\begin{aligned}f_+(x+y) &= f(x+y) - f(0) \leq \max\{f(x), f(y)\} - f(0) \\ &= \max\{\{f(x) - f(0)\}, \{f(y) - f(0)\}\} = \max\{f_+(x), f_+(y)\}\end{aligned}$$

and

$$\widetilde{\mu}_+(xy) = \widetilde{\mu}(xy) + \widetilde{1} - \widetilde{\mu}(0) \supseteq \widetilde{\mu}(y) + \widetilde{1} - \widetilde{\mu}(0) = \widetilde{\mu}_+(y).$$

$$f_+(xy) = f(xy) - f(0) \leq f(y) - f(0) = f_+(y).$$

Hence $< \widetilde{\mu}_+, f_+ >$ is a normal cubic left ideal of S. □

Definition 3.10. Let $A =< \widetilde{\mu}, f >$ and $B =< \widetilde{\nu}, g >$ be cubic subsets of X. The cartesian product of A and B is defined by $(A \times B)(x, y) = (< \widetilde{\mu}, f > \times < \widetilde{\nu}, g >)(x, y) = (< \widetilde{\mu} \times \widetilde{\nu}, f \times g >)(x, y) = [\cap\{\widetilde{\mu}(x), \widetilde{\nu}(y)\}, \max\{f(x), g(y)\}]$ for all $x, y \in X$.

Theorem 3.11. *Let $A =< \widetilde{\mu}, f >$ and $B =< \widetilde{\nu}, g >$ be cubic left ideals of a semiring S. Then $A \times B$ is a cubic left ideal of the semiring $S \times S$.*

Proof. Let $(x_1, x_2), (y_1, y_2) \in S \times S$. Then

$$\begin{aligned}(\widetilde{\mu} \times \widetilde{\nu})((x_1, x_2) + (y_1, y_2)) &= (\widetilde{\mu} \times \widetilde{\nu})(x_1 + y_1, x_2 + y_2) \\ &= \cap\{\widetilde{\mu}(x_1 + y_1), \widetilde{\nu}(x_2 + y_2)\} \\ &\supseteq \cap\{\cap\{\widetilde{\mu}(x_1), \widetilde{\mu}(y_1)\}, \cap\{\widetilde{\nu}(x_2), \widetilde{\nu}(y_2)\}\} \\ &= \cap\{\cap\{\widetilde{\mu}(x_1), \widetilde{\nu}(x_2)\}, \cap\{\widetilde{\mu}(y_1), \widetilde{\nu}(y_2)\}\} \\ &= \cap\{(\widetilde{\mu} \times \widetilde{\nu})(x_1, x_2), (\widetilde{\mu} \times \widetilde{\nu})(y_1, y_2)\}\end{aligned}$$

$$\begin{aligned}(f \times g)((x_1, x_2) + (y_1, y_2)) &= (f \times g)(x_1 + y_1, x_2 + y_2) \\ &= \max\{f(x_1 + y_1), g(x_2 + y_2)\} \\ &\leq \max\{\max\{f(x_1), f(y_1)\}, \max\{g(x_2), g(y_2)\}\} \\ &= \max\{\max\{f(x_1), g(x_2)\}, \max\{f(y_1), g(y_2)\}\} \\ &= \max\{(f \times g)(x_1, x_2), (f \times g)(y_1, y_2)\}\end{aligned}$$

and

$$\begin{aligned}(\widetilde{\mu} \times \widetilde{\nu})((x_1, x_2)(y_1, y_2)) &= (\widetilde{\mu} \times \widetilde{\nu})(x_1 y_1, x_2 y_2) = \cap\{\widetilde{\mu}(x_1 y_1), \widetilde{\nu}(x_2 y_2)\} \\ &\supseteq \cap\{\widetilde{\mu}(y_1), \widetilde{\nu}(y_2)\} = (\widetilde{\mu} \times \widetilde{\nu})(y_1, y_2).\end{aligned}$$

$$\begin{aligned}(f \times g)((x_1, x_2)(y_1, y_2)) &= (f \times g)(x_1 y_1, x_2 y_2) = \max\{f(x_1 y_1), g(x_2 y_2)\} \\ &\leq \max\{f(y_1), g(y_2)\} = (f \times g)(y_1, y_2).\end{aligned}$$

Hence $A \times B$ is a cubic left ideal of $S \times S$. □

4 Cubic bi-ideals and Cubic quasi-ideals

Definition 4.1. Let $A =< \widetilde{\mu}, f >$ and $B =< \widetilde{\theta}, g >$ be two cubic sets of a semiring S. Define composition of A and B by

$$A\Gamma_c B =< \widetilde{\mu}, f > \Gamma_c < \widetilde{\theta}, g >=< \widetilde{\mu}\Gamma_c\widetilde{\theta}, f\Gamma_c g >$$

where

$$\begin{aligned} \widetilde{\mu}\Gamma_c\widetilde{\theta}(x) &= \underset{x=ab}{\cup}[\cap\{\widetilde{\mu}(a), \widetilde{\theta}(b)\}] \\ &= \widetilde{0}, \text{if x cannot be expressed as above} \end{aligned}$$

and

$$\begin{aligned} f\Gamma_c g(x) &= \inf_{x=ab}\{\max\{f(a), g(b)\}\} \\ &= 1, \text{if x cannot be expressed as above} \end{aligned}$$

for $x, a, b \in S$.

Definition 4.2. Let $A =< \widetilde{\mu}, f >$ and $B =< \widetilde{\theta}, g >$ be two cubic sets of a semiring S. Define generalized composition of A and B by

$$Ao_c B =< \widetilde{\mu}, f > o_c < \widetilde{\theta}, g >=< \widetilde{\mu}o_c\widetilde{\theta}, fo_c g >$$

where

$$\begin{aligned} \widetilde{\mu}o_c\widetilde{\theta}(x) &= \underset{x=\sum_{i=1}^{n} a_i b_i}{\cup}[\bigcap_i\{\cap\{\widetilde{\mu}(a_i), \widetilde{\theta}(b_i)\}\}] \\ &= \widetilde{0}, \text{if x cannot be expressed as above} \end{aligned}$$

and

$$\begin{aligned} fo_c g(x) &= \inf_{x=\sum_{i=1}^{n} a_i b_i}[\max_i \{\max\{f(a_i), g(b_i)\}\}] \\ &= 1, \text{if x cannot be expressed as above} \end{aligned}$$

where $x, a_i, b_i \in S$ and $i = 1, \ldots, n$.

Lemma 4.3. *Let $A =< \widetilde{\mu}_1, f >$, $B =< \widetilde{\mu}_2, g >$ be two cubic ideal of a semiring S. Then $A\Gamma_c B \subseteq Ao_c B \subseteq A \cap B \subseteq A, B$, where $A\Gamma_c B =< \widetilde{\mu}_1\Gamma_c\widetilde{\mu}_2, f\Gamma_c g >$ and $Ao_c B =< \widetilde{\mu}_1 o_c\widetilde{\mu}_2, fo_c g >$.*

Proof. Suppose $A =< \widetilde{\mu}_1, f >$, $B =< \widetilde{\mu}_2, g >$ be two cubic ideal of a semiring

S. Then

$$
\begin{aligned}
(\widetilde{\mu}_1 o_c \widetilde{\mu}_2)(x) &= \cup\{\bigcap_i\{\cap_{x=\sum_{i=1}^{n} a_i b_i} \{\widetilde{\mu}_1(a_i), \widetilde{\mu}_2(b_i)\}\}\} \\
&\quad \text{where } x, a_i, b_i \in S \text{ and } i = 1, ..., n. \\
&\supseteq \cup\{\cap_{x=a_1 b_1}\{\widetilde{\mu}_1(a_1), \widetilde{\mu}_2(b_1))\}\} \\
&\quad \text{where } x, a_1, b_1 \in S \\
&= (\widetilde{\mu}_1 \Gamma_c \widetilde{\mu}_2)(x)
\end{aligned}
$$

$$
\begin{aligned}
(f o_c g)(x) &= \inf\{\max_i\{\max_{x=\sum_{i=1}^{n} a_i b_i}\{f(a_i), g(b_i)\}\}\} \\
&\quad \text{where } x, a_i, b_i \in S \text{ and } i = 1, ..., n. \\
&\leq \inf_{x=a_1 b_1}\{\max\{f(a_1), g(b_1)\}\} \\
&\quad \text{where } x, a_1, b_1 \in S \\
&= (f \Gamma_c g)(x)
\end{aligned}
$$

Therefore $A\Gamma_c B \subseteq A o_c B$)

$$
\begin{aligned}
(\widetilde{\mu}_1 o_c \widetilde{\mu}_2)(x) &= \cup\{\bigcap_i\{\cap_{x=\sum_{i=1}^{n} a_i b_i} \{\widetilde{\mu}_1(a_i), \widetilde{\mu}_2(b_i)\}\}\} \\
&\quad \text{where } x, a_i, b_i \in S \text{ and } i = 1, ..., n. \\
&\subseteq \cup\{\bigcap_i\{\widetilde{\mu}_1(a_i)\}\} \\
&\subseteq \cup_{x=\sum_{i=1}^{n} a_i b_i}\{\cap\{\widetilde{\mu}_1(\sum_{i=1}^{n} a_i b_i)\}\} = \widetilde{\mu}_1(x)
\end{aligned}
$$

$$
\begin{aligned}
(f o_c g)(x) &= \inf\{\max_i{}_{x=\sum_{i=1}^{n} a_i b_i} \{\max\{f(a_i), g(b_i)\}\}\} \\
&\quad \text{where } x, a_i, b_i \in S \text{ and } i = 1, ..., n. \\
&= \inf\{\max_i f(a_i)\} \\
&\geq \inf_{x=\sum_{i=1}^{n} a_i b_i}\{\max \{f(\sum_{i=1}^{n} a_i b_i)\}\} = f(x)
\end{aligned}
$$

Since this is true for every representation of x, $Ao_cB \subseteq A$.
Similarly we can prove that $Ao_cB \subseteq B$.
Therefore $Ao_cB \subseteq A \cap B$. Hence the lemma. □

Definition 4.4. A cubic subset $< \widetilde{\mu}, f >$ of a semiring S is called cubic bi-ideal if for all $x, y, z \in S$ we have
(i) $\widetilde{\mu}(x+y) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\},\ f(x+y) \leq \max\{f(x), f(y)\}$
(ii) $\widetilde{\mu}(xy) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\},\ f(xy) \leq \max\{f(x), f(y)\}$
(iii) $\widetilde{\mu}(xyz) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(z)\},\ f(xyz) \leq \max\{f(x), f(z)\}$

Definition 4.5. A cubic subset $< \widetilde{\mu}, f >$ of a semiring S is called cubic quasi-ideal if for all $x, y \in S$ we have
(i) $\widetilde{\mu}(x+y) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\},\ f(x+y) \leq \max\{f(x), f(y)\}$
(ii) $(\widetilde{\mu} o_c \widetilde{\zeta}_{\chi_S}) \cap (\widetilde{\zeta}_{\chi_S} o_c \widetilde{\mu}) \subseteq \widetilde{\mu},\ (f o_c \eta_{\chi_S}) \cup (\eta_{\chi_S} o_c f) \supseteq f$.

Theorem 4.6. *A cubic subset $< \widetilde{\mu}, f >$ of a semiring S is a cubic left ideal of S if and only if for all x,y∈S, we have*
(i) $\widetilde{\mu}(x+y) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\},\ f(x+y) \leq \max\{f(x), f(y)\}$
(ii) $\widetilde{\zeta}_{\chi_S} o_c \widetilde{\mu} \subseteq \widetilde{\mu},\ \eta_{\chi_S} o_c f \supseteq f$.

Proof. Assume that $< \widetilde{\mu}, f >$ is a cubic left ideal of *S*. Then it is sufficient to show that the condition (ii) is satisfied. Let $x \in S$. If x can be expressed as $x = \sum_{i=1}^{n} a_i b_i$, for $a_i, b_i \in S$ and i=1,...,n, then we have

$$\begin{aligned}
(\widetilde{\zeta}_{\chi_S} o_c \widetilde{\mu})(x) &= \bigcup_{x=\sum_{i=1}^{n} a_i b_i} [\bigcap_i \{\cap \{\widetilde{\zeta}_{\chi_S}(a_i), \widetilde{\mu}(b_i)\}\}] \subseteq \bigcup_{x=\sum_{i=1}^{n} a_i b_i} [\bigcap_i \{\cap \{\widetilde{\mu}(a_i \gamma_i b_i)\}\}] \\
&\subseteq \bigcup_{x=\sum_{i=1}^{n} a_i b_i} [\cap \{\widetilde{\mu}(\sum_{i=1}^{n} a_i b_i)\}] = \widetilde{\mu}(x).
\end{aligned}$$

$$\begin{aligned}
(\eta_{\chi_S} o_c f)(x) &= \inf_{x=\sum_{i=1}^{n} a_i b_i} [\max_i \{\max\{\eta_{\chi_S}(a_i) f(b_i)\}\}] = \inf_{x=\sum_{i=1}^{n} a_i b_i} [\max_i \{\max\{f(b_i\}\}] \\
&\geq \inf_{x=\sum_{i=1}^{n} a_i b_i} [\max_i \{\max\{f(a_i b_i)\}\}] \geq \inf_{x=\sum_{i=1}^{n} a_i b_i} [\max\{f(\sum_{i=1}^{n} a_i b_i)\}] = f(x).
\end{aligned}$$

This implies that $\widetilde{\zeta}_{\chi_S} o_c \widetilde{\mu} \subseteq \widetilde{\mu}$, $\eta_{\chi_S} o_c f \supseteq f$.

Conversely, assume that the given conditions hold. Then it is sufficient to show the second condition of the definition of cubic ideal. Let $x, y \in S$. Then we have

$$\begin{aligned}\widetilde{\mu}(xy) &\supseteq (\widetilde{\zeta}_{\chi_S} o_c \widetilde{\mu})(xy) = \bigcup_{xy=\sum_{i=1}^{n} a_i b_i}[\bigcap_i\{\cap\{\widetilde{\zeta}_{\chi_S}(a_i), \widetilde{\mu}(b_i)\}\}] \\ &\supseteq \widetilde{\mu}(y)(\text{since } \ xy = xy).\end{aligned}$$

$$\begin{aligned}f(xy) &\leq (\eta_{\chi_S} o_c f)(xy) = \inf_{xy=\sum_{i=1}^{n} a_i b_i}[\max_i\{\max\{\eta_{\chi_S}(a_i), f(b_i)\}\}] \\ &\leq f(y)(\text{since } xy = xy).\end{aligned}$$

Hence $< \widetilde{\mu}, f >$ is a cubic left ideal of S. □

Theorem 4.7. *Let $A =< \widetilde{\mu}, f >$ and $B =< \widetilde{\nu}, g >$ be a cubic right ideal and a cubic left ideal of a semiring S, respectively. Then $A \cap B$ is a cubic quasi-ideal of S.*

Proof. Let x, y be any element of S. Then

$$\begin{aligned}(\widetilde{\mu} \cap \widetilde{\nu})(x + y) &= \cap\{\widetilde{\mu}(x + y), \widetilde{\nu}(x + y)\} \\ &\supseteq \cap\{\cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}, \cap\{\widetilde{\nu}(x), \widetilde{\nu}(y)\}\} \\ &= \cap\{\cap\{\widetilde{\mu}(x), \widetilde{\nu}(x)\}, \cap\{\widetilde{\mu}(y), \widetilde{\nu}(y)\}\} \\ &= \cap\{(\widetilde{\mu} \cap \widetilde{\nu})(x), (\widetilde{\mu} \cap \widetilde{\nu})(y)\}.\end{aligned}$$

$$\begin{aligned}(f \cup g)(x + y) &= \max\{f(x + y), g(x + y)\} \\ &\leq \max\{\max\{f(x), f(y)\}, \max\{g(x), g(y)\}\} \\ &= \max\{\max\{f(x), g(x)\}, \cap\{f(y), g(y)\}\} \\ &= \max\{(f \cup g)(x), (f \cup g)(y)\}.\end{aligned}$$

On the other hand, we have

$$((A \cap B)o_c\chi_S) \cap (\chi_S o_c(A \cap B)) \subseteq (Ao_c\chi_S) \cap (\chi_S o_c B) \subseteq (A \cap B).$$

This completes the proof. □

Lemma 4.8. *Any cubic quasi-ideal of S is a cubic bi-ideal of S.*

Proof. Let $< \widetilde{\mu}, f >$ be any cubic quasi-ideal of S. It is sufficient to show that $\widetilde{\mu}(xyz) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(z)\}$, $f(xyz) \leq \max\{f(x), f(z)\}$ and $\widetilde{\mu}(xy) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$, $f(xy) \leq \max\{f(x), f(y)\}$ for all $x, y, z \in S$.

In fact, by the assumption,we have

$$\begin{aligned}
\widetilde{\mu}(xyz) &\supseteq ((\widetilde{\mu} o_c \widetilde{\zeta}_{\chi_S}) \cap (\widetilde{\zeta}_{\chi_S} o_c \widetilde{\mu}))(xyz)\\
&= \cap\{(\widetilde{\mu} o_c \widetilde{\zeta}_{\chi_S})(xyz), (\widetilde{\zeta}_{\chi_S} o_c \widetilde{\mu})(xyz)\}\\
&= \cap\{\underset{xyz=\sum_{i=1}^{n} a_i b_i}{\cup}(\cap(\widetilde{\mu}(a_i), \widetilde{\zeta}_{\chi_S}(b_i))), \cup(\cap(\widetilde{\zeta}_{\chi_S}(a_i), \widetilde{\mu}(b_i))\}\\
&\supseteq \cap\{\cap(\widetilde{\mu}(x), \widetilde{\zeta}_{\chi_S}(z)), \cap(\widetilde{\zeta}_{\chi_S}(x), \widetilde{\mu}(z))\}(\text{since } xyz = xyz)\\
&= \cap\{\widetilde{\mu}(x), \widetilde{\mu}(z)\}
\end{aligned}$$

$$\begin{aligned}
f(xyz) &\leq (f o_c \eta_{\chi_S}) \cup (\eta_{\chi_S} o_c f))(xyz)\\
&= \max\{(f o_c \eta_{\chi_S})(xyz), (\eta_{\chi_S} o_c f)(xyz)\}\\
&= \max\{\underset{xyz=\sum_{i=1}^{n} a_i b_i}{\inf}(\max(f(a_i), \eta_{\chi_S}(b_i))), \inf(\max(\eta_{\chi_S}(a_i), f(b_i)))\}\\
&\leq \max\{\max(\eta_{\chi_S}(x), f(z)), \max(f(x), \eta_{\chi_S}(z))\}(\text{since } xyz = xyz)\\
&= \max\{f(x), f(z)\}
\end{aligned}$$

Similarly, we can show that $\widetilde{\mu}(xy) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$, $f(xy) \leq \max\{f(x), f(y)\}$ for all $x, y \in S$.

□

5 Cubic ideals and regular semirings

In this section, we study the concept of regularity in semiring by using cubic ideal, cubic bi-ideal, cubic quasi-ideal.

Definition 5.1. A semiring S is said to be regular if for each $x \in S$, there exist $a \in S$ and such that $x = xax$.

Theorem 5.2. *Let S be a regular semiring. Then for any cubic right ideal $A =< \widetilde{\mu}, f >$ and any cubic left ideal $B =< \widetilde{\nu}, g >$ of S we have $A\Gamma_c B = A \cap B$.*

Proof. Let S be a regular semiring. By Lemma 4.3, we have $A\Gamma_c B = A \cap B$.
For any $a \in S$, there exists $x \in S$ such that $a = axa$. Then

$$\begin{aligned}
(\widetilde{\mu}\Gamma_c\widetilde{\nu})(a) &= \underset{a=eb}{\cup}\{\cap\{\widetilde{\mu}(e), \widetilde{\nu}(b)\}\} \supseteq \cap\{\widetilde{\mu}(ax), \widetilde{\nu}(a)\}\\
&\supseteq \cap\{\widetilde{\mu}(a), \widetilde{\nu}(a)\} = (\widetilde{\mu} \cap \widetilde{\nu})(a).
\end{aligned}$$

$$\begin{aligned}
(f\Gamma_c g)(a) &= \underset{a=eb}{\inf}\{\max\{f(e), g(b)\}\} \leq \max\{f(ax), g(a)\}\\
&\leq \max\{f(a), g(a)\} = (f \cup g)(a).
\end{aligned}$$

Therefore $(A \cap B) \subseteq (A\Gamma_c B)$.
Hence $A\Gamma_c B = A \cap B$.

□

Corollary 5.3. *If S be a regular semiring, then for any cubic right ideal* $A = < \widetilde{\mu}, f >$ *and any cubic left ideal* $B = < \widetilde{\nu}, g >$ *of S we have* $A o_c B = A \cap B$.

Theorem 5.4. *Let S be a regular semiring. Then*
(i) $A \subseteq A o_c \chi_S o_c A$ *for every cubic bi-ideal* $A = < \widetilde{\mu}, f >$ *of S.*
(ii) $A \subseteq A o_c \chi_S o_c A$ *for every cubic bi-ideal* $A = < \widetilde{\mu}, f >$ *of S.*

Proof. Suppose that $A = < \widetilde{\mu}, f >$ be any cubic bi-ideal of S and x be any element of S. Since S is regular there exists $a \in S$ such that $x = xax$. Now

$$\begin{aligned}
(\widetilde{\mu} o_c \widetilde{\zeta}_{\chi_S} o_c \widetilde{\mu})(x) &= \bigcup_{x=\sum_{i=1}^{n} a_i b_i}(\cap \{(\widetilde{\mu} o_c \widetilde{\zeta}_{\chi_S})(a_i), \widetilde{\mu}(b_i)\}) \\
&\supseteq \cap\{(\widetilde{\mu} o_c \widetilde{\zeta}_{\chi_S})(xa), \widetilde{\mu}(x)\} \\
&= \cap\{\bigcup_{xa=\sum_{i=1}^{n} a_i b_i}(\cap \{\widetilde{\mu}(a_i), (\widetilde{\zeta}_{\chi_S})(b_i)\}), \widetilde{\mu}(x)\}\} \\
&\supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(x)\} (\text{since } xa = xaxa). \\
&= \widetilde{\mu}(x)
\end{aligned}$$

$$\begin{aligned}
(f o_c \eta_{\chi_S} o_c f)(x) &= \inf_{x=\sum_{i=1}^{n} a_i b_i}(\max \{(f o_c \eta_{\chi_S})(a_i), f(b_i)\}) \\
&\leq \max\{(f o_c \eta_{\chi_S})(xa), f(x)\} \\
&= \max\{\inf_{xa=\sum_{i=1}^{n} a_i b_i}(\max \{(f(a_i), \eta_{\chi_S}(b_i))\}), f(x)\} \\
&\leq \max\{f(x), f(x)\} (\text{since } xa = xaxa) \\
&= f(x)
\end{aligned}$$

This implies that $A \subseteq A o_c \chi_S o_c A$.
(i)⇒(ii) This is straight forward from Lemma 4.8 □

Theorem 5.5. *Let S be a regular semiring. Then*
(i) $A \cap B \subseteq A o_c B o_c A$ *for every cubic bi-ideal* $A = < \widetilde{\mu}, f >$ *and every cubic ideal* $B = < \widetilde{\nu}, g >$ *of S.*
(ii) $A \cap B \subseteq A o_c B o_c A$ *for every cubic quasi-ideal* $A = < \widetilde{\mu}, f >$ *and every cubic ideal* $B = < \widetilde{\nu}, g >$ *of S.*

Proof. Suppose S is a regular semiring and $A = < \widetilde{\mu}, f >$, $B = < \widetilde{\nu}, g >$ be any cubic bi-ideal and cubic ideal of S, respectively and x be any element of

S. Since S is regular, there exists $a \in S$ such that $x = xax$.

$$\begin{aligned}
(\widetilde{\mu} o_c \widetilde{\nu} o_c \widetilde{\mu})(x) &= \bigcup_{x=\sum_{i=1}^{n} a_i b_i} \left(\cap \{(\widetilde{\mu} o_c \widetilde{\nu})(a_i), \widetilde{\mu}(b_i)\}\right) \\
&\supseteq \cap\{(\widetilde{\mu} o_c \widetilde{\nu})(xa), \widetilde{\mu}(x)\} \\
&= \cap\{\bigcup_{xa=\sum_{i=1}^{n} a_i b_i} (\cap \{(\widetilde{\mu}(a_i), \widetilde{\nu}(b_i))\}, \widetilde{\mu}(x)\} \\
&\supseteq \cap\{\cap\{\widetilde{\mu}(x), \widetilde{\nu}(axa), \widetilde{\mu}(x)\} (\text{since } xa = xaxa) \\
&\supseteq \cap\{\widetilde{\mu}(x), \widetilde{\nu}(x)\} = (\widetilde{\mu} \cap \widetilde{\nu})(x).
\end{aligned}$$

$$\begin{aligned}
(f o_c g o_c f)(x) &= \inf_{x=\sum_{i=1}^{n} a_i b_i} \left(\max \{(f o_c g)(a_i), f(b_i)\}\right) \\
&\leq \max\{(f o_c g)(xa), f(x)\} \\
&= \max\{\inf_{xa=\sum_{i=1}^{n} a_i b_i} (\max\{(f(a_i), g(b_i))\}), f(x)\} \\
&\leq \max\{\max\{f(x), g(axa), f(x)\} (\text{since } xa = xaxa) \\
&\supseteq \max\{f(x), g(x)\} = (f \cup g)(x).
\end{aligned}$$

(i)⇒(ii) This is straight forward from Lemma 4.8. □

References

[1] V. Chinnadurai, K. Bharathivelan, "Cubic bi-ideals in near-rings", International Journal of computer and Mathematical Sciences, Vol. 5, Issue 2(2016) 44 - 52

[2] V. Chinnadurai, K. Bharathivelan, "Cubic Lateral Ideals in Ternary Near - Rings", International Advanced Research Journal in Science, Engineering and Technology, Vol. 3, Issue11(November 2016) 209 - 215

[3] B. Davvaz, "$(\in, \in \vee q)$-fuzzy subnear-rings and ideals", Soft Computing, 10(3) (2006), 206 - 211

[4] Y.B. Jun, S.T. Jung and M.S. Kim, "Cubic subgroups", Annals of Fuzzy Mathematics and Informatics, 2(2011), 9 - 15.

[5] Y.B. Jun, C.S. Kim and K.O. Yang, "Cubic sets", Annals of Fuzzy Mathematics and Informatics, 4(2012), 83 - 98.

[6] O. Kazanci and S. Yamak, "Generalized fuzzy bi-ideals of semigroups", Soft Comput 12(2008), 1119 - 1124.

[7] A. Khan, Y.B. Jun, S.I.A. Shah, M. Ali, "Characterizations of hemirings in terms of cubic h-ideals", Soft Comput, DOI 10.1007/s00500-014-1396-4

[8] A. Rosenfeld, "Fuzzy groups", J. Math. Anal. Appl. 35(1971) 512-517

[9] H.S. Vandiver, Note on a simple type of algebra in which cancellation law of addition does not hold, Bull. Amer. Math. Soc. 40 (1934) 914 - 920.

[10] L.A.Zadeh, "Fuzzy Sets", Information and Control, 8(1965), 338-353.