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DISCRETE NON-COMMUTATIVE GEL'FAND-NAĬMARK DUALITY

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Abstract

We present, in a simplified setting, a non-commutative version of the well-known Gel'fand-Naĭmark duality (between the categories of compact Hausdorff topological spaces and commutative unital C*-algebras), where "geometric spectra" consist of suitable finite bundles of one-dimensional C*-categories equipped with a transition amplitude structure satisfying saturation conditions. Although this discrete duality actually describes the trivial case of finite-dimensional C*-algebras, the structures are here developed at a level of generality adequate for the formulation of a general topological/uniform Gel'fand-Naĭmark duality, fully addressed in a companion work.

1 Introduction

The celebrated Gel'fand-Naĭmark duality theorem (see for example B.Blackadar [7, II.2.2.4, II.2.2.6]) states that there is a duality between the category of unital $*$-homomorphisms between commutative unital C*-algebras and the category of continuous

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maps between compact Hausdorff topological spaces. It is the standard departure point for understanding the transition from classical to quantum mechanics as well as motivating the development of non-commutative geometry [9].

Several (partially successful) attempts have been made to generalize such a duality to the case of non-commutative C*-algebras with different techniques (we will not attempt here to enter into the reconstruction of the long history of this problem).

It is our purpose (fully addressed in its more general form in the companion work [6]) to provide a non-commutative Gel'fand-Naĭmark duality for unital C*-algebras in terms of "non-commutative spaceoids", which are "suitable families" of one-dimensional saturated Fell-bundles[1], hence vindicating the validity of the *spectral conjecture* put forward in previous papers (for example [5, section 6.1]).

In this introductory work, we present the general "algebraic structural setting" on which the full duality theorem is based, avoiding the highly intricate topological and uniformity conditions that are unavoidable for the statement of the theorem in its full generality.

Our approach to non-commutative Gel'fand-Naĭmark duality builds directly on a long tradition of developments (starting from J.M.G.Fell [14, 15], J.Tomiyama-M.Takesaki [29] and, via the celebrated J.Dauns-K.-H.Hofmann theorem [10, 12], culminating in J.Varela duality [30]) on the spectral analysis of C*-algebras via "bundles" and somehow merges it, via a further spectral analysis of the C*-fibers, with the "convolution of pair-groupoid" description of matrix algebras promoted by A.Connes [9, section I.1].

An equally important source of inspiration for our work comes from the many results (initiated by R.V.Kadison [19] and firmly established by E.Alfsen-F.Schultz [28, 1, 2]) on functional representations of C*-algebras via uniformly continuous functions on generalized spectra consisting of their set of pure states equipped with extra structures: projective Kähler uniform bundles, as in R.Cirelli-A.Manià-L.Pizzocchero [8], or Poisson manifolds with transition probabilities, as in N.P.Landsman [21, 22].

For us, all the differential geometric ingredients are subsumed by a "uniform categorical structure": in detail, when (for each of the C*-fibers mentioned above) a horizontal categorification is performed, substituting the trivial $\mathbb{C}$-bundle over the space of pure states with a (uniform) Fell line-bundle (with transition amplitudes) over the pair groupoid, with objects those pure states, a C*-algebra can be recovered essentially as a "convolution C*-algebra".

This can be seen somehow as a modest but precise mathematical implementation (for unital C*-algebras) of the intriguing intuition (supported by the works by R.Feynman, A.Connes and L.Crane) that quantum physics is a byproduct of "categorical features" (1-arrows between points) of the phase space:

a (uniform) categorical structure on the set of pure states seems responsible for the non-commutative features of the C-algebra of observables*

[1]These can be seen as a generalization of the special "spaceoids" already utilized as "spectra" of full Abelian C*-categories in a previous "horizontal categorification" of Gel'fand-Naĭmark duality [3].

of a quantum system and for the differential geometric structures on its corresponding (classical) phase-space.

After these introductory motivations, we proceed now to summarize the content of the present paper.

In section 2 we quickly introduce some basic terminology and notation on unital C*-algebras and their representations, recalling the classification results for finite-dimensional C*-algebras and their unital $*$-homomorphisms. We also describe in remarks 2.1 and 2.3 some quite important structural ingredients that will be used later on in the definition of the spectral spaceoids (without limiting the discussion to finite-dimensional situations).

Section 3 is dedicated to the first stage of the spectral analysis, namely the discrete base duality. This result (theorem 3.2) is essentially nothing more than Varela duality in the special trivial case of finite-dimensional C*-algebras (hence just an intrinsic reformulation of Wedderburn theorem for finite C*-algebras). Every finite-dimensional C*-algebra is spectrally described by a bundle over a finite discrete set (the unitary equivalence classes of its irreps) whose fibers are finite primitive C*-algebras (i.e. they are isomorphic to matrix algebras). All the notations introduced here (although slightly redundant when only finite-dimensional C*-algebras are around) are perfectly adequate for the spectral study of arbitrary C*-algebras.[2] As in the case of Gel'fand duality, Varela duality is actually also an adjoint duality, see proposition 3.6.

The most interesting part of our spectral analysis, namely the fiber equivalence, is the subject of section 4, where each of the previous fibers (non-canonically isomorphic to a matrix C*-algebra) gets further spectrally analyzed, following A.Connes's idea, as a convolution C*-algebra of a finite pair-groupoid. In order to achieve this step intrinsically, we have to deal with some unavoidable complications, typically a gauge freedom in the choice of "orthonormal frames" producing "unitarily conjugated" matrices for the same operator. The solution that we adopt in our definition 4.9 of "discrete propagator" is to generalize S.Gudder's definition of "transition amplitude space" [18, section 4.5] to situations where the transition amplitudes take values in a (one-dimensional) C*-category and recover each "fiber C*-algebra" of the previous bundle (modulo gauge isomorphisms induced between frames by the transition amplitudes) as an enveloping C*-algebra of a one-dimensional finite-object C*-category (one for each orthonormal frame). Essentially every operator in a fiber C*-algebra corresponds to its "fiber Gel'fand transform" as the collection of all of its matrices (one for every orthonormal frame) related by unitary conjugation.

Here we actually made a practically irrelevant, but conceptually important, oversimplification: also in the case of finite non-commutative C*-algebras, (since the family of pure-states is not discrete with the topology induced by the weak*-uniformity) the correct notion of "propagator" would require the usage in definition 4.9 of "uni-

[2] A non-trivial variant of Varela and Dauns-Hofmann results will be developed in [6], in order to spectrally describe unital C*-algebras as bundles, with primitive C*-algebras (with given irreps) as fibers, when the base space is not Hausdorff.

form Fell bundles" with a non-trivial uniformity on the base pair-groupoid of pure states. In the case of finite-dimensional C*-algebras, we can "bypass" this technical complication, only because the restrictions of the Fell bundle to any orthonormal frame becomes a C*-category with finite objects and no topological/uniformity effect is visible in the reconstruction of the C*-algebra (γ-invariant C*-sections are already uniformly continuous).

The core of the fiber-equivalence is the proof of the adjunction 4.23 among spectrum and section functors between the category $\mathscr{R}$ of unitary equivalence classes of primitive C*-algebras and the category 1-$\mathscr{W}$ of one-dimensional total discrete propagators. The reflective subcategory $\overline{\mathscr{R}}$ (on which the Gel'fand transforms are isomorphisms) consists of primitive C*-algebras that are convolution W*-algebras of a pair groupoid and hence include all matrix algebras. The reflective subcategory $\overline{1\text{-}\mathscr{W}}$ (on which the evaluation transforms are isomorphisms) is characterized by a saturation condition (see the discussion in remark 4.25) for the set of frames of each transition amplitude space, that essentially states that the group of unitaries of a certain Hilbert space is acting effectively and transitively on frames.

A "discrete" spectral counterpart of a non-commutative unital C*-algebra, a *discrete non-commutative spaceoid*, is introduced in definition 4.26 as a finite bundle of total discrete propagators. A global fiber-adjunction, between the category $\mathscr{B}_{FD}$ of finite bundles (of irrep-classes) of primitive finite C*-algebras and the category $\mathscr{E}_{FD}$ of discrete spaceoids (both with fiberwise isomorphisms), is stated in theorem 4.30. Imposing fiber by fiber the condition of saturation identifies the reflective subcategory $\overline{\mathscr{E}}_{FD}$ of discrete non-commutative spaceoids in duality with finite C*-algebras.

The main result of the paper is the final "discrete" adjoint duality theorem 4.33 that is obtained by composing the discrete base adjoint duality 3.6 with the discrete fiber adjoint equivalence 4.30.

In the short section 5, we explain how the discrete adjoint duality presented here "extends" the usual commutative Gel'fand-Naĭmark adjoint duality for finite commutative C*-algebras. Furthermore we describe how discrete non-commutative spaceoids (as defined in this work) are related to the topological spaceoids originally introduced as spectra of commutative full C*-categories in [3] (a work that actually constitutes the ideological precursor of the present non-commutative developments).

The final section 6 ventures beyond the discrete case, describing the main steps and obstacles (of topological/uniform nature) encountered in the formulation of the general non-commutative Gel'fand-Naĭmark duality. Specifically, we briefly explain how our techniques are immediately capable of supporting a duality at least for unital C*-algebras $\mathcal{A}$ whose structure space $\widehat{\mathcal{A}}$ is Hausdorff and whose irreducible representations are all finite-dimensional.[3] Then we briefly discuss the topological and uniformity difficulties in the treatment of the fiber equivalence for a general unital

[3] In this Hausdorff situation, for the base duality, Varela result essentially reduces to previous theorems by J.M.G.Fell [14, 15] and the spectra ("scaled" Banach C*-bundles) have been already characterized by A.J.Lazar [23]; for the fiber-equivalence step, since all the fibers are isomorphic to matrix algebras, our present analysis does not need any further topological improvement.

C*-algebra, and the demanding hacks on Varela duality in order to preserve primitive C*-algebras as fibers also in the case of structure spaces that are not Hausdorff. These points will be fully covered in the companion paper [6], where each of the algebraic steps presented here will be suitably "uniformized/topologized".

The complete result reduces[4] to the "discrete case" presented here whenever we limit consideration to unital C*-algebras $\mathcal{A}$ whose structure space $\widehat{\mathcal{A}}$ is topologically discrete and compact (hence a finite set) and whose irreducible representations are all finite-dimensional. The potential reader should not feel disappointed by the fact that, under these quite restrictive conditions, the general non-commutative duality theorem actually collapses to a duality for the completely trivial case of finite-dimensional C*-algebras (for which a perfectly satisfactory classification is well-known as a consequence of Wedderbun theorem): the ingredients here developed are truly capable of supporting a satisfactory non-commutative C*-duality (modulo the introduction of topology and uniformity on the spaceoids). All in all, the time spent on this simplified formulation is indeed rewarding.

For technical purposes, we also limit for now the study to the category of unital $*$-homomorphisms between unital C*-algebras that preserve irreducible representations. This is a class of morphisms that (although quite restrictive in general) already subsumes all the unital $*$-homomorphisms between commutative unital C*-algebras and hence it is sufficient to successfully recover the usual commutative Gel'fand-Naĭmark duality. Again, for the doubtful reader, we anticipate that this technical restriction on morphisms is not crucial for the theorem: as soon as suitable "propagator-bimodules" between non-commutative spaceoids are introduced, one can spectrally treat any unital $*$-homomorphism between unital C*-algebras as well. This is not done here just to avoid a long distracting diversion on Morita theory for C*-categories.

2 Preliminaries on (finite-dimensional) C*-algebras

We recall here basic definitions, terminology, notation and some preliminary results. More details are found, for example in B.Blackadar reference book [7, chapters 2-3].

In this work we will consider only unital C*-algebras over the field of complex numbers $\mathbb{C}$. A complex C*-algebra $\mathcal{A}$ is an associative involutive algebra over $\mathbb{C}$ that is also a Banach space with a norm that is sub-multiplicative: $\|xy\| \leq \|x\| \cdot \|y\|$ for $x, y \in \mathcal{A}$, and satisfies the C*-property: $\|x^*x\| = \|x\|^2$, for $x \in \mathcal{A}$. A C*-algebra is unital if it has a multiplicative identity $1_{\mathcal{A}}$ and in this case we also assume the normalization property: $\|1_{\mathcal{A}}\| = 1$. The cone of positive elements is denoted by $\mathcal{A}_+ := \{x^*x \mid x \in \mathcal{A}\}$. A W*-algebra is a C*-algebra that, as a Banach space, is the dual of a Banach space. A finite-dimensional C*-algebra is a C*-algebra that, as a vector space, has finite dimension.

[4]The exact form of such identification will be explained in [6]: it involves an inclusion as well as a forgetful functor.

A $*$-homomorphism between C*-algebras is a function $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ such that $\phi(xy) = \phi(x)\phi(y)$ and $\phi(x^*) = \phi(x)^*$, for all $x, y \in \mathcal{A}_1$. A unital $*$-homomorphism further satisfies: $\phi(1_{\mathcal{A}_1}) = 1_{\mathcal{A}_2}$.

A representation of a (unital) C*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a (unital) $*$-homomorphism $\mathcal{A} \xrightarrow{\varpi} \mathcal{B}(\mathcal{H})$ into the unital C*-algebra $\mathcal{B}(\mathcal{H})$ of continuous linear maps on $\mathcal{H}$. The representation is irreducible if its commutant

$$\varpi(\mathcal{A})' := \{T \in \mathcal{B}(\mathcal{H}) \mid \forall x \in \mathcal{A} \ : \ T \circ \varpi(x) = \varpi(x) \circ T\}$$

equals $\mathbb{C} \cdot \mathrm{Id}_{\mathcal{H}}$. Two representations $\varpi_1 : \mathcal{A} \to \mathcal{B}(\mathcal{H}_1)$ and $\varpi_2 : \mathcal{A}_2 \to \mathcal{B}(\mathcal{H}_2)$ are unitarily equivalent it there exists a unitary intertwining operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $U \circ \varpi_1(x) = \varpi_2(x) \circ U$, for all $x \in \mathcal{A}$. The set of unitary equivalence classes of irreducible representations of $\mathcal{A}$ will be denoted by $\mathcal{X}_{\mathcal{A}}$.

A unital $*$-homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ is **irrep-preserving** if for every irreducible representation ϖ of $\mathcal{B}$, also $\varpi \circ \phi$ is an irreducible representation of $\mathcal{A}$.

An ideal $\mathcal{I}$ in a unital C*-algebra $\mathcal{A}$ is a primitive ideal if there is at least one irreducible representation $\varpi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that $\mathcal{I} = \mathrm{Ker}\, \varpi$. If $\{0_{\mathcal{A}}\}$ is a primitive ideal, $\mathcal{A}$ is a **primitive C*-algebra**.[5]

A state over the unital C*-algebra $\mathcal{A}$ is linear function $\omega : \mathcal{A} \to \mathbb{C}$ that is positive, $\omega(x^* x) \geq 1_{\mathbb{C}}$, normalized, $\omega(1_{\mathcal{A}}) = 1_{\mathbb{C}}$. The family of states of $\mathcal{A}$ is denoted by $\mathcal{S}_{\mathcal{A}}$.

The well-known Gel'fand-Naĭmark-Segal GNS-representation theorem (see for example B.Blackadar [7, II.6.4]) says that every state ω over a unital C*-algebra $\mathcal{A}$ induces a representation $\varpi_\omega : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\omega)$ on a Hilbert space $\mathcal{H}_\omega$ with a cyclic[6] vector $\xi_\omega \in \mathcal{H}_\omega$ such that $\omega(x) = \langle \xi_\omega \mid \varpi_\omega(x)\xi_\omega \rangle_{\mathcal{H}_\omega}$, for all $x \in \mathcal{A}$.

A pure state is a state ω whose GNS-representation ϖ_ω is irreducible. The family of pure states of the C*-algebra $\mathcal{A}$ is denoted by $\mathcal{P}_{\mathcal{A}} \subset \mathcal{S}_{\mathcal{A}}$.

Composing the GNS-map $\omega \mapsto \varpi_\omega$ restricted to the set of pure states $\mathcal{P}_{\mathcal{A}}$ with the quotient map $\pi \mapsto [\varpi]$ onto $\mathcal{X}_{\mathcal{A}}$, we have a quotient map $\chi_{\mathcal{A}} : \mathcal{P}_{\mathcal{A}} \to \mathcal{X}_{\mathcal{A}}$ given by $\chi_{\mathcal{A}} : \omega \to [\varpi_\omega]$.

Remark 2.1. We recall that, given two GNS-representations that are unitarily equivalent $[\varpi_\omega] = [\varpi_\rho]$, a unitary intertwining operator $U : \mathcal{H}_\rho \to \mathcal{H}_\omega$, i.e. an operator such that $U \circ \varpi_\rho(x) = \varpi_\omega(x) \circ U$, for all $x \in \mathcal{A}$, is not necessarily unique. For pure states ω, ρ, since their GNS-representations $\varpi_\omega, \varpi_\rho$ are irreducible, any two such intertwining unitaries $U, V \in \mathcal{B}(\mathcal{H}_\rho; \mathcal{H}_\omega)$ must satisfy $V^* \circ U \in \varpi_\rho(\mathcal{A})' = \mathbb{C} \cdot \mathrm{Id}_{\mathcal{H}_\rho}$ and $U \circ V^* \in \varpi_\omega(\mathcal{A})' = \mathbb{C} \cdot \mathrm{Id}_{\mathcal{H}_\omega}$ and hence there is a unique spatial unital $*$-isomorphism $\alpha_{\omega\rho} : \varpi_\rho(\mathcal{A})'' \to \varpi_\omega(\mathcal{A})''$ of von Neumann algebras,[7] independent from the choice of intertwiners, given by $\alpha_{\omega\rho}(T) := \mathrm{Ad}_U(T) := U \circ T \circ U^* = \mathrm{Ad}_V(T)$.

[5] By first isomorphism theorem, this is equivalent to $\mathcal{A}$ having a faithful irreducible representation.

[6] The vector $\xi_\omega \in \mathcal{H}_\omega$ is cyclic for the representation $\varpi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\omega)$ if and only if $\{\varpi_\omega(x)(\xi_\omega) \mid x \in \mathcal{A}\}$ is dense in $\mathcal{H}_\omega$.

[7] A von Neumann algebra is a C*-algebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ such that $\mathcal{A} = \mathcal{A}''$ or equivalently, by von Neumann bicommutant theorem [7, I.9.1.1], that is closed in the weak-operator topology.

Since $\alpha_{\omega\rho}\circ\alpha_{\rho\zeta}=\alpha_{\omega\zeta}$, $\alpha_{\omega\rho}^{-1}=\alpha_{\rho\omega}$ and $\alpha_{\omega\omega}=\mathrm{Id}_{\varpi_\omega(\mathcal{A})''}$, for $o\in\mathcal{X}_\mathcal{A}$, for all $\omega,\rho,\zeta\in\chi^{-1}(o)\subset\mathcal{P}_\mathcal{A}$, with the terminology recalled later in definition 4.1 and in footnotes 20 and 33, α is a $*$-functor from the disjoint union of the pair groupoids with objects pure states in $\chi^{-1}(o)$, with values in the groupoid of spatial $*$-isomorphisms of von Neumann algebras. For all $o\in\mathcal{X}_\mathcal{A}$, we can define an intrinsic W*-algebra $\mathcal{A}''_o$ of "orbits" of the pair groupoid $\{\alpha_{\omega\rho}\mid\omega,\rho\in\chi^{-1}(o)\}$ acting on the object von Neumann algebras $\{\varpi_\omega(\mathcal{A})''\mid\omega\in\chi^{-1}(o)\}$ as:

$$\mathcal{A}''_o:=\left\{(T_\omega)_o\mid T_\omega=\alpha_{\omega\rho}(T_\rho),\ \forall\omega,\rho\in\chi^{-1}(o),\ T_\omega\in\varpi_\omega(\mathcal{A})'',\ \forall\omega\in\chi^{-1}(o)\right\},$$

with operations well-defined by

$$(T_\omega)_o^*:=(T_\omega^*)_o,\quad (T_\omega)_o\cdot(S_\omega)_o:=(T_\omega\cdot S_\omega)_o,\quad (T_\omega)_o+(S_\omega)_o:=(T_\omega+S_\omega)_o.$$

For finite-dimensional C*-algebras, $\mathcal{A}''_o$, for $o\in\mathcal{X}_o$, will always be isomorphic to a type I_n factor (matrix algebra), for a certain $n\in\mathbb{N}_0$.

For $o\in\mathcal{X}_\mathcal{A}$, for every $\omega\in\chi^{-1}(o)$, we define $|\,\omega\,\rangle\langle\,\omega\,|:=(|\,\xi_\omega\,\rangle\langle\,\xi_\omega\,|)_o$, the one-dimensional projector $|\,\omega\,\rangle\langle\,\omega\,|\in\mathcal{A}''_o$ that is the "α-orbit" of the one-dimensional projector $|\,\xi_\omega\,\rangle\langle\,\xi_\omega\,|\in\mathcal{B}(\mathcal{H}_\omega)$, onto the one-dimensional subspace generated by ξ_ω, the cyclic vector for the ω-GNS representation ϖ_ω.

Since for all $\omega,\rho\in\mathcal{P}_\mathcal{A}$ such that $[\varpi_\omega]=o=[\varpi_\rho]\in\mathcal{X}_\mathcal{A}$, $\mathrm{Ker}\,\varpi_\omega=\mathrm{Ker}\,\varpi_\rho$ we have a well-defined map $o\mapsto\mathrm{Ker}\,\varpi_o:=\mathrm{Ker}\,\varpi_\omega$, from $\mathcal{X}_\mathcal{A}$ to $\mathrm{Prim}(\mathcal{A})$, the family of primitive ideals of $\mathcal{A}$.

We will also denote by $\varpi_o:\mathcal{A}\to\mathcal{A}''_o$ the unital $*$-homomorphism $x\mapsto(\varpi_\omega(x))_o$, for $x\in\mathcal{A}$, and by $\overline{\varpi}_o:\mathcal{A}_o:=\mathcal{A}/\mathrm{Ker}\,\varpi_o\to\mathcal{A}''_o$ its induced injective unital $*$-homomorphism $x+\mathrm{Ker}\,\varpi_o\mapsto\varpi_o(x)$.

We have a bundle over $\mathcal{X}_\mathcal{A}$ with fibers $\mathcal{X}_{\mathcal{A}_o}$, $o\in\mathcal{X}_\mathcal{A}$, with a canonical section $\kappa_\mathcal{A}:\mathcal{X}_\mathcal{A}\to\biguplus_{o\in\mathcal{X}_\mathcal{A}}\mathcal{X}_{\mathcal{A}_0}$ given by $\kappa_\mathcal{A}:[\varpi_\omega]=:o\mapsto\overline{o}:=[\overline{\varpi}_\omega]\in\mathcal{X}_{\mathcal{A}_o}$. In the finite-dimensional case, since $\mathcal{A}_o$ is isomorphic to a matrix algebra, all the fibers $\mathcal{X}_{\mathcal{A}_o}$ are singletons. ⌟

We recall the following well-known classification theorem for complex unital finite-dimensional C*-algebras (see for example D.Farenick [13, theorem 5.20] or K.Davidson [11, theorem III.1.1]) that is just a special case of Wedderburn's theorem for semi-simple associative algebras.

Theorem 2.2. *Given a finite-dimensional complex C*-algebra $\mathcal{A}$, there exist $N\in\mathbb{N}_0$ and a unique finite sequence $(n_1,\dots,n_N)$ of non-decreasing strictly positive natural numbers such that $\mathcal{A}$ is isomorphic to the direct sum $\bigoplus_{k=1}^N\mathbb{M}_{n_k}(\mathbb{C})$ of complex matrix C*-algebras. It follows that every finite-dimensional (non-trivial) C*-algebra is unital.*

An intrinsic operator-algebraic proof of this theorem is a byproduct of the following remark:

Remark 2.3. The celebrated Gel'fand-Naĭmark representation theorem (see for example B.Blackadar [7, II.6.4.10]) asserts that every (unital) C*-algebra admits an isometric representation. More specifically, if $\mathcal{A}$ is a unital C*-algebra, the map

$$\varpi := \bigoplus_{o\in\mathcal{X}_\mathcal{A}} \varpi_o : \mathcal{A} \to \bigoplus_{o\in\mathcal{X}_\mathcal{A}} \mathcal{A}_o''$$

is an isometric $*$-homomorphism because, making use of [7, corollary II.6.4.9], for every $x \in \mathcal{A}$, there exists an irreducible representation ϖ_x, and hence $[\varpi_x] \in \mathcal{X}_\mathcal{A}$, such that $\|x\| = \|\varpi_x(x)\| = \|\varpi_{[\varpi_x]}(x)\|$. As a consequence, the unital $*$-homomorphism $x \mapsto \varpi(x) := \bigoplus_{o\in\mathcal{X}_\mathcal{A}} \varpi_o(x) \subset \bigoplus_{o\in\mathcal{X}_\mathcal{A}} \mathcal{A}_o''$ induces an isomorphism of C*-algebras $\mathcal{A} \simeq \varpi(\mathcal{A})$.

Since, the W*-algebras $\mathcal{A}_o''$, for $o \in \mathcal{X}_\mathcal{A}$, are type I factors (they are all isomorphic to $\mathcal{B}(\mathcal{H}_\rho)$, whenever $[\varpi_\rho] = o$), they admit a unique trace $\mathrm{Tr}_o : \mathcal{A}_o'' \to \mathbb{C}$ and they act (reducibly) on the canonical Hilbert-Schmidt space

$$L^2(\mathcal{A}_o'') := \{T \in \mathcal{A}_o'' \mid \mathrm{Tr}(T^*T) < +\infty\}.$$

With a slight abuse of notation, denoting by $\varpi : \mathcal{A} \to \mathcal{B}\left(\bigoplus_{o\in\mathcal{X}_\mathcal{A}} L^2(\mathcal{A}_o'')\right)$ the induced representation of the C*-algebra $\mathcal{A}$ on the Hilbert space $\bigoplus_{o\in\mathcal{X}_\mathcal{A}} L^2(\mathcal{A}_o'')$, we obtain $\varpi(\mathcal{A})'' = \bigoplus_{o\in\mathcal{X}_\mathcal{A}} \mathcal{A}_o''$.

Since by von Neumann bicommutant theorem [7, I.1.9.1] $\varpi(\mathcal{A})$ is weakly dense in $\varpi(\mathcal{A})''$, in the finite-dimensional case we have $\mathcal{A} \simeq \varpi(\mathcal{A}) = \varpi(\mathcal{A})'' = \bigoplus_{o\in\mathcal{X}_\mathcal{A}} \mathcal{A}_o''$ and hence $\mathcal{X}_\mathcal{A}$ must be finite. ⌟

There is also a characterization, via Bratteli diagrams (up to the previous isomorphisms with direct sums of matrix algebras), of all the unital $*$-homomorphisms between finite-dimensional C*-algebras (see K.Davidson [11, lemma III.2.1]).

Proposition 2.4. *There is a bijective correspondence between the family of equivalence classes under inner automorphisms*[8] *of unital $*$-homomorphisms of finite-dimensional C*-algebras $\phi : \mathcal{A} \to \mathcal{B}$, where $\mathcal{A} \simeq \bigoplus_{j=1}^N \mathbb{M}_{n_j}(\mathbb{C})$ and $\mathcal{B} \simeq \bigoplus_{k=1}^M \mathbb{M}_{m_k}(\mathbb{C})$, for $(m_k)_{k=1}^M, (n_j)_{j-1}^N$ non-decreasing finite sequences, and the family of multiplicity matrices $[\phi_{kj}] \in \mathbb{M}_{M\times N}(\mathbb{C})$ of non-negative integers such that $m_k = \sum_{j=1}^N \phi_{kj} \cdot n_j$, for all $j = 1, \dots, M$. Irrep-preserving unital $*$-homomorphisms correspond to multiplicity matrices with rows containing a single non-zero entry equal to 1.*

Both of the previous results can actually be intrinsically reformulated into the subsequent duality result 3.2 that is an immediate byproduct (in the case of finite-dimensional C*-algebras and irrep-preserving $*$-homomorphisms) of Varela duality for arbitrary C*-algebras [30], itself an application of the celebrated Dauns-Hofmann theorems [10, 12].

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[8] An **automorphism** (i.e. an invertible unital $*$-homomorphism) $\phi : \mathcal{A} \to \mathcal{A}$ of a unital C*-algebra $\mathcal{A}$ is **inner** if there exists $u \in \mathcal{A}$ unitary (i.e. $u^*u = 1_\mathcal{A} = uu^*$) such that $\phi(x) = u^*xu$, for all $x \in \mathcal{A}$.

3 Discrete Base Duality

For all the basic notions of category theory used in this paper: category, functor, natural transformation, adjunction, equivalence, we refer to any standard text (for example T.Leinster [24] or E.Riehl [27]).

We denote by $\mathscr{A}_{FD}$ the **category of irrep-preserving unital $*$-homomorphisms of finite-dimensional C*-algebras**, whose objects are finite-dimensional (non-trivial) C*-algebras and whose morphisms are the unital $*$-homomorphisms that are irrep-preserving (composition being the usual composition of functions and identities morphisms the usual identity $*$-isomorphisms).

We denote by $\mathscr{B}_{FD}$ the **category of fibrewise $*$-isomorphisms of bundles of finite-dimensional primitive C*-algebras over finite sets** defined as follows:

- objects of $\mathscr{B}_{FD}$ are bundles $\mathcal{F} \xrightarrow{\theta} \mathcal{X}$ with a finite set $\mathcal{X}$ as base space and fibers $\mathcal{F}_o := \theta^{-1}(o)$, for $o \in \mathcal{X}$, that are primitive finite-dimensional C*-algebras (hence isomorphic to matrix algebras).

- morphisms of $\mathscr{B}_{FD}$ consist of pairs of maps $(\mathcal{F}^1, \theta^1, \mathcal{X}^1) \xrightarrow{(\lambda,\Lambda)} (\mathcal{F}^2, \theta^2, \mathcal{X}^2)$, where $\lambda : \mathcal{X}^1 \to \mathcal{X}^2$ and $\Lambda : \lambda^\bullet(\mathcal{F}^2) \to \mathcal{F}^1$ is a fiber-preserving map, defined on the total space of the λ-pull-back[9] $(\lambda^\bullet(\mathcal{F}^2), (\theta^2)_\lambda, \mathcal{X}^1)$ of $(\mathcal{F}^2, \theta^2, \mathcal{X}^2)$, such that, for all $o \in \mathcal{X}^1$, its o-fiber restriction $\Lambda_o : \mathcal{F}^2_{\lambda(o)} \to \mathcal{F}^1_o$ is a $*$-isomorphism.[10]

- given two morphisms $(\mathcal{F}^1, \theta^1, \mathcal{X}^1) \xrightarrow{(\lambda_2,\Lambda_2)} (\mathcal{F}^2, \theta^2, \mathcal{X}^2) \xrightarrow{(\lambda_1,\Lambda_1)} (\mathcal{F}^3, \theta^3, \mathcal{X}^3)$, their composition is defined by $(\lambda_1, \Lambda_1) \circ (\lambda_2, \Lambda_2) := (\lambda_1 \circ \lambda_2, \Lambda_2 \circ \lambda_2^\bullet(\Lambda_1) \circ \zeta^{\theta^3}_{\lambda_1,\lambda_2})$, where $\zeta^{\theta^3}_{\lambda_1,\lambda_2} : (\lambda_1 \circ \lambda_2)^\bullet(\mathcal{F}^3) \to \lambda_2^\bullet \circ \lambda_1^\bullet(\mathcal{F}^3)$ is the usual canonical isomorphism of pull-backs and $\lambda_2^\bullet(\Lambda_1) : \lambda_2^\bullet \circ \lambda_1^\bullet(\mathcal{F}^3) \to \lambda_2^\bullet(\mathcal{F}^2)$ is the λ_2-pull-back of the morphism $\Lambda_1 : \lambda_1^\bullet(\mathcal{F}^3) \to \mathcal{F}^2$ of bundles over $\mathcal{X}^2$.

- identity morphisms, for every $(\mathcal{F}, \theta, \mathcal{X})$, are given by $\iota(\theta) := (\mathrm{Id}_\mathcal{X}, \zeta_\mathcal{F})$, where $\zeta_\mathcal{F} : \mathrm{Id}^\bullet_\mathcal{X}(\mathcal{F}) \to \mathcal{F}$ is the canonical isomorphism of $\mathcal{F}$ with its $\mathrm{Id}_\mathcal{X}$-pull-back.

[9] A λ-pull-back of the bundle $(\mathcal{F}^2, \theta^2, \mathcal{X}^2)$, with $\lambda : \mathcal{X}^1 \to \mathcal{X}^2$, is by definition a commuting square $\lambda \circ \theta = \theta^2 \circ \Lambda$, where $(\mathcal{F}, \theta, \mathcal{X}^1)$ is a bundle over $\mathcal{X}^1$ and $\Lambda : \mathcal{F} \to \mathcal{F}^2$ is a fibrewise morphism, such that the following universal factorization property is satisfied: for any other such commuting square $\lambda \circ \theta' = \theta^2 \circ \Lambda'$, for a bundle $(\mathcal{F}', \theta', \mathcal{X}^1)$ and a fibrewise morphism $\Lambda' : \mathcal{F}' \to \mathcal{F}^2$, there exists a unique fibrewise morphism $\Theta : \mathcal{F}' \to \mathcal{F}$ such that $\theta \circ \Theta = \theta'$ and $\Lambda \circ \Theta = \Lambda'$.

The **standard λ-pull-back** $\lambda^\bullet(\theta^2)$ of the bundle θ^2 is here the bundle $(\lambda^\bullet(\mathcal{F}^2), (\theta^2)_\lambda, \mathcal{X}^1)$ having total space $\lambda^\bullet(\mathcal{F}^2) := \bigcup_{o \in \mathcal{X}^1} \mathcal{F}^2_{\lambda(o)} \times \{o\}$, with fibers $\mathcal{F}^2_{\lambda(o)} \times \{o\}$, for $o \in \mathcal{X}^1$, and with fibrewise morphism $\lambda^{(\theta^2)} : \lambda^\bullet(\mathcal{F}^2) \to \mathcal{F}^2$ given by $\lambda^{(\theta^2)}_o : (f, o) \mapsto f$, for all $f \in \mathcal{F}^2_{\lambda(o)}$, for $o \in \mathcal{X}^1$; hence we have the commuting square $\lambda^{(\theta^2)} \circ \theta^2 = \lambda \circ (\theta^2)_\lambda$.

We make free use of the fact the standard λ-pull-back gives a covariant functor $\lambda^\bullet$ from the category of fibrewise morphisms of bundles over $\mathcal{X}^2$ to the category of fibrewise morphisms of bundles over $\mathcal{X}^1$.

[10] Although in the specific case of fibrewise isomorphisms the two notions are indistinguishable, in view of subsequent generalizations and direct continuity with previous works, we prefer to use here this notion of *geometric morphism* between bundles (a term motivated by the similarity with the situation in topos theory), rather than the more familiar definition of morphism via commuting squares i.e. pairs (F, f) with $\theta^2 \circ F = \theta^1 \circ f$.

Remark 3.1. Given a bundle $(\mathcal{F}, \theta, \mathcal{X}) \in \mathscr{B}^0_{FD}$, we can define the bundle $(\mathcal{X}_\theta, [\theta], \mathcal{X})$ whose o-fibers $(\mathcal{X}_\theta)_o := [\theta]^{-1}(o)$, for all $o \in \mathcal{X}$, are the equivalence classes $\mathcal{X}_{\mathcal{F}_o}$ of irreducible representations of $\mathcal{F}_o$. Since the fiber $\mathcal{F}_o$ is a primitive finite-dimensional C*-algebra, we have a unique equivalence class of irreducible representations of $\mathcal{F}_o$ and hence a canonical bijection $\kappa_\theta : \mathcal{X} \to \mathcal{X}_\theta$. ⌟

We now define a contravariant **base section functor** $\underline{\Gamma}^{FD} : \mathscr{B}_{FD} \to \mathscr{A}_{FD}$:

- to every object $(\mathcal{F}, \theta, \mathcal{X})$ in $\mathscr{B}_{FD}$ we associate the family

  $$\underline{\Gamma}^{FD}(\theta) := \{\sigma : \mathcal{X} \to \mathcal{F} \mid \theta \circ \sigma = \mathrm{Id}_\mathcal{X}\}$$

  of sections of the bundle θ and we note that $\underline{\Gamma}^{FD}(\theta)$ is a finite-dimensional C*-algebra with the following supremum norm and pointwise-defined operations, for all $o \in \mathcal{X}$, $\alpha \in \mathbb{C}$, $\sigma, \tau \in \underline{\Gamma}^{FD}(\theta)$:

  $$(\sigma + \tau)_o := \sigma_o +_{\mathcal{F}_o} \tau_o, \quad (\alpha \cdot \sigma)_o := \alpha \cdot_{\mathcal{F}_o} \sigma_o, \quad (\sigma^*)_o := (\sigma_o)^{*_{\mathcal{F}_o}}$$
  $$(\sigma \bullet \tau)_o := \sigma_o \bullet_{\mathcal{F}_o} \tau_o, \quad \|\sigma\| := \max_{o \in \mathcal{X}} \|\sigma_o\|_{\mathcal{F}_o}.$$

  The C*-algebra $\underline{\Gamma}^{FD}(\theta)$ is naturally isomorphic to $\bigoplus_{o \in \mathcal{X}} \mathcal{F}_o$ and, for all $o \in \mathcal{X}$, $\mathcal{F}_o$ is (non canonically) isomorphic to a unique matrix algebra.

- to a morphism $(\mathcal{F}^1, \theta^1, \mathcal{X}^1) \xrightarrow{(\lambda,\Lambda)} (\mathcal{F}^2, \theta^2, \mathcal{X}^2)$ in $\mathscr{B}_{FD}$ we associate the map of finite-dimensional C*-algebras $\underline{\Gamma}^{FD}_{(\lambda,\Lambda)} : \underline{\Gamma}^{FD}(\theta^2) \to \underline{\Gamma}^{FD}(\theta^1)$ defined by

  $$\underline{\Gamma}^{FD}_{(\lambda,\Lambda)}(\sigma) := \Lambda \circ \lambda^\bullet(\sigma) \circ (\mathrm{Id}_{\mathcal{X}^2})^{-1}_\lambda,$$

  for all $\sigma \in \underline{\Gamma}^{FD}(\theta^2)$, where $\lambda^\bullet(\sigma) : \lambda^\bullet(\mathcal{X}^2) \to \lambda^\bullet(\mathcal{F}^2)$ is the λ-pull-back of the section $\sigma : \mathcal{X}^2 \to \mathcal{F}^2$ and $(\mathrm{Id}_{\mathcal{X}^2})_\lambda : \lambda^\bullet(\mathcal{X}^2) \to \mathcal{X}^1$ is a homeomorphism (as total spaces of bundles over $\mathcal{X}^1$).

  A straightforward calculation shows that $\underline{\Gamma}^{FD}_{(\lambda,\Lambda)} : \underline{\Gamma}^{FD}(\theta^2) \to \underline{\Gamma}^{FD}(\theta^1)$ is a unital $*$-homomorphism of finite-dimensional C*-algebras and is irrep-preserving.[11]

A direct calculation gives the contravariant functoriality of $\underline{\Gamma}^{FD}$:

$$\underline{\Gamma}^{FD}_{\iota(\theta)} = \underline{\Gamma}^{FD}_{(\mathrm{Id}_\mathcal{X}, \eta_\mathcal{F})} = \mathrm{Id}_{\underline{\Gamma}^{FD}(\theta)} \quad \text{and} \quad \underline{\Gamma}^{FD}_{(\lambda_1,\Lambda_1)\circ(\lambda_2,\Lambda_2)} = \underline{\Gamma}^{FD}_{(\lambda_2,\Lambda_2)} \circ \underline{\Gamma}^{FD}_{(\lambda_1,\Lambda_1)}.$$

We introduce now a contravariant **base spectrum functor** $\underline{\Sigma}^{FD} : \mathscr{A}_{FD} \to \mathscr{B}_{FD}$:

- to a finite-dimensional C*-algebra $\mathcal{A}$ in $\mathscr{A}_{FD}$, we associate its spectral bundle of finite-dimensional primitive C*-algebras $\underline{\Sigma}^{FD}(\mathcal{A}) := (\mathcal{F}_\mathcal{A}, \theta_\mathcal{A}, \mathcal{X}_\mathcal{A})$, where:

[11] Since $\underline{\Gamma}^{FD}(\theta^1) = \bigoplus_{o\in\mathcal{X}} \mathcal{F}^1_o$, an irreducible representation $\pi : \underline{\Gamma}^{FD}(\theta^1) \to \mathcal{B}(\mathcal{H})$ provides an isomorphism between the simple finite-dimensional algebra $\mathcal{B}(\mathcal{H})$ and a unique fiber $\mathcal{F}^1_{o_\pi}$ and hence $\pi \circ \underline{\Gamma}^{FD}_{(\lambda,\Lambda)}$ is an isomorphism between $\mathcal{B}(\mathcal{H})$ and $\mathcal{F}^2_{\lambda(o_\pi)}$.

  - $\mathcal{X}_{\mathcal{A}}$ is the set[12] of unitary equivalence classes of irreducible representations of $\mathcal{A}$
  - $\mathcal{F}_{\mathcal{A}} := \biguplus_{[\varpi]\in\mathcal{X}_{\mathcal{A}}} \frac{\mathcal{A}}{\operatorname{Ker}\varpi}$ is the disjoint union of finite-dimensional primitive C*-algebras of the form $(\mathcal{F}_{\mathcal{A}})_o := \mathcal{A}/(\operatorname{Ker}\varpi)$, for $o \in \mathcal{X}_{\mathcal{A}}$, where $\varpi \in o$ is an irreducible representation.[13]
  - $\theta_{\mathcal{A}} : \mathcal{F}_{\mathcal{A}} \to \mathcal{X}_{\mathcal{A}}$ is the projection map $x + \operatorname{Ker}\pi \mapsto [\pi]$, with fibers $(\mathcal{F}_{\mathcal{A}})_o$, for all $o \in \mathcal{X}_{\mathcal{A}}$.

- to every morphism (irrep-preserving unital $*$-homomorphism of finite-dimensional C*-algebras) $\mathcal{A}_1 \xrightarrow{\phi} \mathcal{A}_2$ in $\mathscr{A}_{FD}$, we associate the morphism of spectral bundles $\underline{\Sigma}^{FD}_\phi := \underline{\Sigma}^{FD}(\mathcal{A}_2) \to \underline{\Sigma}^{FD}(\mathcal{A}_1)$ in $\mathscr{B}_{FD}$, given by $\underline{\Sigma}^{FD}_\phi := (\lambda_\phi, \Lambda_\phi)$, where:
  - $\lambda_\phi : \mathcal{X}_{\mathcal{A}_2} \to \mathcal{X}_{\mathcal{A}_1}$ is the quotient[14] (under unitary equivalence relation) of the ϕ-pull-back of irreps $\varpi \mapsto \varpi \circ \phi$, for ϖ irrep of $\mathcal{A}_2$.
  - $\Lambda_\phi : \lambda^\bullet_\phi(\mathcal{F}_{\mathcal{A}_1}) \to \mathcal{F}_{\mathcal{A}_2}$ is fibrewise defined, as follows:[15]

$$(\Lambda_\phi)_o : (\mathcal{F}_{\mathcal{A}_1})_{\lambda_\phi(o)} \to (\mathcal{F}_{\mathcal{A}_2})_o, \quad \text{for all } o \in \mathcal{X}_{\mathcal{A}_2},$$
$$(\Lambda_\phi)_o : x + \operatorname{Ker}(\varpi\circ\phi) \mapsto \phi(x) + \operatorname{Ker}\varpi, \quad \text{for any } x \in \mathcal{A}_1.$$

A direct calculation shows that $\underline{\Sigma}^{FD}_{\mathrm{Id}_{\mathcal{A}}} = (\mathrm{Id}_{\mathcal{X}_{\mathcal{A}}}, \eta_{\mathcal{F}_{\mathcal{A}}})$ and $\underline{\Sigma}^{FD}_{\phi\circ\psi} = \underline{\Sigma}^{FD}_\psi \circ \underline{\Sigma}^{FD}_\phi$, hence establishing the contravariant functoriality of $\underline{\Sigma}^{FD}$.

The following result is just a very trivial restriction of J.Varela's duality [30] for C*-algebras,[16] for convenience of the reader we recall a short proof.

Theorem 3.2. *There is duality*[17] *between the pair of contravariant functors*

$$\mathscr{A}_{FD} \underset{\underline{\Gamma}^{FD}}{\overset{\underline{\Sigma}^{FD}}{\rightleftarrows}} \mathscr{B}_{FD}\ .$$

---

[12]Notice that for a finite-dimensional C*-algebra this set is always finite (see remark 2.3).

[13]Notice that if $[\varpi_1] = [\varpi_2] \in \mathcal{X}_{\mathcal{A}}$, we have $\operatorname{Ker}\varpi_1 = \operatorname{Ker}\varpi_2$ hence the fiber $(\mathcal{F}_{\mathcal{A}})_o = \mathcal{A}/\operatorname{Ker}\varpi$, for $\varpi \in o$ is well-defined. Furthermore, since $\mathcal{A}$ is finite-dimensional, for every irrep $\varpi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ is necessarily finite-dimensional and we have $\varpi(\mathcal{A}) = \varpi(\mathcal{A})'' = (\mathbb{C}\cdot\mathrm{Id}_{\mathcal{H}})' = \mathcal{B}(\mathcal{H})$ and hence we see that $\mathcal{A}/\operatorname{Ker}\varpi \simeq \mathcal{B}(\mathcal{H})$ is primitive finite-dimensional.

[14]Notice that the quotient map is well-defined: every unitary U intertwining the irreducible representations ϖ_2 and ϖ_2 of $\mathcal{A}_2$, is also a unitary intertwining the irreducible representations $\varpi_1 \circ \phi$ and $\varpi_2 \circ \phi$.

[15]The map $(\Lambda_\phi)_o$ is well-defined, since $\phi(\operatorname{Ker}(\varpi\circ\phi)) \subset \operatorname{Ker}\varpi$, for all irrep $\varpi \in o$. Furthermore $(\Lambda_\phi)_o$ is a unital $*$-isomorphism of primitive finite-dimensional C*-algebras, because ϕ is irrep-preserving and hence the fibers $\mathcal{A}_1/(\operatorname{Ker}\varpi\circ\phi)$ and $\mathcal{A}_2/(\operatorname{Ker}\varpi)$ are both isomorphic to $\mathcal{B}(\mathcal{H}_\varpi)$, for a finite-dimensional Hilbert space $\mathcal{H}_\varpi$, irreducible under ϖ.

[16]Varela duality holds without any restriction to irrep-preserving and fibrewise $*$-isomorphic morphisms. These conditions are here imposed in view of a further spectral analysis of the primitive C*-algebras fibers.

[17]A **duality** is a contravariant equivalence: a pair of contravariant functors that are inverses, modulo natural isomorphisms.

The ***base Gel'fand transform*** $\underline{\mathfrak{G}}^{FD}_{\mathcal{A}} : \mathcal{A} \to \underline{\Gamma}^{FD} \circ \underline{\Sigma}^{FD}(\mathcal{A})$, *is the natural isomorphism of C*-algebras given by* $x \mapsto \underline{\hat{x}} \in \underline{\Gamma}^{FD}(\theta_{\mathcal{A}})$, *for* $x \in \mathcal{A}$, *where we define* $\underline{\hat{x}} : o \mapsto x + \operatorname{Ker} \varpi \in (\mathcal{F}_{\mathcal{A}})_o$, *with* $[\varpi] = o \in \mathcal{X}_{\mathcal{A}}$.

The ***base evaluation transform*** $\underline{\mathfrak{E}}^{FD}_{\theta} := (\underline{\eta}^{\theta}, \underline{\Omega}^{\theta})$, *is the natural isomorphism of bundles of primitive finite-dimensional C*-algebras*

$$\underline{\mathfrak{E}}^{FD}_{\theta} : (\mathcal{F}, \theta, \mathcal{X}) \xrightarrow{(\underline{\eta}^{\theta},\, \underline{\Omega}^{\theta})} \underline{\Sigma}^{FD} \circ \underline{\Gamma}^{FD}(\mathcal{F}, \theta, \mathcal{X}) := (\mathcal{F}_{\underline{\Gamma}^{FD}(\theta)}, \theta_{\underline{\Gamma}^{FD}(\theta)}, \mathcal{X}_{\underline{\Gamma}^{FD}(\theta)}) \quad where:$$

- $\underline{\eta}^{\theta} : \mathcal{X} \to \mathcal{X}_{\underline{\Gamma}^{FD}(\theta)}$ *is defined, for* $o \in \mathcal{X}$, *as* $\underline{\eta}^{\theta} : o \mapsto [\mathrm{ev}_o]$, *with* $\mathrm{ev}_o : \sigma \mapsto \sigma_o$, *for all* $\sigma \in \underline{\Gamma}^{FD}(\theta)$, *where* $[\mathrm{ev}_o] \in \mathcal{X}_{\underline{\Gamma}^{FD}(\theta)}$ *denotes the unique unitary equivalence class of irreps determined by the irrep-preserving unital ∗-homomorphism* $\mathrm{ev}_o : \underline{\Gamma}^{FD}(\theta) \to \mathcal{F}_o$;[18]
- $\underline{\Omega}^{\theta} := \biguplus_{o\in X} \underline{\Omega}^{\theta}_o : (\underline{\eta}^{\theta})^{\bullet}(\mathcal{F}_{\underline{\Gamma}^{FD}(\theta)}) \to \mathcal{F}$, *with* $\underline{\Omega}^{\theta}_o : (\underline{\eta}^{\theta})^{\bullet}(\mathcal{F}_{\underline{\Gamma}^{FD}(\theta)})_o \to \mathcal{F}_o$ *is given by the map* $\underline{\Omega}^{\theta}_o : \frac{\underline{\Gamma}^{FD}(\theta)}{\operatorname{Ker}(\underline{\eta}^{\theta}(o))} \to \mathcal{F}_o$, *well-defined as follows:* $\sigma + \operatorname{Ker}(\underline{\eta}^{\theta}(o)) \mapsto \sigma_o$, *for all* $o \in \mathcal{X}$.

Proof. The base Gel'fand transform $\underline{\mathfrak{G}}^{FD}_{\mathcal{A}} : \mathcal{A} \to \underline{\Gamma}^{FD} \circ \underline{\Sigma}^{FD}(\mathcal{A})$ is a unital ∗-homomorphism in $\mathscr{A}^1_{FD}$:

$$(\underline{\widehat{x+y}})_o = x + y + \operatorname{Ker}\varpi_o = (x + \operatorname{Ker}\varpi_o) + (y + \operatorname{Ker}\varpi_o) = \underline{\widehat{x}}_o + \underline{\widehat{y}}_o = \left(\underline{\widehat{x}} + \underline{\widehat{y}}\right)_o,$$
$$(\underline{\widehat{xy}})_o = xy + \operatorname{Ker}\varpi_o = (x + \operatorname{Ker}\varpi_o)(y + \operatorname{Ker}\varpi_o) = \underline{\widehat{x}}_o\underline{\widehat{y}}_o = \left(\underline{\widehat{x}} \bullet \underline{\widehat{y}}\right)_o,$$
$$(\underline{\widehat{x^*}})_o = x^* + \operatorname{Ker}\varpi_o = (x + \operatorname{Ker}\varpi_o)^* = ((\underline{\widehat{x}})_o)^* = ((\underline{\widehat{x}})^*)_o,$$
$$(\underline{\widehat{1}}_{\mathcal{A}})_o = 1_{\mathcal{A}} + \operatorname{Ker}\varpi_o = 1_{\mathcal{A}_o} = (1_{\underline{\Gamma}^{FD}\circ\underline{\Sigma}^{FD}(\mathcal{A})})_o, \quad \forall x, y \in \mathcal{A},\ \forall o \in \mathcal{X}_{\mathcal{A}}.$$

By Gel'fand-Naĭmark theorem (see [7, corollaries II.6.4.9, II.6.4.10]), as detailed in remark 2.3: $\|\underline{\widehat{x}}\|_{\underline{\Gamma}^{FD}\circ\underline{\Sigma}^{FD}(\mathcal{A})} = \max_{o\in\mathcal{X}_{\mathcal{A}}} \|x + \operatorname{Ker}\varpi_o\|_{\mathcal{A}_o} = \max_{o\in\mathcal{X}_{\mathcal{A}}} \|\varpi_o(x)\|_{\mathcal{A}_o} = \|x\|_{\mathcal{A}}$, so $\underline{\mathfrak{G}}^{FD}_{\mathcal{A}}$ is norm preserving and hence injective.

Again, from remark 2.3, for finite-dimensional C*-algebras, $\varpi : \mathcal{A} \to \bigoplus_{o\in\mathcal{X}_{\mathcal{A}}} \mathcal{A}''_o$ is the isometric ∗-isomorphism $\varpi : x \mapsto \oplus_{o\in\mathcal{X}_{\mathcal{A}}} \varpi_o(x) = \oplus_{o\in\mathcal{X}_{\mathcal{A}}} \underline{\widehat{x}}_o$, $x \in \mathcal{A}$.

The surjectivity of $\underline{\mathfrak{G}}^{FD}$ follows immediately considering, for an arbitrary section $\sigma \in \underline{\Gamma}^{FD} \circ \underline{\Sigma}^{FD}(\mathcal{A})$, the corresponding $\oplus_{o\in\mathcal{X}_{\mathcal{A}}} \sigma_o \in \bigoplus_{o\in\mathcal{X}_{\mathcal{A}}} \mathcal{A}''_o$ and checking that $x := \varpi^{-1}(\oplus_{o\in\mathcal{X}_{\mathcal{A}}} \sigma_o) \in \mathcal{A}$ has base Gel'fand transform σ, since $\underline{\widehat{x}}_o = \sigma_o$ for all $o \in \mathcal{X}_{\mathcal{A}}$.

Finally, the naturality of $\underline{\mathfrak{G}}^{FD}$ is obtained, for $\phi : \mathcal{A}_1 \to \mathcal{A}_2$, $x \in \mathcal{A}_1$, $o \in \mathcal{X}_{\mathcal{A}_2}$, from this calculation:

$$\begin{aligned}(\underline{\Gamma}^{FD} \circ \underline{\Sigma}^{FD})_{\phi}(\underline{\widehat{x}})_o &= \underline{\Gamma}^{FD}_{(\lambda^{\phi},\Lambda^{\Phi})}(\underline{\widehat{x}})_o = \Lambda^{\Phi}_o \circ (\lambda^{\phi})^{\bullet}(\underline{\widehat{x}}) \circ (\mathrm{Id}_{\mathcal{X}_{\mathcal{A}_1}})^{-1}_{\lambda^{\phi}}(o)\\ &= \Lambda^{\Phi}_o(x + \operatorname{Ker}(\varpi_o \circ \phi)) = \phi(x) + \operatorname{Ker}(\varpi_o) = \underline{\widehat{\phi(x)}}.\end{aligned}$$

[18] Since $\mathcal{F}_o$ is a primitive finite-dimensional C*-algebra, it uniquely determines a unitary equivalence class of irreps.

For a bundle $(\mathcal{F}, \theta, \mathcal{X})$ of finite dimensional primitive C*-algebras over a finite set, the unital C*-algebra $\underline{\Gamma}^{FD}(\theta)$ coincides with the direct sum $\bigoplus_{o\in\mathcal{X}} \mathcal{F}_o$. Since, for all $o \in \mathcal{X}$, $\mathcal{F}_o$ is simple (being isomorphic to a finite-dimensional matrix algebra) the only $*$-ideals of $\underline{\Gamma}^{FD}(\theta)$ are of the form $\bigoplus_{o\in\mathcal{Y}} \mathcal{F}_o$, for $\mathcal{Y} \subset \mathcal{X}$. For any irreducible representation ϖ of $\underline{\Gamma}^{FD}(\theta)$, we have that $\operatorname{Ker}\varpi$ is a maximal ideal and hence there exists an $o \in \mathcal{X}$ such that $\frac{\underline{\Gamma}^{FD}(\theta)}{\operatorname{Ker}\varpi}$ is isomorphic to $\mathcal{F}_o$. Hence $[\pi] = \underline{\eta}^\theta(o)$ and the surjectivity of $\underline{\eta}^\theta : \mathcal{X} \to \mathcal{X}_{\underline{\Gamma}^{FD}(\theta)}$ is obtained. Since, for $o_1, o_2 \in \mathcal{X}$, $[\mathrm{ev}_{o_1}] = [\mathrm{ev}_{o_2}]$ if and only if there is an isomorphism $\psi : \mathcal{F}_{o_1} \to \mathcal{F}_{o_2}$ such that $\psi \circ \mathrm{ev}_{o_1}(\sigma) = \mathrm{ev}_{o_2}(\sigma)$, for all $\sigma \in \underline{\Gamma}^{FD}(\theta)$, we obtain $o_1 = o_2$ (otherwise any section σ with $\sigma_{o_1} = 0_{\mathcal{F}_{o_1}}$ and $\sigma_{o_2} = 1_{\mathcal{F}_{o_2}}$ would lead to contradiction) and hence the injectivity and bijectivity of $\underline{\eta}^\theta$.

For all $o \in \mathcal{X}$, the map $\underline{\Omega}^\theta_o : (\mathcal{F}_{\underline{\Gamma}^{FD}(\theta)})_{\underline{\eta}^\theta(o)} \to \mathcal{F}_o$ is a unital $*$-homomorphism: for all $\sigma, \tau \in \underline{\Gamma}^{FD}(\theta)$,

$$
\begin{aligned}
&\underline{\Omega}^\theta_o(\sigma + \tau + \operatorname{Ker}\underline{\eta}^\theta(o)) = (\sigma + \tau)_o = \sigma_o + \tau_o = \underline{\Omega}^\theta_o(\sigma) + \underline{\Omega}^\theta_o(\tau),\\
&\underline{\Omega}^\theta_o(\sigma \bullet \tau + \operatorname{Ker}\underline{\eta}^\theta(o)) = (\sigma \bullet \tau)_o = \sigma_o \bullet_{\mathcal{F}_o} \tau_o = \underline{\Omega}^\theta_o(\sigma) \bullet_{\mathcal{F}_o} \underline{\Omega}^\theta_o(\tau),\\
&\underline{\Omega}^\theta_o(\sigma^* + \operatorname{Ker}\underline{\eta}^\theta(o)) = (\sigma^*)_o = (\sigma_o)^{*_{\mathcal{F}_o}} = \underline{\Omega}^\theta_o(\sigma)^{*_{\mathcal{F}_o}},\\
&\underline{\Omega}^\theta_o(1_{\underline{\Gamma}^{FD}(\theta)} + \operatorname{Ker}\underline{\eta}^\theta(o)) = (1_{\underline{\Gamma}^{FD}(\theta)})(o) = 1_{\mathcal{F}_o}.
\end{aligned}
$$

Since, for all $f \in \mathcal{F}_o$, we have the section $\delta^f \in \underline{\Gamma}^{FD}(\theta)$, defined as $\delta^f_{o'} = 0_{\mathcal{F}_{o'}}$ whenever $o' \neq o$ and $\delta^f_o := f$, and $\underline{\Omega}^\theta_o(\delta^f + \operatorname{Ker}\underline{\eta}^\theta(o)) = f$, we obtain the surjectivity of $\underline{\Omega}^\theta_o$. Since $\underline{\Omega}^\theta_o(\sigma + \operatorname{Ker}\underline{\eta}^\theta(o)) = 0_{\mathcal{F}_o}$ implies $\sigma_o = 0_{\mathcal{F}_o}$ and hence $\sigma \in \operatorname{Ker}(\underline{\eta}^\theta)$, the injectivity follows and we see that $\underline{\Omega}^\theta : (\underline{\eta}^\theta)^\bullet(\mathcal{F}_{\underline{\Gamma}^{FD}(\theta)}) \to \mathcal{F}$ is a fibrewise unital $*$-isomorphism. As a consequence $(\underline{\eta}^\theta, \underline{\Omega}^\theta)$ is an isomorphism in $\mathscr{B}_{FD}$.

Finally, for the naturality of $\underline{\mathfrak{E}}^{FD}$, for any morphism $(\mathcal{F}^1, \theta^1, \mathcal{X}^1) \xrightarrow{(\lambda,\Lambda)} (\mathcal{F}^2, \theta^2, \mathcal{X}^2)$, we must prove that $(\lambda^{\underline{\Gamma}_{(\lambda,\Lambda)}}, \Lambda^{\underline{\Gamma}_{(\lambda,\Lambda)}}) \circ (\underline{\eta}^{\theta^1}, \underline{\Omega}^{\theta^1}) = (\underline{\eta}^{\theta^2}, \underline{\Omega}^{\theta^2}) \circ (\lambda, \Lambda)$ i.e.:

$$
\begin{aligned}
&\lambda^{\underline{\Gamma}_{(\lambda,\Lambda)}} \circ \underline{\eta}^{\theta^1} = \underline{\eta}^{\theta^2} \circ \lambda,\\
&\underline{\Omega}^{\theta^1} \circ (\underline{\eta}^{\theta^1})^\bullet(\Lambda^{\underline{\Gamma}_{(\lambda,\Lambda)}}) \circ \zeta^{\underline{\Sigma}\circ\underline{\Gamma}(\theta^2)}_{\lambda^{\underline{\Gamma}_{(\lambda,\Lambda)}},\, \underline{\eta}^{\theta^1}} = \Lambda \circ \lambda^\bullet(\underline{\Omega}^{\theta^2}) \circ \zeta^{\underline{\Sigma}\circ\underline{\Gamma}(\theta^2)}_{\underline{\eta}^{\theta^2},\, \lambda}.
\end{aligned}
$$

For the first equation, for every $o \in \mathcal{X}^1$, $\underline{\eta}^{\theta^2}_{\lambda(o)}$ is the unique irreducible representation $[\mathrm{ev}^{\theta^2}_{\lambda(o)}]$ associated with the unital $*$-homomorphism of $\underline{\Gamma}^{FD}(\theta^2)$ onto the fiber $\mathcal{F}^{\underline{\Gamma}(\theta^2)}_{\lambda(o)}$; $\underline{\eta}^{\theta^1}_o$ is the unique irreducible representation $[\mathrm{ev}^{\theta^1}_o]$ associated with the unital $*$-homomorphism of $\underline{\Gamma}^{FD}(\theta^1)$ onto the fiber $\mathcal{F}^{\underline{\Gamma}(\theta^1)}_o$. Since $\lambda^{\underline{\Gamma}_{(\lambda,\Lambda)}}_o : \mathcal{F}^{\underline{\Gamma}(\theta^1)}_o \to \mathcal{F}^{\underline{\Gamma}(\theta^2)}_{\lambda(o)}$ is a unital $*$-isomorphism, these two classes of irreps necessarily coincide.

For the second equation, for $o \in \mathcal{X}^1$ and for all $\sigma \in \Gamma^{FD}(\theta^2)$, evaluating both terms on the element $\sigma + \operatorname{Ker}(\lambda^{\underline{\Gamma}_{(\lambda,\Lambda)}} \circ \underline{\eta}^{\theta^1}) = \sigma + \operatorname{Ker}(\underline{\eta}^{\theta^2} \circ \lambda)$ we obtain: $\Lambda \circ \sigma \circ \lambda(o) \in \mathcal{E}^1_o$. $\square$

Remark 3.3. For a finite-dimensional C*-algebra $\mathcal{A}$, the center $Z(\mathcal{A})$ is isomorphic (via the base Gel'fand transform) to the commutative unital C*-algebra of sections of the trivial complex line-bundle over $\mathcal{X}_{\mathcal{A}}$, hence the usual Gel'fand spectrum $\mathrm{Sp}(Z(\mathcal{A}))$ is homeomorphic to the (discrete compact) space $\mathcal{X}_{\mathcal{A}}$ that also coincides with the usual structure space $\widehat{\mathcal{A}}$ (the set $\mathcal{X}_{\mathcal{A}}$ equipped with the quotient topology induced by the weak*-topology of $\mathcal{P}_{\mathcal{A}}$ under the map $\omega \mapsto [\omega]$) and with the primitive spectrum $\mathrm{Prim}(\mathcal{A})$ (the set of primitive ideals) equipped with the hull kernel topology. ⌟

Varela duality, like the original Gel'fand-Naĭmak duality, is an adjoint duality.

Remark 3.4. We recall that, in the case of contravariant functors, $\mathscr{C} \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \mathscr{D}$ the usual notions of left-right $\Psi \dashv \Phi$ and right-left $\Psi \vdash \Gamma$ adjuctions are replaced by right-right $\dashv \Psi\ \Phi \vdash$ and left-left $\vdash \Psi\ \Phi \dashv$ adjunctions (see for example [27, section 4.3]).

More specifically, if $\mathscr{C} \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \mathscr{D}$ is a covariant adjunction $\Psi \dashv \Phi$ with unit $\eta : \mathrm{Id}_{\mathscr{C}} \to \Phi \circ \Psi$ and co-unit $\epsilon : \Psi \circ \Phi \to \mathrm{Id}_{\mathscr{D}}$, passing to the opposite category $\mathscr{D}^\circ$, we get a contravariant right-right adjunction $\mathscr{C} \underset{\Phi^\circ}{\overset{{}^\circ\Psi}{\rightleftarrows}} \mathscr{D}^\circ$, $\dashv {}^\circ\Psi\ \Phi^\circ \vdash$, with two units $\eta : \mathrm{Id}_{\mathscr{C}} \to \Phi^\circ \circ {}^\circ\Psi$ and $\epsilon^\circ : \mathrm{Id}_{\mathscr{D}^\circ} \to {}^\circ\Psi \circ \Phi^\circ$.

Similarly, passing to the dual category $\mathscr{C}^\circ$, we obtain a contravariant left-left adjunction $\mathscr{C}^\circ \underset{{}^\circ\Phi}{\overset{\Psi^\circ}{\rightleftarrows}} \mathscr{D}$, $\vdash \Psi^\circ\ {}^\circ\Phi \dashv$ with two co-units $\eta^\circ : {}^\circ\Phi \circ \Psi^\circ \to \mathrm{Id}_{\mathscr{C}^\circ}$ and $\epsilon : \Psi^\circ \circ {}^\circ\Phi \to \mathrm{Id}_{\mathscr{D}}$. Notice, that considering both the opposite categories we obtain a covariant adjunction $\mathscr{C}^\circ \underset{{}^\circ\Phi^\circ}{\overset{{}^\circ\Psi^\circ}{\rightleftarrows}} \mathscr{D}^\circ$, ${}^\circ\Psi^\circ \vdash {}^\circ\Phi^\circ$ with co-unit $\eta^\circ : {}^\circ\Phi^\circ \circ {}^\circ\Psi^\circ \to \mathrm{Id}_{\mathscr{C}^\circ}$ and unit $\epsilon^\circ : \mathrm{Id}_{\mathscr{D}^\circ} \to {}^\circ\Psi^\circ \circ {}^\circ\Phi^\circ$.

In an adjoint equivalence[19] we necessarily have $(\Sigma \dashv \Gamma) \Leftrightarrow (\Sigma \vdash \Gamma)$.

A duality is a contravariant equivalence, hence by an **adjoint duality** we mean a duality between contravariant functors whose natural isomorphisms satisfy adjunction triangle identities. In an adjoint duality $(\dashv \Gamma\ \Sigma \vdash) \Leftrightarrow (\vdash \Gamma\ \Sigma \dashv)$. ⌟

Remark 3.5. Notice that the actual choice of the "direction" for geometric morphisms $(\lambda, \Lambda) \in \mathscr{B}^1_{FD}$ of bundles in $\mathscr{B}_{FD}$ is essentially dictated by backward compatibility with the usual Gel'fand-Naĭmark duality in the commutative C*-algebraic case. Taking the opposite direction (hence using the opposite category $\mathscr{B}^\circ_{FD}$) would result in covariant functors $\Sigma^{FD\circ}$ and ${}^\circ\Gamma^{FD}$ and an equivalence of categories. ⌟

[19] An adjoint equivalence is an equivalence between covariant functors whose natural isomorphisms satisfy adjunction triangle identities.

Proposition 3.6. *The Varela duality in theorem 3.2 is an adjoint right-right duality* $(\dashv \underline{\Sigma}^{FD}\ \underline{\Gamma}^{FD} \vdash)$ *with units the base Gel'fand and base evaluation transform isomorphisms. Equivalently we have an adjoint left-left duality* $(\vdash \underline{\Sigma}^{FD}\ \underline{\Gamma}^{FD} \dashv)$ *with co-units* $(\underline{\mathfrak{G}}^{FD})^{-1} : \underline{\Gamma}^{FD} \circ \underline{\Sigma}^{FD} \to \mathrm{Id}_{\mathscr{A}_{FD}}$ *and* $(\underline{\mathfrak{E}}^{FD})^{-1} : \underline{\Sigma}^{FD} \circ \underline{\Gamma}^{FD} \to \mathrm{Id}_{\mathscr{B}_{FD}}$ *the inverses of the base Gel'fand and base evaluation transform isomorphisms.*

Proof. In order to avoid adjoint triangle identities for contravariant cases, we make use of remark 3.5 and the notation in remark 3.4, passing to the opposite category $\mathscr{B}^{\circ}_{FD}$ and considering the covariant functors $\underline{\Gamma}^{FD\circ} : \mathscr{B}^{\circ}_{FD} \to \mathscr{A}_{FD}$ and ${}^{\circ}\underline{\Sigma}^{FD} : \mathscr{A}_{FD} \to \mathscr{B}^{\circ}_{FD}$. The induced natural transformation isomorphisms $\underline{\mathfrak{G}}^{FD} : \mathrm{Id}_{\mathscr{A}_{FD}} \to \underline{\Gamma}^{FD\circ} \circ {}^{\circ}\underline{\Sigma}^{FD}$ and $\underline{\mathfrak{E}}^{FD\circ} : {}^{\circ}\underline{\Sigma}^{FD} \circ \underline{\Gamma}^{FD\circ} \to \mathrm{Id}_{\mathscr{B}^{\circ}_{FD}}$ satisfy the adjunction triangle identities for the covariant adjoint equivalence ${}^{\circ}\underline{\Sigma}^{FD} \dashv \underline{\Gamma}^{FD\circ}$ and hence, by remark 3.4 $\underline{\mathfrak{G}}^{FD}$ and $\underline{\mathfrak{E}}^{FD}$ are the two unit isomorphisms of the right-right adjoint duality $(\dashv \underline{\Sigma}^{FD}\ \underline{\Gamma}^{FD} \vdash)$.

From remark 3.4 we also have that $(\underline{\mathfrak{G}}^{FD})^{-1} : \underline{\Gamma}^{FD} \circ \underline{\Sigma}^{FD} \to \mathrm{Id}_{\mathscr{A}_{FD}}$ and that $(\underline{\mathfrak{E}}^{FD})^{-1} : \underline{\Sigma}^{FD} \circ \underline{\Gamma}^{FD} \to \mathrm{Id}_{\mathscr{B}_{FD}}$ are the two co-unit isomorphisms for the left-left adjoint equivalence $(\vdash \underline{\Sigma}^{FD}\ \underline{\Gamma}^{FD} \dashv)$. □

4 Discrete Fiber Equivalence and Discrete Duality

In the next step, we proceed beyond Varela duality, making a further spectral analysis of the fibers of our bundles (that for objects in $\mathscr{B}_{FD}$ are just primitive finite-dimensional C*-algebras and hence type I_n factors, for some fiber-depending $n \in \mathbb{N}_0$) in terms of 1-C*-categories. This essentially amounts to an intrinsic description of each fiber (a matrix algebra) as a convolution C*-algebra of a finite pair groupoid, an idea that is quite well-known in non-commutative geometry [9, section 1.1].

Although everything is still treated in the discrete case (without topologies and uniformities), most of the material here presented is developed at a level of generality well beyond the immediate requirements for a duality for finite-dimensional C*-algebras; the assumption of finite-dimensionality becomes relevant starting from definition 4.26.

Before proceeding, we recall some basic properties of C*-categories (for further details, we refer to P.Ghez-R.Lima-J.Roberts [17] and P.Mitchener [26]).

Definition 4.1. *A* ***C*-category*** $\mathcal{C}$ *is a strict involutive (dagger) 1-category*[20] $(\mathcal{C}, \circ, *, \iota)$ *equipped with:*

[20] An **involutive category** is a 1-quiver $\mathcal{C}^0 \xleftarrow{s} \mathcal{C}^1 \xrightarrow{t} \mathcal{C}^0$ (having source map s and target map t), with identity $\iota : \mathcal{C}^0 \to \mathcal{C}^1$, composition $\circ : \mathcal{C}^1 \times_\circ \mathcal{C}^1 := \{(x,y) \in \mathcal{C}^1 \times \mathcal{C}^1 \mid s(x) = t(y)\} \to \mathcal{C}^1$, and involution $* : \mathcal{C}^1 \to \mathcal{C}^1$, that satisfy the structural requirements $s(\iota(A)) = A = t(\iota(A))$, for all $A \in \mathcal{C}^0$, $s(x^*) = t(x)$, $t(x^*) = s(x)$, for all $x \in \mathcal{C}^1$, $s(x \circ y) = s(y)$, $t(x \circ y) = t(x)$, for all $(x,y) \in \mathcal{C}^1 \times_\circ \mathcal{C}^1$, and the following algebraic axioms of: associativity $x \circ (y \circ z) = (x \circ y) \circ z$, for all $(x,y),(y,z) \in \mathcal{C}^1 \times_\circ \mathcal{C}^1$, unitality $x \circ \iota(s(x)) = x = \iota(t(x)) \circ x$, for all $x \in \mathcal{C}^1$, involutivity $(x^*)^* = x$, for all $x \in \mathcal{C}^1$, and anti-multiplicativity $(x \circ y)^* = y^* \circ x^*$, for all $(x,y) \in \mathcal{C}^1 \times_\circ \mathcal{C}^1$. A **groupoid** is an involutive category $\mathcal{C}$ where all morphisms $x \in \mathcal{C}$ are isomorphisms (invertible) with $x^* = x^{-1}$ and a **pair groupoid** is a groupoid with all the hom-sets $\mathcal{C}_{AB} := \mathrm{Hom}_{\mathcal{C}}(B,A)$ of cardinality one. The bundle map $(t,s) : \mathcal{C} \to \mathcal{C}^0 \times \mathcal{C}^0$, with fibers $\mathcal{C}_{AB}$, is a $*$-functor from the involutive category $\mathcal{C}$ to the pair groupoid $\mathcal{C}^0 \times \mathcal{C}^0$ with objects $\mathcal{C}^0$.

- *an addition* $+ : \mathcal{C}^1 \times_+ \mathcal{C}^1 := \bigcup_{A,B\in\mathcal{C}^0} \mathcal{C}_{AB} \to \mathcal{C}^1$ *and a scalar multiplication* $\cdot : \mathbb{C}\times\mathcal{C}^1 \to \mathcal{C}^1$ *and such that* $s(\alpha\cdot x) = s(x)$, $t(\alpha\cdot x) = t(x)$, *for all* $(\alpha, x) \in \mathbb{C}\times\mathcal{C}^1$, $s(x) = s(x + y) = s(y)$, $t(x) = t(x + y) = t(y)$, *for all* $(x, y) \in \mathcal{C}^1 \times_+ \mathcal{C}^1$, *making the hom-sets* $\mathcal{C}_{AB} := \mathrm{Hom}_{\mathcal{C}}(B, A)$, *for all* $A, B \in \mathcal{C}^0$, $\mathbb{C}$*-vector spaces* $(\mathcal{C}_{AB}, +, \cdot)$ *in such a way that involutions are conjugate-linear:*

  $$(\alpha \cdot x + y)^* = \overline{\alpha} \cdot x^* + y^*,$$

  for all $(x, y) \in \mathcal{C}^1 \times_+ \mathcal{C}^1$, *for all* $\alpha \in \mathbb{C}$, *and compositions bilinear:*

  $$(x + y) \circ (\alpha \cdot z) = \alpha \cdot (x \circ z) + \alpha \cdot (y \circ z),$$
  $$(\alpha \cdot w) \circ (x + y) = \alpha \cdot (w \circ x) + \alpha \cdot (z \circ y),$$

  for all $(w, (x, y), z) \in \mathcal{C}^1 \times_\circ (\mathcal{C}^1 \times_+ \mathcal{C}^1) \times_\circ \mathcal{C}^1$, *for all* $\alpha \in \mathbb{C}$.

- *a norm function* $\|\cdot\| : \mathcal{C}^1 \to \mathbb{R}$ *such that, for every pair of objects* $A, B \in \mathcal{C}^0$, *the hom-sets* $\mathcal{C}_{AB} := \mathrm{Hom}_{\mathcal{C}}(B, A)$ *are Banach spaces; the norm is sub-multiplicative:* $\|x \circ y\| \leq \|x\| \cdot \|y\|$, *for all* $(x, y) \in \mathcal{C}^1 \times_\circ \mathcal{C}^1$*; satisfies the C*-property:* $\|x^* \circ x\| = \|x\|^2$, *for all* $x \in \mathcal{C}^1$*; and finally the positivity holds:* $x^* \circ x$ *is positive in the C*-algebra*[21] $\mathcal{C}_{s(x)s(x)}$, *for all* $x \in \mathcal{C}^1$.

A C-category is* ***full*** *if* $\overline{\mathcal{C}_{AB} \circ \mathcal{C}_{BC}} = \mathcal{C}_{AC}$, *for all* $A, B, C \in \mathcal{C}^0$. [22] *The C*-category* $\mathcal{C}$ *is* ***one-dimensional*** *if* $\mathcal{C}_{AB}$ *is 1-dimensional, for all* $A, B \in \mathcal{C}^0$.[23] *The C*-category* $\mathcal{C}$ *is* ***finite-dimensional*** *if all the hom-sets* $\mathcal{C}_{AB}$ *are finite-dimensional, for all* $A, B \in \mathcal{C}^0$. *A finite-dimensional C*-category* $\mathcal{C}$ *is said to be* ***finite*** *if its family of objects* $\mathcal{C}^0$ *is finite. A* ***W*-category*** *is a C*-category* $\mathcal{C}$ *such that, for all* $A, B \in \mathcal{C}^0$, *the hom-set* $\mathcal{C}_{AB}$, *as a Banach space, is the dual of a Banach space (its pre-dual* $\mathcal{C}_{AB*}$*). A finite-dimensional C*-category is a W*-category.*

A covariant ***$*$-functor*** $\mathcal{C}_1 \xrightarrow{(\Phi^0,\Phi^1)} \mathcal{C}_2$ *between two C*-categories* $(\mathcal{C}_j, \circ_j, *_j, \iota_j)$, $j = 1, 2$, *is a covariant functor*[24] *such that* $\Phi^1(x^{*_1}) = \Phi^1(x)^{*_2}$, *for every* $x \in \mathcal{C}^1_1$.

Remark 4.2. There is functor $\mathscr{A} \xrightarrow{\mathfrak{U}} \mathscr{C}$, from the category $\mathscr{A}$ of unital $*$-homomorphisms between unital C*-algebras to the category $\mathscr{C}$ of $*$-functors between small C*-categories, that to every unital C*-algebra $\mathcal{A}$ associates the C*-category $\mathfrak{U}(\mathcal{A})$

[21] From the previous axioms it already follows that, for all $A, B \in \mathcal{C}^0$, $\mathcal{C}_{AA}$ is a C*-algebra, and $\mathcal{C}_{AB}$ is a $\mathcal{C}_{AA} - \mathcal{C}_{BB}$-bimodule.

[22] This condition is equivalent to require that all the bimodules $\mathcal{C}_{AB}$ are $\mathcal{C}_{AA} - \mathcal{C}_{BB}$, imprimitivity Hilbert C*-bimodules (see footnote 25), when equipped with the two left/right inner products ${}_\bullet\langle x \mid y\rangle := x \circ y^* \in \mathcal{C}_{AA}$ and $\langle x \mid y\rangle_\bullet := x^* \circ y \in \mathcal{C}_{BB}$, for all $x, y \in \mathcal{C}_{AB}$, for all $A, B \in \mathcal{C}^0$. This means that ${}_\bullet\langle x \mid y\rangle \cdot z = x \cdot \langle y \mid z\rangle_\bullet$, for all $x, y, z \in \mathcal{C}_{AB}$ (a condition that is always satisfied for hom-sets in a C*-category) and the following fullness condition: $\overline{\mathcal{C}_{AB} \circ \mathcal{C}_{BA}} = \mathcal{C}_{AA}$, $\overline{\mathcal{C}_{BA} \circ \mathcal{C}_{AB}} = \mathcal{C}_{BB}$.

[23] A one-dimensional C*-category is always full, since $\|x^* \circ x\| = \|x\|^2 \neq 0$, for any $0 \neq x \in \mathcal{C}_{AB}$ implies that $\mathcal{C}_{AB} \circ \mathcal{C}_{BA} = \mathcal{C}_{AA}$.

[24] Recall that a covariant functor between two categories consists of a pair of maps $\Phi^0 : \mathcal{C}^0_1 \to \mathcal{C}^0_2$ and $\Phi^1 : \mathcal{C}^1_1 \to \mathcal{C}^1_2$ such that $s_2 \circ \Phi^1 = \Phi^0 \circ s_1$, $t^2 \circ \Phi^1 = \Phi^0 \circ t_1$, $\Phi^1 \circ \iota_1 = \iota_2 \circ \Phi^0$ and $\Phi^1 \circ \circ_1 = \circ_2 \circ \Phi^1$.

specified as follows: the objects of $\mathfrak{U}(\mathcal{A})$ are the Hermitian projections $p \in \mathcal{A}$, with $p = p^2 = p^*$; given two objects p, q, morphisms in $\mathfrak{U}(\mathcal{A})$ from p to q are given by (q, x, p), with $x \in \mathcal{A}$; identities in $\mathfrak{U}(\mathcal{A})$ are given by $\iota(p) := (p, 1_{\mathcal{A}}, p)$, composition in $\mathfrak{U}(\mathcal{A})$ is defined as $(p_3, x, p_2) \circ (p_2, y, p_1) := (p_3, xp_2y, p_1)$; involutions in $\mathfrak{U}(\mathcal{A})$ are $(p, x, q)^* := (q, x^*, p)$ and $\|(q, x, p)\| := \|qxp\|$. ⌟

Definition 4.3. *A* ***unital C*-enveloping algebra*** *of a C*-category $\mathcal{C}$, is a unital C*-algebra* C*($\mathcal{C}$) *with a $*$-functor $\eta^{\mathcal{C}} : \mathcal{C} \to \mathfrak{U}(\mathrm{C}^*(\mathcal{C}))$ that satisfies the following universal factorization property: for any $*$-functor $\phi : \mathcal{C} \to \mathfrak{U}(\mathcal{A})$ for a unital C*-algebra $\mathcal{A}$, there exists a unique unital $*$-homomorphism $\overline{\phi} : \mathrm{C}^*(\mathcal{C}) \to \mathcal{A}$ such that $\phi = \mathfrak{U}(\overline{\phi}) \circ \eta^{\mathcal{C}}$.*

Two different C*-enveloping algebras of the same C*-category are necessarily isomorphic via a unique $*$-isomorphism factorizing their defining $*$-functors. We provide here a sketch of a proof of the existence of a C*-enveloping algebra for a C*-category $\mathcal{C}$, (for other constructions see [17, 26]).

Proposition 4.4. *A small C*-category $\mathcal{C}$ has a canonically associated unital enveloping C*-algebra* C*($\mathcal{C}$).

Proof. Consider $\bigoplus_{A\in\mathcal{C}^0} \mathcal{C}_{AA} := \{(x_{AA}) \in \bigtimes_{A\in\mathcal{C}^0} \mathcal{C}_{AA} \mid \sup_{A\in\mathcal{C}^0} \|x_{AA}\| < \infty\}$, the direct sum of the diagonal C*-algebras of the C*-category, that is a C*-algebra with componentwise operations and norm defined by $\|(x_{AA})\| := \sup_{A\in\mathcal{C}^0} \|x_{AA}\|_{\mathcal{C}_{AA}}$. Considering $\mathcal{H}^{\mathcal{C}} := \{(x_{AB}) \in \bigtimes_{A,B\in\mathcal{C}^0} \mathcal{C}_{AB} \mid (\sum_{B\in\mathcal{C}^0} (x^*)_{AB} \circ x_{BA})_{A\in\mathcal{C}^0} \in \bigoplus_{A\in\mathcal{C}^0} \mathcal{C}_{AA}\}$, we see that $\mathcal{H}^{\mathcal{C}}$ is a right Hilbert C*-module[25] over the C*-algebra $\bigoplus_{A\in\mathcal{C}^0} \mathcal{C}_{AA}$ when equipped with the right inner product given by $\langle x \mid y \rangle_\bullet := \sum_{B\in\mathcal{C}^0} (x^*)_{AB} \circ y_{BA}$. Every element $x \in \mathcal{C}$ has a well-defined left action on $\mathcal{H}^{\mathcal{C}}$ given by $\eta^{\mathcal{C}}_x(\xi) := x \circ \xi$. The map $\eta^{\mathcal{C}} : \mathcal{C} \to \mathcal{L}(\mathcal{H}^{\mathcal{C}})$ is a $*$-functor into the C*-algebra of adjointable operators on the Hilbert C*-module $\mathcal{H}^{\mathcal{C}}$. We define the C*-algebra C*($\mathcal{C}$) to be the unital C*-algebra generated by $\eta^{\mathcal{C}}(\mathcal{C}) \subset \mathcal{L}(\mathcal{H}^{\mathcal{C}})$ and we still denote by $\eta^{\mathcal{C}} : \mathcal{C} \to \mathrm{C}^*(\mathcal{C})$ the $*$-functor with values into C*($\mathcal{C}$). We can finally verify that since $\phi : \mathcal{C} \to \mathcal{A}$ is a $*$-functor, and hence $\|\phi(x)\| \leq \|x\|$, there is a unique unital $*$-homomorphism $\overline{\phi} : \mathrm{C}^*(\mathcal{C}) \to \mathcal{A}$ that satisfies the universal factorization property. □

Remark 4.5. There is a perfectly parallel "W*-version" of the previous constructions.

Exactly as in remark 4.2, we have a functor $\mathscr{A}'' \xrightarrow{\mathfrak{U}''} \mathscr{C}''$, from the category $\mathscr{A}''$ of σ-weakly continuous[26] unital $*$-homomorphism between W*-algebras to the category $\mathscr{C}''$ of σ-weakly continuous $*$-functors between small W*-categories.

[25] We recall (see for example [7, section II.7]) that a right Hilbert C*-module $\mathcal{M}_{\mathcal{A}}$ over a unital C*-algebra $\mathcal{A}$ is a right unital $\mathcal{A}$-module equipped with an $\mathcal{A}$-valued inner product $\langle \cdot \mid \cdot \rangle_{\mathcal{A}} : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$ that is $\mathcal{A}$-linear in the second variable, Hermitian $\langle x \mid y \rangle_{\mathcal{A}}^* = \langle y \mid x \rangle_{\mathcal{A}}$ for $x, y \in \mathcal{M}$, non-degenerate $\langle x \mid x \rangle_{\mathcal{A}} = 0_{\mathcal{A}} \Rightarrow x = 0_{\mathcal{M}}$, positive $\langle x \mid x \rangle_{\mathcal{A}} \in \mathcal{A}_+ := \{a^*a \in \mathcal{A} \mid a \in \mathcal{A}\}$ for $x \in \mathcal{M}$, and complete in the norm $\|x\|_{\mathcal{M}} := \|\langle x \mid x \rangle_{\mathcal{A}}\|_{\mathcal{A}}^{1/2}$. An map $T : \mathcal{M} \to \mathcal{M}$ is adjointable if, for a certain (necessarily unique) map $T^* : \mathcal{M} \to \mathcal{M}$, for all $x, y \in \mathcal{M}$, $\langle x \mid T(y) \rangle_{\mathcal{A}} = \langle T^*(x) \mid y \rangle_{\mathcal{A}}$. Adjointable maps are necessarily linear continuous and they constitute a C*-algebra $\mathcal{L}(\mathcal{M}_{\mathcal{A}})$.

[26] The **σ-weak topology** on a W*-algebra $\mathcal{R}$ is the weak*-topology induced by its pre-dual space $\mathcal{R}_*$.

A **W*-enveloping algebra** of a W*-category $\mathcal{R}$, is a W*-algebra W*($\mathcal{R}$) with a σ-weakly continuous $*$-functor $\eta^{\mathcal{R}} : \mathcal{R} \to \mathfrak{U}''(\mathrm{W}^*(\mathcal{R}))$ that satisfies the following universal factorization property: for any σ-weakly continuous $*$-functor $\phi : \mathcal{R} \to \mathfrak{U}''(\mathcal{S})$ for a W*-algebra $\mathcal{S}$, there exists a unique σ-weakly continuous unital $*$-homomorphism $\overline{\phi} : \mathrm{W}^*(\mathcal{R}) \to \mathcal{S}$ such that $\phi = \mathfrak{U}''(\overline{\phi}) \circ \eta^{\mathcal{R}}$.

A construction of the W*-enveloping algebra of a W*-category via inductive limits is given in [17]. An explicit construction of the W*-enveloping algebra of a W*-category can also be obtained along the same lines of proposition 4.4, considering the W*-algebra $\mathcal{W}^{\mathcal{R}} := \bigoplus_{A\in\mathcal{R}^0} \mathcal{R}_{AA}$, its self-dual $\mathcal{W}^{\mathcal{R}}$-Hilbert W*-module $(\mathcal{H}^{\mathcal{R}})'$ of the $\mathcal{W}^{\mathcal{R}}$-module morphisms from $\mathcal{H}^{\mathcal{R}}$ to $\mathcal{W}^{\mathcal{R}}$ (see for example [25, section 3] for the relevant definitions for Hilbert W*-modules), and taking the closures with respect to the σ-weak topology. ⌟

Remark 4.6. Consider the directed set $\mathscr{Q}$ of finite subsets $\mathcal{Q} \subset \mathcal{C}^0$ of objects of $\mathcal{C}$, under inclusion. Every section $\sigma : \mathcal{C}^0 \times \mathcal{C}^0 \to \mathcal{C}^1$ of the bundle $\mathcal{C}^1 \xrightarrow{(t,s)} \mathcal{C}^0 \times \mathcal{C}^0$ with support in $\mathcal{Q}\times\mathcal{Q}$ determines an adjointable operator acting via "line-by-column" multiplication on $\mathcal{H}^{\mathcal{C}}$: $\sigma(\xi)_{AB} := \sum_{J\in\mathcal{Q}} \sigma_{AJ}\circ\xi_{JB}$. In a perfectly similar way, making use of the construction of the W*-enveloping algebra in remark 4.5, every section with finite support will determine an operator on $(\mathcal{H}^{\mathcal{C}})'$. The family of finitely supported sections is an involutive subalgebra of both C*($\mathcal{C}$) and W*($\mathcal{C}$) with multiplication given by **matrix convolution** $(\sigma^1\odot\sigma^2)_{AB} = \sum_J \sigma^1_{AJ}\circ\sigma^2_{JB}$ and involution given by the **matrix adjoint** $(\sigma^\star)_{AB} = (\sigma_{BA})^*$.

Consider the **canonical resolution of the identity** $\mathcal{Q} \mapsto I_{\mathcal{Q}}$, where $(I_{\mathcal{Q}})_{AA} := 1_{\mathcal{C}_{AA}}$, for all $A \in \mathcal{Q}$, and $(I_{\mathcal{Q}})_{BC} = 0_{\mathcal{C}_{BC}}$ otherwise. For every operator $T \in \mathcal{L}(\mathcal{H}^{\mathcal{C}})$, consider its induced net of finite-objects truncations $(T_{\mathcal{Q}})_{\mathscr{Q}}$ with $T_{\mathcal{Q}} := I_{\mathcal{Q}} \circ T \circ I_{\mathcal{Q}}$.

Define $\mathcal{K}(\mathcal{H}^{\mathcal{C}}) \subset \mathcal{L}(\mathcal{H}^{\mathcal{C}})$ as the (not necessarily unital) C*-algebra generated by finite rank operators and notice that, whenever $T \in \mathcal{K}(\mathcal{H}^{\mathcal{C}})$, we have $T = \lim_{\mathcal{Q}\to+\infty} T_{\mathcal{Q}}$, where the limit is taken in the norm topology.

It follows that $\mathcal{K}(\mathcal{H}^{\mathcal{C}})$ consists of sections $\sigma : \mathcal{C}^0 \times \mathcal{C}^0 \to \mathcal{C}^1$ (not necessarily with finite support) whose nets of finite truncations $(\sigma_{\mathcal{Q}})_{\mathscr{Q}}$ are convergent in the operator norm of $\mathcal{L}(\mathcal{H}^{\mathcal{C}})$. The multiplication operation in $\mathcal{K}(\mathcal{H}^{\mathcal{C}})$ coincides with the well-defined convolution $(\sigma^1\odot\sigma^2)_{AB} := \sum_J \sigma^1_{AJ}\circ\sigma^2_{JB}$ and the involution operation coincides with the well-defined adjunction $(\sigma^\star)_{AB} = (\sigma_{BA})^*$. The enveloping unital C*-algebra C*($\mathcal{C}$) is isomorphic to the canonical unitization of of the non-unital enveloping C*-algebra $\mathcal{K}(\mathcal{C})$ and hence for every $\sigma \in$ C*($\mathcal{C}$) there is a constant $k_\sigma \in \mathbb{C}$ such that $(\sigma_{\mathcal{Q}} - k_\sigma \cdot I_{\mathcal{Q}})$ converges in operator norm to $\sigma - k_\sigma \cdot I$.

The convergence in operator norm property of the net $(T_{\mathcal{Q}})_{\mathscr{Q}}$ of finite truncations of $T \in \mathcal{L}(\mathcal{H})$ is stronger than the request of "blockwise" norm convergence of the net of truncations, that in the case of W*-categories (and hence finite-dimensional C*-categories) $\mathcal{C}$, actually identifies the sections in the W*-enveloping algebra W*($\mathcal{C}$). ⌟

Proposition 4.7. *A 1-dimensional C*-category $\mathcal{C}$ canonically determines a class of unitary equivalent irreps of its (unital) C*-enveloping algebra and of its W*-envelop-*

ing algebra, on Hilbert spaces whose dimension is equal to the cardinality of the class of objects $\mathcal{C}^0$. *The enveloping W*-algebra* $W^*(\mathcal{C})$ *is a type I factor and we have a canonical injective unital* σ*-weakly dense* $*$*-homomorphism* $\iota_{\mathcal{C}} : C^*(\mathcal{C}) \to W^*(\mathcal{C})$ *preserving the previous unitary equivalence classes of irreps.*

Proof. For any 1-dimensional C*-category $\mathcal{C}$, making use of Gel'fand-Mazur theorem [7, II.1.4.3] for the diagonal C*-algebras $\mathcal{C}_{AA}$, for every object $A \in \mathcal{C}^0$, we see that there is a unique state on $\bigoplus_{A\in\mathcal{C}^0} \mathcal{C}_{AA}$, defined as a family $(\phi_A)_{A\in\mathcal{C}^0}$ of (necessarily unique) unital $*$-isomorphisms $\phi_A : \mathcal{C}_{AA} \to \mathbb{C}$. Following the proof of proposition 4.4, the enveloping C*-algebra $C^*(\mathcal{C})$ has been defined as a unital C*-subalgebra of the adjointable operators on the left-$\bigoplus_{A\in\mathcal{C}^0} \mathcal{C}_{AA}$ Hilbert C*-module $\mathcal{H}^{\mathcal{C}} = \bigoplus_{A\in\mathcal{C}^0} \mathcal{H}^{\mathcal{C}}_A$, (external) direct sum of the left-$\mathcal{C}_{AA}$ Hilbert C*-modules

$$\mathcal{H}^{\mathcal{C}}_A := \left\{ (x_{BA}) \in \bigtimes_{B\in\mathcal{C}^0} \mathcal{C}_{BA} \mid \left(\sum_{B\in\mathcal{C}^0} (x^*)_{AB} \circ x_{BA} \right)_{A\in\mathcal{C}^0} \in \mathcal{C}_{AA} \right\}$$

that, via the canonical C*-isomorphisms $\phi_A : \mathcal{C}_{AA} \to \mathbb{C}$, are actually Hilbert spaces invariant and irreducible[27] under the action of the C*-algebra $C^*(\mathcal{C})$. We only need to show that all the irreducible representations of $C^*(\mathcal{C})$ on $\mathcal{H}^{\mathcal{C}}_A$, for $A \in \mathcal{C}^0$, are unitarily equivalent. For this purpose, we observe that $\mathcal{H}^{\mathcal{C}} = \bigoplus_{A\in\mathcal{C}^0} \mathcal{H}^{\mathcal{C}}_A$ is also a right module for the original C*-category $\mathcal{C}$, via the standard (line by column) right action $R_x(\xi) := \xi \circ x \in \mathcal{H}^{\mathcal{C}}_B$, for all $\xi \in \mathcal{H}^{\mathcal{C}}_A$, $x \in \mathcal{C}_{AB}$, hence $\mathcal{R}_x(\mathcal{H}^{\mathcal{C}}_A) \subset \mathcal{H}^{\mathcal{C}}_B$ and, by associativity of composition in $\mathcal{C}$, we see that R_x is an intertwiner of representations $T(R_x(\xi)) = R_x(T(\xi))$, for $T \in C^*(\mathcal{C})$ and $\xi \in \mathcal{H}^{\mathcal{C}}_A$. For two arbitrary objects $A, B \in \mathcal{C}^0$, let $x \in \mathcal{C}_{AB}$ be a non-zero element, considering $u := x/\|x\|$, since $u \circ u^* = 1_{\mathcal{C}_{AA}}$ and $u^* \circ u = 1_{\mathcal{C}_{BB}}$, we obtain that $R_u : \mathcal{H}^{\mathcal{C}}_A \to \mathcal{H}^{\mathcal{C}}_B$ is a unitary intertwiner between the irreps of $C^*(\mathcal{C})$ on $\mathcal{H}^{\mathcal{C}}_A$ and $\mathcal{H}^{\mathcal{C}}_B$, as desired.

A 1-C*-category is necessarily a 1-W*-category; by remark 4.5, since Hilbert spaces are self-dual, also $W^*(\mathcal{C})$ is irreducibly faithfully represented, in an equivalent way, on each of the previous Hilbert spaces $\mathcal{H}^{\mathcal{C}}_A$, for $A \in \mathcal{C}^0$. It follows that $W^*(\mathcal{C})$ is necessarily a type I factor, isomorphic to $\mathcal{B}(\mathcal{H}^{\mathcal{C}}_A)$. Our direct constructions ensure the σ-weakly dense inclusion $C^*(\mathcal{C}) \subset W^*(\mathcal{C}) \subset \mathcal{L}(\mathcal{H}^{\mathcal{C}})$ and also show that under pullback by $\iota_{\mathcal{C}}$ the canonical irreducible representation of $W^*(\mathcal{C})$ restricts to the canonical irreducible representation of $C^*(\mathcal{C})$. □

As a standard byproduct of abstract category theory (see for example [24, theorem 2.3.6, corollary 2.3.7]), the previous construction of C*-enveloping algebras is functorial and underlies an adjunction.

Corollary 4.8. *There is an adjunction* $C^* \dashv \mathfrak{U}$ *of functors* $\mathscr{A} \underset{C^*}{\overset{\mathfrak{U}}{\rightleftarrows}} \mathscr{C}$ *with unit*

[27] Since $\mathcal{C}$ is full, for any pair of objects $C, D \in \mathcal{C}^0$, we can choose a unitary element $e^{CD} \in C^*(\mathcal{C})$ such that $e^{CD}_{C'D'} := 0_{\mathcal{C}_{C'D'}}$ whenever $(C', D') \neq (C, D)$; if an operator $T \in \mathcal{B}(\mathcal{H}_A)$ is commuting with all the e^{CD}, we necessarily have that $T \in \mathbb{C} \cdot \mathrm{Id}_{\mathcal{H}_A}$.

$\eta : \mathrm{Id}_{\mathscr{C}} \to \mathfrak{U} \circ \mathrm{C}^*$ defined as $\mathcal{C} \mapsto \eta^{\mathcal{C}}$, for $\mathcal{C} \in \mathscr{C}^0$, and co-unit $\epsilon : \mathrm{C}^* \circ \mathfrak{U} \to \mathrm{Id}_{\mathscr{A}}$ given, for $\mathcal{A} \in \mathscr{A}^0$, by $\mathcal{A} \mapsto \epsilon^{\mathcal{A}} := \overline{\mathfrak{U}(\iota_{\mathcal{A}})}$, the unique unital $*$-homomorphism such that $\mathfrak{U}(\epsilon^{\mathcal{A}}) \circ \eta^{\mathfrak{U}(\mathcal{A})} = \mathfrak{U}(\iota_{\mathcal{A}})$, where $\iota_{\mathcal{A}}$ is the identity map of the unital C-algebra $\mathcal{A}$.*

There is a perfectly parallel adjunction $\mathrm{W}^* \dashv \mathfrak{U}''$ of functors $\mathscr{A}'' \underset{\mathrm{W}^*}{\overset{\mathfrak{U}''}{\rightleftarrows}} \mathscr{C}''$ between the categories $\mathscr{A}''$ of W-algebras and $\mathscr{C}''$ of W*-categories, as defined in remark 4.5.*

We also need to define a "categorical-valued generalization" of the notion of transition amplitude space (a structure reminiscent of a "square root" of a K.Landsman transition probability space [21, 22]) that was originally introduced and studied by S.Gudder [18, section 4.5].[28]

Definition 4.9. *Given a C*-category $\mathcal{C}$ with objects $\mathcal{P} := \mathcal{C}^0$, and morphisms $\mathcal{E} := \mathcal{C}^1$, an **$\mathcal{E}$-valued C*-propagator on** $\mathcal{P}$ consists of a section $\gamma : \mathcal{P} \times \mathcal{P} \to \mathcal{E}$ of the bundle $\pi : \mathcal{E} \to \mathcal{P} \times \mathcal{P}$, where $\pi(x) := (t(x), s(x))$, for $x \in \mathcal{C}$, such that:*

- *$\gamma(p, p) = \iota(p) \in \mathcal{E}_{pp}$, for all $p \in \mathcal{P}$,*
- *$\gamma(p, q)^* = \gamma(q, p)$, for all $p, q \in \mathcal{P}$,*
- *there exists at least one **γ-frame**,[29] i.e. a subset $\mathcal{O} \subset \mathcal{P}$ such that*

$$\gamma(p, q) = \sum_{t \in \mathcal{O}} \gamma(p, t) \circ \gamma(t, q), \quad \forall p, q \in \mathcal{P}.$$

*A subset $\mathcal{I} \subset \mathcal{P}$ is **γ-orthonormal** if $\gamma(p, q) = 0_{\mathcal{C}_{pq}}$, whenever $p \neq q$. We say that the propagator γ is **total** if every maximal orthonormal set is a γ-frame.[30]*

Whenever $\mathcal{C}$ is a W-category, $(\mathcal{E}, \pi, \gamma, \mathcal{P})$ is a **W*-propagator**.*

*A propagator $(\mathcal{E}, \pi, \gamma, \mathcal{P})$ is **finite-dimensional** (one-dimensional) whenever its C*-category $(\mathcal{E}, \pi, \mathcal{P})$ is.[31] A finite-dimensional propagator $(\mathcal{E}, \pi, \gamma, \mathcal{P})$ is **finite** if it has a finite frame.[32] The family of total $\mathcal{E}$-valued propagators on $\mathcal{P}$ is denoted by $\mathscr{W}^0(\pi)$.*

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[28] We warn again the reader that here we are "oversimplifying" the discussion: in reality, also for the case of finite-dimensional C*-algebras, the "correct" notion of propagator requires the introduction of suitable topologies and uniformities. In this paper, we can proceed ignoring such complications, only because, in the finite-dimensional case, the restriction of the propagators to γ-frames becomes a one-dimensional C*-category.

[29] It follows that every C*-category $\mathcal{C}$ admitting a propagator must be **connected** i.e. $\dim_{\mathbb{C}} \mathcal{C}_{pq} > 0$, for all $p, q \in \mathcal{C}^0$.

[30] Every frame is always a maximal orthonormal set [18, lemma 4.20], hence for total transition amplitude spaces the two notions coincide.

[31] Any finite-dimensional C*-propagator is a W*-propagator.

[32] Notice that a finite propagator is not necessarily a finite C*-category (its family of objects is not necessarily a finite set) although it is always a finite-dimensional C*-category.

A ***geometric morphism of C*-propagators***[33] $(\mathcal{E}^1, \pi^1, \gamma^1, \mathcal{P}^1) \xrightarrow{(\xi,\Xi)} (\mathcal{E}^2, \pi^2, \gamma^2, \mathcal{P}^2)$ *consists of a function* $\mathcal{P}^2 \xrightarrow{\xi} \mathcal{P}^1$ *and a covariant* $*$*-functor*[34] $\xi^\bullet(\pi^1) \xrightarrow{\Xi} \pi^2$*, such that* $\Xi_0 = \mathrm{Id}_{\mathcal{P}^2}$ *and* $\gamma^2 = \Xi_1 \circ \xi^\bullet(\gamma^1) \circ (\mathrm{Id}_{\mathcal{P}^1})^{-1}_{\xi}$*;* ***geometric morphisms of W*-propagators*** *are similarly defined requiring the* σ*-weak continuity of the* $*$*-functor* Ξ_1*. A geometric morphism of propagators is* ***frame-preserving*** *if for any* γ^2*-frame* $\mathcal{O}$ *the image* $\xi(\mathcal{O})$ *is a* γ^1*-frame.*[35]

We denote by $\mathscr{W}$ *the category of frame-preserving geometric morphisms of total propagators with compositions and identities as previously defined in section 3 for the category* $\mathscr{B}_{FD}$*. The full subcategory of 1-dimensional total propagators is here denoted by* 1*-*$\mathscr{W}$*. The subcategory of frame-preserving geometric isomorphisms in* $\mathscr{W}$ *between finite 1-dimensional total propagators is denoted by* $\mathscr{W}_{FD}$*.*

Given a (finite) propagator $(\mathcal{E}, \pi, \gamma, \mathcal{P})$*, a section* $\sigma : \mathcal{P} \to \mathcal{E}$ *of the bundle* π *is* γ***-invariant*** *if, for any two pairs of* γ*-frames* $\mathcal{O}_1 \times \mathcal{O}_2, \mathcal{O}_3 \times \mathcal{O}_4$*, we have:*

$$\sigma(p,q) = \sum_{(t,r)\in\mathcal{O}_3\times\mathcal{O}_4} \gamma(p,r) \circ \sigma(r,t) \circ \gamma(t,q), \quad \forall (p,q) \in \mathcal{O}_1 \times \mathcal{O}_2.$$

A ***C*-section*** *is a section of a C*-propagator that, restricted to a* γ*-frame* $\mathcal{O}$*, belongs to the unital C*-enveloping algebra of the full C*-subcategory* $\mathcal{E}|_{\mathcal{O}} := \pi^{-1}(\mathcal{O}\times\mathcal{O}) \subset \mathcal{E}$ *with objects* $\mathcal{O}$*:*

$$\sigma|_{\mathcal{O}\times\mathcal{O}} \in \mathrm{C}^*(\mathcal{E}|_{\mathcal{O}}), \quad \textit{for every } \gamma\textit{-frame } \mathcal{O}.$$

A ***W*-section*** *of a W*-propagator, is a section that restricted to a* γ*-frame* $\mathcal{O}$*, belongs to the W*-enveloping algebra of the full W*-subcategory* $\mathcal{E}|_{\mathcal{O}} := \pi^{-1}(\mathcal{O}\times\mathcal{O}) \subset \mathcal{E}$ *with objects* $\mathcal{O}$*:*

$$\sigma|_{\mathcal{O}\times\mathcal{O}} \in \mathrm{W}^*(\mathcal{E}|_{\mathcal{O}}), \quad \textit{for every } \gamma\textit{-frame } \mathcal{O}.$$

A γ-invariant section is a C*-section σ of a C*-propagator (or a W*-section of a W*-propagator) if and only there exists a γ-frame $\mathcal{O}$ for which $\sigma|_{\mathcal{O}\times\mathcal{O}} \in \mathrm{C}^*(\mathcal{E}|_{\mathcal{O}})$ (respectively $\sigma|_{\mathcal{O}\times\mathcal{O}} \in \mathrm{W}^*(\mathcal{E}|_{\mathcal{O}})$ for a γ-frame $\mathcal{O}$).

[33] Again, in the case of isomorphisms, the notion here introduced could be equivalently replaced by a covariant $*$-functor (this means that we have two maps $\Xi_1 : \mathcal{E}^1 \to \mathcal{E}^2$ and $\Xi_0 : \mathcal{P}^1 \to \mathcal{P}^2$, such that $\pi^2 \circ \Xi_1 = \Xi_0 \circ \pi^1$, that satisfy $\Xi_1(\iota^1(\omega)) = \iota^2(\Xi_0(\omega))$, for all $\omega \in \mathcal{P}^1$, $\Xi_1(x^{*_1}) = \Xi_1(x)^{*_2}$, for all $x \in \mathcal{E}^1$ and $\Xi_1(x \circ^1 y) = \Xi_1(x) \circ^2 \Xi_1(y)$, for all $(x,y) \in \mathcal{E}^1 \times_{\circ^1} \mathcal{E}^1$), such that $\Xi_1 \circ \gamma^1 = \gamma^2 \circ \Xi_0$.

[34] Here, we consider C*-categories as Fell bundles over discrete pair-groupoids and Ξ is a covariant $*$-functor, that preserves the transition amplitudes, from the C*-category $\xi^\bullet(\pi^1)$, the (ξ,ξ)-pull-back of the C*-category π^1, to the C*-category π^2.

[35] In the case of 1-dimensional propagators, since 1-C*-categories are full, every $*$-functor between them is necessarily full and faithful and hence a fibrewise linear isomorphism. As a consequence, for geometric morphisms of 1-dimensional propagators, $\xi : \mathcal{P}^2 \to \mathcal{P}^1$ preserves orthogonality, but in general does not necessarily send frames in frames (the surjectivity of ξ is a sufficient condition). Frame-preserving geometric morphisms of 1-dimensional propagators are necessarily isomorphisms when restricted between corresponding frames.

We proceed now to establish an adjunction between a category of propagators and a category of irreps-classes of unital C*-algebras. In view of further future applications, we will develop this material at a level of generality that is much higher than strictly required for our finite-dimensional situation.

Proposition 4.10. *The family of γ-invariant C*-sections*

$$\Gamma_\gamma(\pi) := \{\sigma : \mathcal{P}\times\mathcal{P} \to \mathcal{E} \mid \pi\circ\sigma = \mathrm{Id}_{\mathcal{P}},\ \gamma\text{-invariant C*-section}\}$$

of a (finite) C-propagator $(\mathcal{E},\pi,\gamma,\mathcal{P})$, is a unital (finite-dimensional) C*-algebra with the following operations and norm:*

$$(\sigma+\tau)(p,q) := \sigma(p,q) +_{\mathcal{E}} \tau(p,q), \quad (\alpha\cdot\sigma)(p,q) := \alpha\cdot_{\mathcal{E}}\sigma(p,q),$$
$$(\sigma^\star)(p,q) := (\sigma(q,p))^{*_{\mathcal{E}}} \quad (\sigma\odot\tau)(p,q) := \sum_{t\in\mathcal{O}}\sigma(p,t)\circ_{\mathcal{E}}\tau(t,q),$$
$$\|\sigma\|_{\Gamma_\gamma(\pi)} := \|\sigma|_{\mathcal{O}\times\mathcal{O}}\|_{C^*(\mathcal{E}|_{\mathcal{O}})},$$

where $\alpha\in\mathbb{C}$, $\sigma,\tau\in\Gamma_\gamma(\pi)$, $p,q\in\mathcal{P}$, $\mathcal{O}$ is a γ-frame.
The C-algebra $\Gamma_\gamma(\pi)$ is the* ***convolution C*-algebra of the C*-propagator*** $(\mathcal{E},\pi,\gamma,\mathcal{P})$.

Similarly, the family of γ-invariant W-sections of a (finite) W*-propagator*

$$\Gamma''_\gamma(\pi) := \{\sigma : \mathcal{P}\times\mathcal{P} \to \mathcal{E} \mid \pi\circ\sigma = \mathrm{Id}_{\mathcal{P}},\ \gamma\text{-invariant W*-section}\}$$

is a (finite-dimensional) W-algebra with the operations and norms defined above. The W*-algebra $\Gamma''_\gamma(\pi)$ is the* ***convolution W*-algebra of the W* propagator****.*

Proof. From the definition of propagator, we have $\gamma|_{\mathcal{O}_1\times\mathcal{O}_2}\odot\gamma|_{\mathcal{O}_2\times\mathcal{O}_3} = \gamma|_{\mathcal{O}_1\times\mathcal{O}_3}$, $\gamma|_{\mathcal{O}_1\times\mathcal{O}_2} = \gamma|^\star_{\mathcal{O}_2\times\mathcal{O}_1}$ for all γ-frames $\mathcal{O}_1,\mathcal{O}_2,\mathcal{O}_3$, and $\gamma|_{\mathcal{O}\times\mathcal{O}}$ is an "identity matrix" on the γ-frame $\mathcal{O}$, hence $\gamma\in\Gamma_\gamma(\pi)$. A direct computation shows that the "convolution" multiplication $\odot$ and the norm $\|\cdot\|_{\Gamma_\gamma(\pi)}$ do not depend on the choice of the γ-frame $\mathcal{O}$. Using remark 4.6, given a γ-frame $\mathcal{O}$, we see that the map $\sigma\mapsto\sigma|_{\mathcal{O}\times\mathcal{O}}$ is actually a bijective map between $\Gamma_\gamma(\pi)$ and the C*-enveloping algebra $C^*(\mathcal{E}|_{\mathcal{O}})$, that is also an isometric $*$-homomorphism, hence $\Gamma_\gamma(\pi)$, with the given operations and norm, is a C*-algebra with identity γ. Similarly, for a W*-propagator, since $\Gamma_\gamma(\pi)$ is σ-weakly dense in $\Gamma''_\gamma(\pi)$ and $C^*(\mathcal{E}|_{\mathcal{O}})$ is σ-weakly dense in $W^*(\mathcal{E}|_{\mathcal{O}})$, the previously defined isomorphism of C*-algebras $\sigma\mapsto\sigma|_{\mathcal{O}\times\mathcal{O}}$ uniquely extends to an isomorphism of W*-algebras from $\Gamma''_\gamma(\pi)$ to the W*-enveloping algebra $W^*(\mathcal{E}|_{\mathcal{O}})$. □

Proposition 4.11. *A 1-dimensional propagator $(\mathcal{E},\pi,\gamma,\mathcal{P})$ uniquely determines a unitary equivalence class $[\pi,\gamma]\in\mathcal{X}_{\Gamma_\gamma(\pi)}$ of irreps of its convolution C*-algebra $\Gamma_\gamma(\pi)$ and hence a unique unitary equivalence class $[\pi,\gamma]''\in\mathcal{X}_{\Gamma''_\gamma(\pi)}$ of the W*-algebra $\Gamma''_\gamma(\pi)$.*[36]

There is canonical unital injective $*$-homomorphism $\iota_{(\pi,\gamma)} : \Gamma_\gamma(\pi)\to\Gamma''_\gamma(\pi)$ with σ-weakly dense image, from the convolution C-algebra to the convolution W*-algebra of the 1-dimensional propagator. The map $\iota_{(\pi,\gamma)}$ satisfies $\iota^\bullet_{(\pi,\gamma)}([\pi,\gamma]'')\subset[\pi,\gamma]$.*

[36] For finite propagators, since the convolution C*-algebra is simple finite-dimensional, it has a unique irreducible representation up to equivalence.

Proof. By proposition 4.7, for every γ-frame $\mathcal{O}$, we already have a canonical unitary equivalence class of irreps for the enveloping C*-algebra $C^*(\mathcal{E}|_{\mathcal{O}})$ of the 1-dimensional C*-category $\mathcal{E}|_{\mathcal{O}}$.

Since $\Gamma_\gamma(\pi)$ is isomorphic to $C^*(\mathcal{E}_{\mathcal{O}})$, for any γ-frame $\mathcal{O}$, via the isomorphism $|_{\mathcal{O}} : \sigma \mapsto \sigma|_{\mathcal{O}\times\mathcal{O}}$, and such isomorphisms are equivariant under the adjoint action of the pair-groupoid with objects the γ-frames and morphisms $\mathcal{O}_2 \xrightarrow{\gamma|_{\mathcal{O}_1\times\mathcal{O}_2}} \mathcal{O}_1$ with convolution composition, $|_{\mathcal{O}_1} = \mathrm{Ad}_{\gamma_{\mathcal{O}_1\times\mathcal{O}_2}} \circ |_{\mathcal{O}_2}$, we have a unique unitary equivalence class of irreps for $\Gamma_\gamma(\pi)$.

Similarly, by proposition 4.7, for every γ-frame $\mathcal{O}$, we also have a canonical unitary equivalence class of irreps of the eveloping W*-algebra $W^*(\mathcal{E}|_{\mathcal{O}})$. Since $\Gamma''_\gamma(\pi)$ is canonically isomorphic to $W^*(\mathcal{E}|_{\mathcal{O}})$, and by the previous argument $\mathrm{Ad}_{\gamma_{\mathcal{O}_1\times\mathcal{O}_2}}$ is a canonical W*-isomorphism between $W^*(\mathcal{E}|_{\mathcal{O}_2})$ and $W^*(\mathcal{E}|_{\mathcal{O}_2})$, for any pair of γ-frames $\mathcal{O}_1, \mathcal{O}_2$, we have a unique unitary equivalence class of irreps for $\Gamma''_\gamma(\pi)$.

The σ-weakly dense inclusion $\iota_{\mathcal{E}|_{\mathcal{O}}} : C^*(\mathcal{E}|_{\mathcal{O}}) \to W^*(\mathcal{E}|_{\mathcal{O}})$ in proposition 4.7 is $\mathrm{Ad}_{\gamma_{\mathcal{O}_1\times\mathcal{O}_2}}$-covariant and hence it uniquely induces the map $\iota_{(\pi,\gamma)}$ that "preserves the canonical irrep classes" i.e. $\iota_{(\pi,\gamma)} \circ \varpi \in [\pi, \gamma]$, for all $\varpi \in [\pi, \gamma]''$. □

Remark 4.12. For a 1-dimensional propagator $(\mathcal{E}.\pi, \gamma, \mathcal{P})$, we have a natural W*-isomorphism between the W*-algebra $\Gamma''_\gamma(\pi)$ of γ-invariant W*-sections and $(\Gamma_\gamma(\pi))''_{[\pi,\gamma]}$, the W*-algebra of "orbits", of $\Gamma_\gamma(\pi)$ under unitary equivalence of the irreducible representations $\omega \in [\pi, \gamma]$.

By the previous proposition, we know that $\Gamma_\gamma(\pi)$ is σ-weakly dense in $\Gamma''_\gamma(\pi)$ with $\iota^\bullet_{(\pi,\gamma)}([\pi,\gamma]'') \subset [\pi,\gamma]$. Furthermore, following the arguments in remark 2.1 and later in remark 4.16, we have a canonical unital $*$-homomorphism $\varpi_{[\pi,\gamma]} : \Gamma_\gamma(\pi) \to (\Gamma_\gamma(\pi))''_{[\pi,\gamma]}$ with σ-weak dense image such that $\varpi^\bullet_{[\pi,\gamma]}([\pi,\gamma]'') \subset [\pi,\gamma]$. As a consequence, the two W*-algebras $\Gamma''_\gamma(\pi) \simeq (\Gamma_\gamma(\pi))''_{[\pi,\gamma]}$ are canonically W*-isomorphic. ⌟

The construction of the convolution C*-algebra and W*-algebra of a 1-dimensional propagator and their unitary equivalent classes of irreps is functorial.

Proposition 4.13. *Let $\mathscr{R}$ denote the category whose objects are pairs $(\mathcal{A}, o)$, where $\mathcal{A}$ is a unital C*-algebra and $o \in \mathcal{X}_{\mathcal{A}}$; and whose morphisms $(\mathcal{A}^1, o_1) \xrightarrow{\phi} (\mathcal{A}^2, o_2)$ are unital $*$-homomorphisms $\phi : \mathcal{A}^1 \to \mathcal{A}^2$ such that $\phi^\bullet(o_2) \subset o_1$.*[37]

There is a covariant functor $\mathbb{\Gamma} : 1\text{-}\mathscr{W} \to \mathscr{R}$ that:

- *to every one-dimensional total propagator (π, γ) associates the pair:*

$$\mathbb{\Gamma}(\pi,\gamma) := (\Gamma''_\gamma(\pi), [\pi,\gamma]''),$$

[37] If ϖ, ϖ' are unitarily equivalent irreps of $\mathcal{A}^2$, we have that $\phi^\bullet(\varpi)$ and $\phi^\bullet(\varpi')$ are unitarily equivalent representations of $\mathcal{A}^1$ and if one of them is irreducible they are both unitarily equivalent irreps of $\mathcal{A}_1$. The condition $\phi^\bullet(o_2) \subset o_1$ is equivalent to the existence of an irrep $\varpi \in o_2$ such that $\phi^\bullet(\varpi) = \varpi \circ \phi \in o_1$. An irrep-preserving unital $*$-homomorphism $\phi : \mathcal{A}^1 \to \mathcal{A}^2$ also preserves unitary equivalence, as a consequence the ϕ-pull-back map between irreps induces a well-defined quotient map $[\varpi] \mapsto [\varpi \circ \phi]$.

- *to a frame-preserving geometric morphism of propagators* $(\pi^1,\gamma^1)\xrightarrow{(\xi,\Xi)}(\pi^2,\gamma^2)$ *in* $\mathscr{W}$ *associates the morphism* $\mathbb{\Gamma}(\pi^1,\gamma^1)\xrightarrow{\mathbb{\Gamma}_{(\xi,\Xi)}}\mathbb{\Gamma}(\pi^2,\gamma^2)$ *in* $\mathscr{R}$ *given by:*

$$\mathbb{\Gamma}_{(\xi,\Xi)}:\sigma\mapsto\Xi\circ\xi^\bullet(\sigma)\circ(\mathrm{Id}_{\mathcal{P}^1})_\xi^{-1},\quad\forall\sigma\in\Gamma''_{\gamma^1}(\pi^1).$$

Proof. By footnote 35, a frame-preserving geometric morphism of 1-dimensional propagators necessarily restricts to an isomorphism between the γ_2-frames $\mathcal{O}$ and the γ^1-frames $\xi(\mathcal{O})$. As a consequence, we have $\mathbb{\Gamma}_{(\xi,\Xi)}(\sigma)\in\Gamma''_{\gamma^2}(\pi^2)$ for $\sigma\in\Gamma''_{\gamma^1}(\pi^1)$, i.e. the section $\Xi\circ\xi^\bullet(\sigma)\circ(\mathrm{Id}_{\mathcal{P}^1})_\xi^{-1}$ is a W*-section that is γ^2-invariant.

A direct computation shows that $\mathbb{\Gamma}_{(\xi,\Xi)}:\Gamma''_{\gamma^1}(\pi^1)\to\Gamma''_{\gamma^2}(\pi^2)$ is a unital $*$-homomorphism of unital C*-algebras (and actually an isomorphism of W*-algebras) and that $(\mathbb{\Gamma}_{(\xi,\Xi)})^\bullet([\pi^2,\gamma^2]'')=[\pi^1,\gamma^1]''$.

Finally, $\mathbb{\Gamma}$ is covariant: $\mathbb{\Gamma}_{(\xi_1,\Xi_1)\circ(\xi_2,\Xi_2)}=\mathbb{\Gamma}_{(\xi_1,\Xi_1)}\circ\mathbb{\Gamma}_{(\xi_2,\Xi_2)}$, $\quad\mathbb{\Gamma}_{(\mathrm{Id}_\mathcal{P},(\mathrm{Id}_\mathcal{P})_\pi)}=\mathrm{Id}_{\Gamma_\gamma(\pi)}$. □

We will denote by $\mathbb{\Gamma}^{FD}:\mathscr{W}_{FD}\to\mathscr{R}_{FD}$ the "restriction" of the functor $\mathbb{\Gamma}$ from the sub-category $\mathscr{W}_{FD}$ of isomorphisms of finite 1-dimensional total propagators and the subcategory $\mathscr{R}_{FD}$ of isomorphisms in $\mathscr{R}$ for finite-dimensional C*-algebras.

Making full usage of the notation introduced in remark 2.1 we have the following spectral construction:

Proposition 4.14. *To a pair* $(\mathcal{A},o)$*, with* $\mathcal{A}$ *a unital C*-algebra, and* $o\in\mathcal{X}_\mathcal{A}$ *a unitary equivalence class of irreducible representations of* $\mathcal{A}$*, there is an associated* ***spectral transition amplitude propagator*** $\Sigma_o(\mathcal{A}):=((\mathcal{E}_\mathcal{A})_o,(\pi_\mathcal{A})_o,(\gamma_\mathcal{A})_o,(\mathcal{P}_\mathcal{A})_o)$ *specified as follows (see remark 2.1 for the notation):*

- $(\mathcal{P}_\mathcal{A})_o:=\chi_\mathcal{A}^{-1}(o)=\{\omega\in\mathcal{P}_\mathcal{A}\mid[\varpi_\omega]=o\}$ *is the family of pure states* $\omega\in\mathcal{P}_\mathcal{A}$ *whose GNS-irrep* ϖ_ω *belongs to the equivalence class* $o\in\mathcal{X}_\mathcal{A}$*,*
- $(\mathcal{E}_\mathcal{A})_o:=\biguplus_{(\omega,\rho)\in(\mathcal{P}_\mathcal{A})_o\times(\mathcal{P}_\mathcal{A})_o}(\mathcal{E}_\mathcal{A})_{\omega\rho}$*, where* $(\mathcal{E}_\mathcal{A})_{\omega\rho}:=\mid\omega\rangle\langle\omega\mid\mathcal{A}''_o\mid\rho\rangle\langle\rho\mid$*,*
- $(\pi_\mathcal{A})_o:(\mathcal{E}_\mathcal{A})_o\to(\mathcal{P}_\mathcal{A})_o\times(\mathcal{P}_\mathcal{A})_o$*, with* $(\pi_\mathcal{A})_o^{-1}((\omega,\rho)):=(\mathcal{E}_\mathcal{A})_{\omega\rho}$*,*
- $(\gamma_\mathcal{A})_o:(\mathcal{P}_\mathcal{A})_o\times(\mathcal{P}_\mathcal{A})_o\to(\mathcal{E}_\mathcal{A})_o$ *with* $(\gamma_\mathcal{A})_o(\omega,\rho):=\mid\omega\rangle\langle\omega\mid 1_{\mathcal{A}''_o}\mid\rho\rangle\langle\rho\mid$*, for* $\omega,\rho\in(\mathcal{P}_\mathcal{A})_o$*.*

The spectral transition amplitude propagator is one-dimensional and total.

Proof. The "bundle" $((\mathcal{E}_\mathcal{A})_o,(\pi_\mathcal{A})_o,(\mathcal{P}_\mathcal{A})_o\times(\mathcal{P}_\mathcal{A})_o)$ over the pair groupoid base-space $(\mathcal{P}_\mathcal{A})_o\times(\mathcal{P}_\mathcal{A})_o$ becomes a C*-category, with hom-sets consisting of the normed spaces $((\mathcal{E}_\mathcal{A})_o)_{\omega\rho}\subset\mathcal{A}''_o$, with the following operations:
$\Big(\mid\omega\rangle\langle\omega\mid T\mid\zeta\rangle\langle\zeta\mid\Big)\circ\Big(\mid\zeta\rangle\langle\zeta\mid S\mid\rho\rangle\langle\rho\mid\Big):=\mid\omega\rangle\langle\omega\mid\Big(T\mid\zeta\rangle\langle\zeta\mid S\Big)\mid\rho\rangle\langle\rho\mid$,
$\Big(\mid\omega\rangle\langle\omega\mid T\mid\zeta\rangle\langle\zeta\mid\Big)^*:=\mid\rho\rangle\langle\rho\mid T^*\mid\omega\rangle\langle\omega\mid,\quad\forall\omega,\zeta,\rho\in(\mathcal{P}_\mathcal{A})_o,\quad\forall T,S\in\mathcal{A}''_o$,
$\iota(\omega):=1_{((\mathcal{E}_\mathcal{A})_o)_{\omega\omega}}$, the identity of C*-algebra $((\mathcal{E}_\mathcal{A})_o)_{\omega\omega}\simeq\mathbb{C}$, $\forall\omega\in(\mathcal{P}_\mathcal{A})_o$.

We need only to show that $(\gamma_{\mathcal{A}})_o$ satisfies the propagator axioms:

$$(\gamma_{\mathcal{A}})_o(\omega,\omega) = |\,\omega\,\rangle\langle\,\omega\,|\,1_{\mathcal{A}''_o}\,|\,\omega\,\rangle\langle\,\omega\,| = 1_{((\mathcal{E}_{\mathcal{A}})_o)_{\omega\omega}} = \iota(\omega),$$

$$(\gamma_{\mathcal{A}})_o(\omega,\rho)^* = \Big(\,|\,\omega\,\rangle\langle\,\omega\,|\,1_{\mathcal{A}''_o}\,|\,\rho\,\rangle\langle\,\rho\,|\,\Big)^* = |\,\rho\,\rangle\langle\,\rho\,|\,1_{\mathcal{A}''_o}\,|\,\omega\,\rangle\langle\,\omega\,| = (\gamma_{\mathcal{A}})_o(\rho,\omega),$$

since $o \in \mathcal{X}_{\mathcal{A}}$ and for $\omega \in o$ we have $\mathcal{A}''_o \simeq \mathcal{B}(\mathcal{H}_\omega)$, frames are in bijective correspondence with orthonormal sets in $\mathcal{H}_\omega$ and hence $((\pi_{\mathcal{A}})_o, (\gamma_{\mathcal{A}})_o)$ is a 1-dimensional total propagator. □

The previous construction of the spectral transition amplitude propagators of irrep-classes of unital C*-algebras can be made into a covariant functor.

Proposition 4.15. *There is a covariant functor $\mathbb{\Sigma} : \mathscr{R} \to 1\text{-}\mathscr{W}$ defined as follows:*

- *to every pair $(\mathcal{A}, o) \in \mathscr{R}$, $\mathbb{\Sigma}$ associates the spectral transition amplitude propagator constructed in proposition 4.14, $\mathbb{\Sigma}(\mathcal{A}, o) := \Sigma_o(\mathcal{A})$,*
- *to every morphism $(\mathcal{A}_1, o_1) \xrightarrow{\phi} (\mathcal{A}_2, o_2)$ in $\mathscr{R}$, $\mathbb{\Sigma}$ associates a morphism in $1\text{-}\mathscr{W}$ $\mathbb{\Sigma}(\mathcal{A}_1, o_1) \xrightarrow{\mathbb{\Sigma}_\phi} \mathbb{\Sigma}(\mathcal{A}_2, o_2)$ given by $\mathbb{\Sigma}_\phi := (\xi^\phi, \Xi^\phi)$ where: $\xi^\phi : (\mathcal{P}_{\mathcal{A}_2})_{o_2} \to (\mathcal{P}_{\mathcal{A}_1})_{o_1}$ is the ϕ-bull-back of pure states $\xi^\phi(\omega) := \omega \circ \phi$, for $\omega \in (\mathcal{P}_{\mathcal{A}_2})_{o_2}$; and the map $\Xi^\phi : (\xi^\phi)^\bullet((\mathcal{E}_{\mathcal{A}_1})_{o_1}) \to (\mathcal{E}_{\mathcal{A}_2})_{o_2}$ is fiberwise defined for all $\omega, \rho \in (\mathcal{P}_{\mathcal{A}_2})_{o_2}$ as $(\Xi^\phi)_{\omega\rho} : (\xi^\phi)^\bullet((\mathcal{E}_{\mathcal{A}_1})_{o_1})|_{\omega\rho} = ((\mathcal{E}_{\mathcal{A}_1})_{o_1})|_{\xi^\phi(\omega)\xi^\phi(\rho)} \to ((\mathcal{E}_{\mathcal{A}_2})_{o_2})|_{\omega\rho}$ via the map $|\,\omega\circ\phi\,\rangle\langle\,\omega\circ\phi\,|\,\varpi_{o_1}(x)\,|\,\rho\circ\phi\,\rangle\langle\,\rho\circ\phi\,| \mapsto |\,\omega\,\rangle\langle\,\omega\,|\,\varpi_{o_2}\circ\phi(x)\,|\,\rho\,\rangle\langle\,\rho\,|$, for $x \in \mathcal{A}_1$.*

Proof. Since ϕ satisfies $\phi^\bullet(o_2) \subset o_1$, the map $\xi^\phi : (\mathcal{P}_{\mathcal{A}_2})_{o_2} \to (\mathcal{P}_{\mathcal{A}_1})_{o_1}$ and the fibrewise linear map $\Xi^\phi : (\xi^\phi)^\bullet((\mathcal{E}_{\mathcal{A}_1})_{o_1}) \to (\mathcal{E}_{\mathcal{A}_2})_{o_2}$ are well-defined.

The fact that Ξ^ϕ is a $*$-functor and that $\Xi^\phi \circ (\xi^\phi)^\bullet((\gamma_{\mathcal{A}_1})_{o_1}) \circ (\mathrm{Id}_{(\mathcal{P}_{\mathcal{A}_1})_{o_1}})^{-1}_{\xi^\phi} = (\gamma_{\mathcal{A}_2})_{o_2}$ follow, by direct computation, since $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ is a unital $*$-homomorphism.

Since $\phi^\bullet(o_2) \subset o_1$, we see that (ξ^ϕ, Ξ^ϕ) is a frame-preserving morphism in $1\text{-}\mathscr{W}$.

The covariance of $\mathbb{\Sigma}$: $\mathbb{\Sigma}_{\phi\circ\psi} = \mathbb{\Sigma}_\phi \circ \mathbb{\Sigma}_\psi$, $\mathbb{\Sigma}_{\mathrm{Id}_{\mathcal{A}}} = (\mathrm{Id}_{(\mathcal{P}_{\mathcal{A}})_o}, (\mathrm{Id}_{(\mathcal{P}_{\mathcal{A}})_o})_{(\pi_{\mathcal{A}})_o} = \mathrm{Id}_{\mathbb{\Sigma}_o(\mathcal{A})}$ follows from a direct computation. □

We will denote by $\mathbb{\Sigma}^{FD} : \mathscr{R}_{FD} \to \mathscr{W}_{FD}$ the "restriction" of the functor $\mathbb{\Sigma}$ from the sub-category $\mathscr{R}_{FD}$ of isomorphisms in $\mathscr{R}$ for finite-dimensional C*-algebras to the subcategory $\mathscr{W}_{FD}$ of isomorphism of finite 1-dimensional total propagators.

Remark 4.16. The unital $*$-homomorphism $\varpi_o : \mathcal{A} \to \mathcal{A}''_o$ defined at the end of remark 2.1 has image $\varpi_o(\mathcal{A})$ that is σ-weakly dense in $\mathcal{A}''_o$ since $\varpi_\omega(\mathcal{A})'' = \mathcal{B}(\mathcal{H}_\omega)$, for all pure states $\omega \in o$, and $\mathcal{A}''_o \simeq \mathcal{B}(\mathcal{H}_\omega)$ via the W*-isomorphism $(T_\omega) \mapsto T_\omega$. As a consequence every pure state $\omega \in (\mathcal{P}_{\mathcal{A}})_o$ uniquely extends to a pure state $\omega'' \in \mathcal{P}_{\mathcal{A}''_o}$; furthermore, for all $\omega \in (\mathcal{P}_{\mathcal{A}})_o$, the states ω'' induce unitarily equivalent irreps of $\mathcal{A}''_o$ and hence there is a unique $o'' \in \mathcal{X}_{\mathcal{A}''_o}$ such that $\varpi_o^\bullet(o'') \subset o$ and we have a bijective the map $(\mathcal{P}_{\mathcal{A}})_o \ni \omega \mapsto \omega'' \in (\mathcal{P}_{\mathcal{A}''_o})_{o''}$.[38] It also follows that there is a canonical W*-isomorphism $\varpi''_o : \mathcal{A}''_o \to (\mathcal{A}''_o)''_{o''}$.

[38] Although there is a bijective map between $(\mathcal{P}_{\mathcal{A}})_o$ and $(\mathcal{P}_{\mathcal{A}''_o})_{o''}$, in general the map $(\mathcal{P}_{\mathcal{A}})_o \to \mathcal{P}_{\mathcal{A}''_o}$ can be very far from surjective and $\mathcal{X}_{\mathcal{A}''_o}$ can contain many other points apart from o''.

The construction of the spectral transition amplitude propagator of $(\mathcal{A}, o)$ in proposition 4.14 is "naturally invariant" under the map $(\ )'' : (\mathcal{A}, o) \mapsto (\mathcal{A}''_o, o'')$, that is actually an endo-functor of $\mathscr{R}$. This means that the functor $\mathbb{\Sigma} : (\mathcal{A}, o) \mapsto \Sigma_o(\mathcal{A})$ is naturally isomorphic to the functor $\mathbb{\Sigma} \circ (\ )'' : (\mathcal{A}''_o, o'') \mapsto \Sigma_{o''}(\mathcal{A}''_o)$ via the natural isomorphism $\Sigma_{o''}(\mathcal{A}''_o) \xrightarrow{(\theta_{(\mathcal{A},o)}, \Theta_{(\mathcal{A},o)})} \Sigma_o(\mathcal{A})$ where $\theta_{(\mathcal{A},o)} : (\mathcal{P}_\mathcal{A})_o \to (\mathcal{P}_{\mathcal{A}''_o})_{o''}$ is defined as $\theta_{(\mathcal{A},o)} : \omega \mapsto \omega''$ and $\Theta_{(\mathcal{A},o)} : \theta^\bullet_{(\mathcal{A},o)}((\pi_{\mathcal{A}''_o})_{o''}) \to (\pi_\mathcal{A})_o$ is the $\theta_{(\mathcal{A},o)}$-pull-back fibrewise identification between $(\mathcal{E}_{\mathcal{A}''_o})_{o''}$ and $(\mathcal{E}_\mathcal{A})_o$ given by

$$| \, \omega'' \rangle\langle \, \omega'' \mid \varpi''_o(T) \mid \rho'' \, \rangle\langle \, \rho'' \, | \mapsto | \, \omega \, \rangle\langle \, \omega \mid T \mid \rho \, \rangle\langle \, \rho \, |, \quad \forall T \in \mathcal{A}''_o,$$

where $\varpi''_o : \mathcal{A}''_o \to (\mathcal{A}''_o)''_{o''}$ is above mentioned canonical W*-isomorphism. ⌟

Definition 4.17. *The* ***algebraic Gel'fand transform*** *is the natural transformation* $\mathfrak{G} : \mathrm{Id}_\mathscr{R} \to \mathbb{\Gamma} \circ \mathbb{\Sigma}$ *that to a unitary equivalence class of irreps of a unital C*-algebra* $(\mathcal{A}, o) \in \mathscr{R}^0$ *associates the morphism*

$$\mathfrak{G}_{(\mathcal{A},o)} : (\mathcal{A}, o) \to (\Gamma''_{(\gamma_\mathcal{A})_o}((\pi_\mathcal{A})_o), [(\pi_\mathcal{A})_o, (\gamma_\mathcal{A})_o]'')$$

in $\mathscr{R}$ *given by* $\mathfrak{G}_{(\mathcal{A},o)}(x) := \hat{\boldsymbol{x}}$*, for all* $x \in \mathcal{A}$*, where:*

$$\hat{\boldsymbol{x}} : (\omega, \rho) \mapsto | \, \omega \, \rangle\langle \, \omega \mid \varpi_o(x) \mid \rho \, \rangle\langle \, \rho \, |, \quad \textit{for all } \omega, \rho \in \mathcal{P}_\mathcal{A} \textit{ such that } [\varpi_\omega] = o = [\varpi_\rho].$$

The previous definition is fully justified by the following lemma.

Lemma 4.18. *For* $(\mathcal{A}, o) \in \mathscr{R}^0$*, the algebraic Gel'fand transform* $\hat{\boldsymbol{x}}$ *of* $x \in \mathcal{A}$ *satisfies* $\hat{\boldsymbol{x}} \in \Gamma''_{(\gamma_\mathcal{A})_o}((\pi_\mathcal{A})_o)$*, and hence* $\mathfrak{G}_{(\mathcal{A},o)} : x \mapsto \hat{\boldsymbol{x}}$ *gives a well-defined function* $\mathfrak{G}_{(\mathcal{A},o)} : \mathcal{A} \to \Gamma''_{(\gamma_\mathcal{A})_o}((\pi_\mathcal{A})_o)$ *that is also a morphism in* $\mathscr{R}$*. Furthermore the function* $\mathfrak{G} : (\mathcal{A}, o) \mapsto \mathfrak{G}_{(\mathcal{A},o)}$ *is a natural transformation* $\mathfrak{G} : \mathrm{Id}_\mathscr{R} \to \mathbb{\Gamma} \circ \mathbb{\Sigma}$.

Proof. For all $x \in \mathcal{A}$ and for every $(\gamma_\mathcal{A})_o$-frame $\mathcal{O}$, $\hat{\boldsymbol{x}}$ is a W*-section in $W^*((\mathcal{E}_\mathcal{A})_o|_\mathcal{O})$, because, making use of remark 4.6, given any finite subset $\mathcal{Q} \subset \mathcal{O}$, the net of $\mathcal{Q}$-truncations $\hat{\boldsymbol{x}}|_\mathcal{Q}$ σ-weakly converges to the operator norm bounded section $\hat{\boldsymbol{x}}$.

From the proof of proposition 4.10, we know that $(\gamma_\mathcal{A})_o = \hat{\mathbf{1}}_\mathcal{A}$ is the identity of the unital W*-algebra $\Gamma''_{(\gamma_\mathcal{A})_o}((\pi_\mathcal{A})_o)$ and hence, for every $(\gamma_\mathcal{A})_o$-frame $\mathcal{O}$, we have that $(\gamma_\mathcal{A})_o \in W^*((\mathcal{E}_\mathcal{A})_o|_\mathcal{O})$ is the identity element of the enveloping W*-algebra of the C*-category $(\mathcal{E}_\mathcal{A})_o|_\mathcal{O}$. Since $1_{W^*(\mathcal{E}_\mathcal{A})_o|_\mathcal{O}} = \sum_{\rho \in \mathcal{O}} | \, \rho \, \rangle\langle \, \rho \, |$, by direct computation, we get $\hat{\boldsymbol{x}}|_{\mathcal{O}_2\mathcal{O}_2} = (\gamma_\mathcal{A})_o|_{\mathcal{O}_2\mathcal{O}_1} \odot \hat{\boldsymbol{x}}|_{\mathcal{O}_1\mathcal{O}_1} \odot (\gamma_\mathcal{A})_o|_{\mathcal{O}_1\mathcal{O}_2}$, that is the $(\gamma_\mathcal{A})_o$-invariance of $\hat{\boldsymbol{x}}$.

Showing that $\mathfrak{G}_{(\mathcal{A},o)} : \mathcal{A} \to \mathbb{\Gamma} \circ \mathbb{\Sigma}(\mathcal{A})$ is a unital $*$-homomorphism, requires the proof of the following properties $\widehat{\boldsymbol{x} \cdot \boldsymbol{y}} = \hat{\boldsymbol{x}} \cdot \hat{\boldsymbol{y}}$, $\widehat{\boldsymbol{x} + \boldsymbol{y}} = \hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}$, $\widehat{\boldsymbol{x}^*} = (\hat{\boldsymbol{x}})^*$ and $\hat{\mathbf{1}}_\mathcal{A} = 1_{\mathbb{\Gamma} \circ \mathbb{\Sigma}(\mathcal{A},o)}$, for all $x, y \in \mathcal{A}$, that, just evaluating all the Gel'fand transforms on pairs $\omega, \rho \in (\mathcal{P}_\mathcal{A})_o$, are immediate consequence of the definition of the operations in the spectral transition amplitude propagator given in the proof of proposition 4.14.

First of all, notice that, using the notation in proposition 4.11 and remark 4.16, given a propagator $(\pi, \gamma) \in 1\text{-}\mathscr{W}$, if $\varpi : \Gamma_{(\gamma_\mathcal{A})_o}((\pi_\mathcal{A})_o) \to \mathcal{B}(\mathcal{H}_\varpi)$ is an irreducible representation $\varpi \in [\pi, \gamma]$, we have a type I factor $\varpi(\Gamma_\gamma(\pi))'' = \mathcal{B}(\mathcal{H}_\varpi)$ and a unique normal extension $\varpi'' : \Gamma''_{(\gamma_{\mathcal{A}})_o}((\pi_{\mathcal{A}})_o) \to \mathcal{B}(\mathcal{H}_\varpi)$ to the W*-algebra $\Gamma''_{(\gamma_{\mathcal{A}})_o}((\pi_{\mathcal{A}})_o)$ with $\varpi''(\Gamma''_\gamma(\pi)) = \mathcal{B}(\mathcal{H}_\varpi)$. In this way we obtain a bijective correspondence of irreducible representations $[\pi,\gamma] \ni \varpi \mapsto \varpi'' \in [\pi,\gamma]''$. Making use of the notation introduced in 4.20 and lemma 4.21, we have $\boldsymbol{\eta}^{((\pi_{\mathcal{A}})_o,(\gamma_{\mathcal{A}})_o)} : (\mathcal{P}_{\mathcal{A}})_o \to \mathcal{P}_{\Gamma''_{(\gamma_{\mathcal{A}})_o}((\pi_{\mathcal{A}})_o)}$ and $\boldsymbol{\eta}^{((\pi_{\mathcal{A}})_o,(\gamma_{\mathcal{A}})_o)}_\omega(\hat{\mathbf{x}}) = \zeta_\omega(\hat{\mathbf{x}}(\omega,\omega)) = \omega(x)$, for all $x \in \mathcal{A}$ and $\omega \in o$, and hence $\boldsymbol{\eta}^{((\pi_{\mathcal{A}})_o,(\gamma_{\mathcal{A}})_o)}_\omega \circ \mathfrak{G}_{(\mathcal{A},o)} = \omega$; furthermore $\boldsymbol{\eta}^{((\pi_{\mathcal{A}})_o,(\gamma_{\mathcal{A}})_o)}_\omega \in [(\pi_{\mathcal{A}})_o,(\gamma_{\mathcal{A}})_o]$. Considering the irreducible $\boldsymbol{\eta}^{((\pi_{\mathcal{A}})_o,(\gamma_{\mathcal{A}})_o)}_\omega$-GNS representation $\varpi_{\boldsymbol{\eta}_\omega}$ of $\Gamma_{(\gamma_{\mathcal{A}})_o}((\pi_{\mathcal{A}})_o)$, with GNS-vector ξ, we see that $\langle \xi \mid \varpi''_{\boldsymbol{\eta}_\omega} \circ \mathfrak{G}_{(\mathcal{A},o)}(x)\xi\rangle = \boldsymbol{\eta}_\omega(\hat{\mathbf{x}}) = \omega(x) = \langle \xi_\omega \mid \varpi_\omega(x)\xi_\omega\rangle$. It follows that there exists a unique unitary operator mapping ξ_ω to ξ and intertwining the two GNS-representations above and hence $[\varpi''_{\boldsymbol{\eta}_\omega} \circ \mathfrak{G}_{(\mathcal{A},o)}] = [\varpi_\omega] = [\omega] = o$.

To prove that $\mathfrak{G}$ is a natural transformation, we need to check that, for all morphisms $(\mathcal{A}^1,o_1) \xrightarrow{\phi} (\mathcal{A}^2,o_2)$ in the category $\mathscr{R}$, we have $\mathfrak{G}_{(\mathcal{A}^2,o_2)} \circ \phi = \mathbb{\Gamma}_{(\mathbb{\Sigma}_\phi)} \circ \mathfrak{G}_{(\mathcal{A}^1,o_1)}$, i.e. $\mathbb{\Gamma}_{(\xi^\phi,\Xi^\phi)}(\hat{\mathbf{x}}) = \widehat{\boldsymbol{\phi}(\mathbf{x})}$, for all $x \in \mathcal{A}^1$, and this just means that for all $\omega,\rho \in (\mathcal{P}_{\mathcal{A}^2})_{o_2}$,

$$\Xi^\phi_{\omega\rho}(\mid \omega\circ\phi\,\rangle\langle\, \omega\circ\phi \mid \varpi_{o_1}(x) \mid \rho\circ\phi\,\rangle\langle\,\rho\circ\phi \mid) = \mid \omega\,\rangle\langle\,\omega \mid \varpi_{o_2}\circ\phi(x) \mid \rho\,\rangle\langle\,\rho \mid,$$

which is the defining property of $\mathbb{\Sigma}_\phi = (\xi^\phi,\Xi^\phi)$ in proposition 4.15. □

Remark 4.19. In general the algebraic Gel'fand transform $\mathcal{A} \xrightarrow{\mathfrak{G}_{(\mathcal{A},o)}} \Gamma''_{(\gamma_{\mathcal{A}})_o}((\pi_{\mathcal{A}})_o)$ is neither injective nor surjective. If the unital C*-algebra $\mathcal{A}$ is not primitive there is for sure $o \in \mathcal{X}_{\mathcal{A}}$ for which $\mathfrak{G}_{(\mathcal{A},o)}$ is not injective since, for $\pi \in o$, $\operatorname{Ker}\pi$ is non-trivial. The surjectivity of the algebraic Gel'fand transform is present exactly when $\mathcal{A}_o$ is isomorphic to a W*-enveloping algebra of a pair-groupoid. The full subcategory $\overline{\mathscr{R}}$ of $\mathscr{R}$, with objects unitary equivalent classes or irreps of unital C*-algebras whose Gel'fand transform is an isomorphism in $\mathscr{R}$, necessarily contains all the primitive finite-dimensional C*-algebras (hence $\overline{\mathscr{R}}_{FD} = \mathscr{R}_{FD}$), since a primitive finite-dimensional C*-algebra is necessarily isomorphic to a matrix algebra. ⌟

Definition 4.20. *The* ***algebraic evaluation transform*** $\mathfrak{E} : \mathbb{\Sigma} \circ \mathbb{\Gamma} \to \mathrm{Id}_{1\text{-}\mathscr{W}}$ *is the natural transformation that to every one-dimensional total propagator* $(\pi,\gamma) \in 1\text{-}\mathscr{W}^0$ *associates the morphism of propagators* $\mathfrak{E}_{(\pi,\gamma)} : ((\pi_{\Gamma''_\gamma(\pi)})_{[\pi,\gamma]''}, (\gamma_{\Gamma''_\gamma(\pi)})_{[\pi,\gamma]''}) \to (\pi,\gamma)$ *in* $1\text{-}\mathscr{W}$ *given by* $\mathfrak{G}_{(\pi,\gamma)} := (\boldsymbol{\eta}^{(\pi,\gamma)},\boldsymbol{\Omega}^{(\pi,\gamma)})$ *where:*

$\boldsymbol{\eta}^{(\pi,\gamma)} : \mathcal{P} \to (\mathcal{P}_{\Gamma''_\gamma(\pi)})_{[\pi,\gamma]''}$, *with* $\boldsymbol{\eta}^{(\pi,\gamma)}_p(\sigma) := \zeta^\pi_p \circ \sigma(p,p)$, $\forall p \in \mathcal{P}$, $\forall \sigma \in \Gamma''_\gamma(\pi)$,

with $\zeta^\pi_p : \mathcal{E}_{pp} \to \mathbb{C}$ *denoting the unique Gel'fand-Mazur isomorphism*;

$\boldsymbol{\Omega}^{(\pi,\gamma)} : (\boldsymbol{\eta}^{(\pi,\gamma)})^\bullet((\mathcal{E}_{\Gamma''_\gamma(\pi)})_{[\pi,\gamma]''}) \to \mathcal{E}$, *fibrewise given*, $\forall p,q \in \mathcal{P}$, $\forall \sigma \in \Gamma''_\gamma(\pi)$, *by*

$\boldsymbol{\Omega}^{(\pi,\gamma)}_{pq} : ((\mathcal{E}_{\Gamma''_\gamma(\pi)})_{[\pi,\gamma]''})_{\boldsymbol{\eta}^{(\pi,\gamma)}_p \boldsymbol{\eta}^{(\pi,\gamma)}_q} \to \mathcal{E}_{pq}$,

$\boldsymbol{\Omega}^{(\pi,\gamma)}_{pq} : \mid \boldsymbol{\eta}^{(\pi,\gamma)}_p\,\rangle\langle\,\boldsymbol{\eta}^{(\pi,\gamma)}_p \mid \varpi_{[\pi,\gamma]''}(\sigma) \mid \boldsymbol{\eta}^{(\pi,\gamma)}_q\,\rangle\langle\,\boldsymbol{\eta}^{(\pi,\gamma)}_q \mid \mapsto \sigma_{pq}$.

Again, the previous definition is fully justified by the following lemma.

Lemma 4.21. *For* $(\mathcal{E}, \pi, \gamma, \mathcal{P}) \in 1\text{-}\mathscr{W}^0$ *and* $p \in \mathcal{P}$*, we have* $\boldsymbol{\eta}_p^{(\pi,\gamma)} \in (\mathcal{P}_{\Gamma''_\gamma(\pi)})_{[\pi,\gamma]''}$*, hence* $\boldsymbol{\eta}^{(\pi,\gamma)} : \mathcal{P} \to (\mathcal{P}_{\Gamma''_\gamma(\pi)})_{[\pi,\gamma]''}$ *is well-defined.*
If $\sigma, \tau \in \Gamma''_\gamma(\pi)$ *and* $|\,\boldsymbol{\eta}_p\,\rangle\langle\,\boldsymbol{\eta}_p\,|\,\varpi_{[\pi,\gamma]''}(\sigma)\,|\,\boldsymbol{\eta}_q\,\rangle\langle\,\boldsymbol{\eta}_q\,| = |\,\boldsymbol{\eta}_p\,\rangle\langle\,\boldsymbol{\eta}_p\,|\,\varpi_{[\pi,\gamma]''}(\tau)\,|\,\boldsymbol{\eta}_q\,\rangle\langle\,\boldsymbol{\eta}_q\,|$*, for any* $p, q \in \mathcal{P}$*, we have* $\sigma_{pq} = \tau_{pq}$*, hence* $\boldsymbol{\Omega}^{(\pi,\gamma)} : (\boldsymbol{\eta}^{(\pi,\gamma)})^\bullet((\mathcal{E}_{\Gamma''_\gamma(\pi)})_{[\pi,\gamma]''}) \to \mathcal{E}$ *is a well-defined fibrewise linear map that is also an object-preserving* $*$*-functor such that* $\gamma = \boldsymbol{\Omega}^{(\pi,\gamma)} \circ (\boldsymbol{\eta}^{(\pi,\gamma)})^\bullet(\gamma_{\Gamma''_\gamma(\pi)}) \circ (\mathrm{Id}_{\mathcal{P}_{\Gamma''_\gamma(\pi)}})^{-1}_{\boldsymbol{\eta}^{(\pi,\gamma)}}$ *and hence* $\mathfrak{E}_{(\pi,\gamma)} := (\boldsymbol{\eta}^{(\pi,\gamma)}, \boldsymbol{\Omega}^{(\pi,\gamma)})$ *is a geometric morphism of propagators in* $1\text{-}\mathscr{W}$*. Furthermore, the map* $\mathfrak{E} : (\pi, \gamma) \mapsto \mathfrak{E}_{(\pi,\gamma)}$ *is a natural transformation* $\mathfrak{E} : \mathbb{\Sigma} \circ \mathbb{\Gamma} \to \mathrm{Id}_{1\text{-}\mathscr{W}}$*.*

Proof. For every $p \in \mathcal{P}$, the fiber $\mathcal{E}_{pp}$ is a one-dimensional C*-algebra and there is a unique Gel'fand-Mazur isomorphism $\zeta_p^\pi : \mathcal{E}_{pp} \to \mathbb{C}$ and hence $\boldsymbol{\eta}_p^{(\pi,\gamma)} : \Gamma''_\gamma(\pi) \to \mathbb{C}$ is a well-defined linear map. The map $\boldsymbol{\eta}_p^{(\pi,\gamma)}$ is unital:

$$\boldsymbol{\eta}_p^{(\pi,\gamma)}(1_{\Gamma''_\gamma(\pi)}) = \boldsymbol{\eta}_p^{(\pi,\gamma)}(\gamma) = \zeta_p^\pi(\gamma(p,p)) = \zeta_p^\pi(1_{\mathcal{E}_{pp}}) = 1_{\mathbb{C}};$$

and positive, since for any γ-frame $\mathcal{O}$, we have:

$$\boldsymbol{\eta}_p^{(\pi,\gamma)}(\sigma^\star \odot \sigma) = \zeta_p^\pi\left(\sum_{q\in\mathcal{O}} \sigma(q,p)^* \circ \sigma(q,p)\right) \geq 0_{\mathbb{C}}.$$

Since (π, γ) is total, by Zorn's lemma, for every $p \in \mathcal{X}$, there is a γ-frame $\mathcal{O}$ such that $p \in \mathcal{O}$. Let $\psi : \sigma \mapsto \sigma|_{\mathcal{O}}$ be the isomorphism between $\Gamma''_\gamma(\pi)$ and $\mathrm{W}^*(\mathcal{E}|_{\mathcal{O}})$, the enveloping W*-algebra of the W*-category $\mathcal{E}|_{\mathcal{O}}$. Since compactly supported sections of $\mathcal{E}|_{\mathcal{O}}$ correspond to finite rank operators on the $\boldsymbol{\eta}_p^{(\pi,\gamma)} \circ \psi^{-1}$-GNS Hilbert space, the image $\varpi_{\boldsymbol{\eta}_p^{(\pi,\gamma)}\circ\psi^{-1}}(\mathrm{W}^*(\mathcal{E}|_{\mathcal{O}}))$ contains all the compact operators and hence $\varpi_{\boldsymbol{\eta}_p^{(\pi,\gamma)}\circ\psi^{-1}}$ is an irreducible representation i.e. $\boldsymbol{\eta}_p^{(\pi,\gamma)} \circ \psi^{-1}$ and $\boldsymbol{\eta}_p^{(\pi,\gamma)}$ are pure states belonging to the canonical irrep $[\pi, \gamma]''$.

From its definition, $\boldsymbol{\Omega}_{pq}^{(\pi,\gamma)}$ is a linear relation. In order to show that it is a well-defined linear map, we need to show, for $\sigma \in \Gamma''_\gamma(\pi)$, $p, q \in \mathcal{P}$, that $\widehat{\boldsymbol{\sigma}}(\boldsymbol{\eta}_p^{(\pi,\gamma)}, \boldsymbol{\eta}_q^{(\pi,\gamma)}) = 0$ implies $\sigma_{pq} = 0$. For this purpose, consider the γ-invariant C*-sections defined by $\gamma_p(p', q') := \gamma(p', p) \circ \gamma(p, q')$ and similarly $\gamma_q(p', q') := \gamma(p', q) \circ \gamma(q, q')$ and notice that $\varpi_{[\pi,\gamma]}(\gamma_p) = |\,\boldsymbol{\eta}_p^{(\pi,\gamma)}\,\rangle\langle\,\boldsymbol{\eta}_p^{(\pi,\gamma)}\,|$ and similarly for γ_q. Since

$$0 = \widehat{\boldsymbol{\sigma}}(\boldsymbol{\eta}_p^{(\pi,\gamma)}, \boldsymbol{\eta}_q^{(\pi,\gamma)}) = |\,\boldsymbol{\eta}_p^{(\pi,\gamma)}\,\rangle\langle\,\boldsymbol{\eta}_p^{(\pi,\gamma)}\,|\,\varpi_{[\pi,\gamma]}(\sigma)\,|\,\boldsymbol{\eta}_p^{(\pi,\gamma)}\,\rangle\langle\,\boldsymbol{\eta}_p^{(\pi,\gamma)}\,| = \varpi_{[\pi,\gamma]}\left(\gamma_p \circ \sigma \circ \gamma_q\right),$$

using the fact that $\varpi_{[\pi,\gamma]}$ is faithful, we obtain: $\sigma_{pq} = \gamma_p \circ \sigma \circ \gamma_q = 0$.

The linearity of $\boldsymbol{\Omega}^{(\pi,\gamma)}$ is explicit in its definition. Since it is acting fiberwise as $\boldsymbol{\Omega}_{pq}^{(\pi,\gamma)} : (\boldsymbol{\eta}^{(\pi,\gamma)})^\bullet(\mathcal{E}_{\Gamma''_\gamma(\pi)})_{pq} \to \mathcal{E}_{pq}$, it preserves objects, i.e. $\pi \circ \boldsymbol{\Omega}^{(\pi,\gamma)} = (\pi_{\Gamma''_\gamma(\pi)})_{\boldsymbol{\eta}^{(\pi,\gamma)}}$.

Let $e \in \mathcal{E}_{\Gamma''_\gamma(\pi)}|_{pt}$ and $f \in \mathcal{E}_{\Gamma''_\gamma(\pi)}|_{tq}$; consider any two sections $\sigma, \tau \in \Gamma''_\gamma(\pi)$, such that $\widehat{\boldsymbol{\sigma}}(\boldsymbol{\eta}_p^{(\pi,\gamma)}, \boldsymbol{\eta}_t^{(\pi,\gamma)}) = e$ and $\widehat{\boldsymbol{\tau}}(\boldsymbol{\eta}_t^{(\pi,\gamma)}, \boldsymbol{\eta}_q^{(\pi,\gamma)}) = f$. We have $\boldsymbol{\Omega}^{(\pi,\gamma)}(e) = \sigma_{pt}$ and $\boldsymbol{\Omega}^{(\pi,\gamma)}(f) = \tau_{tq}$. Let us define the new γ-invariant C*-sections $\sigma'(p', q') := \gamma(p', p) \circ \sigma(p, t) \circ \gamma(t, q')$ and similarly $\tau'(p', q') := \gamma(p', p) \circ \tau(p, t) \circ \gamma(t, q')$, for all $p', q' \in \mathcal{P}$. The new sections satisfy $\sigma'_{pt} = \sigma_{pt}$ and $\tau'_{tq} = \tau_{tq}$. Since the map $\mathbf{\Omega}^{(\pi,\gamma)}$ is a fibrewise isomorphism, we have $\widehat{\boldsymbol{\sigma}'}(\boldsymbol{\eta}_p^{(\pi,\gamma)}, \boldsymbol{\eta}_t^{(\pi,\gamma)}) = e$ and $\widehat{\boldsymbol{\tau}'}(\boldsymbol{\eta}_t^{(\pi,\gamma)}, \boldsymbol{\eta}_q^{(\pi,\gamma)}) = f$. Since $(\sigma' \odot \tau')_{pq} = \sigma'_{pt} \circ \tau'_{tq}$, we obtain $\mathbf{\Omega}_{pq}^{(\pi,\gamma)}(e \circ f) = \mathbf{\Omega}_{pt}^{(\pi,\gamma)}(e) \circ \mathbf{\Omega}_{tq}^{(\pi,\gamma)}(f)$ and hence the functoriality of $\mathbf{\Omega}^{(\pi,\gamma)}$.

Since $\left(| \boldsymbol{\eta}_p \rangle\langle \boldsymbol{\eta}_p | \varpi_{[\pi,\gamma]}(\sigma) | \boldsymbol{\eta}_q \rangle\langle \boldsymbol{\eta}_q |\right)^* = | \boldsymbol{\eta}_q \rangle\langle \boldsymbol{\eta}_q | \varpi_{[\pi,\gamma]}(\sigma^\star) | \boldsymbol{\eta}_p \rangle\langle \boldsymbol{\eta}_p |$, we have that $\mathbf{\Omega}^{(\pi,\gamma)}$ is involutive: $\mathbf{\Omega}^{(\pi,\gamma)} \circ * = * \circ \mathbf{\Omega}^{(\pi,\gamma)}$. Hence $\mathbf{\Omega}^{(\pi,\gamma)}$ is a $*$-functor.

Since we have $\mathbf{\Omega}^{(\pi,\gamma)}_{\boldsymbol{\eta}_p^{(\pi,\gamma)} \boldsymbol{\eta}_q^{(\pi,\gamma)}} \left(| \boldsymbol{\eta}_p^{(\pi,\gamma)} \rangle\langle \boldsymbol{\eta}_p^{(\pi,\gamma)} | \varpi_{[\pi,\gamma]}(\gamma) | \boldsymbol{\eta}_q^{(\pi,\gamma)} \rangle\langle \boldsymbol{\eta}_q^{(\pi,\gamma)} |\right) = \gamma(p,q)$ for all $p, q \in \mathcal{P}$, it follows that $(\boldsymbol{\eta}^{(\pi,\gamma)}, \mathbf{\Omega}^{(\pi,\gamma)})$ is a geometric morphism of propagators.

As explained in footnote 35, since $(\boldsymbol{\eta}^{(\pi,\gamma)}, \mathbf{\Omega}^{(\pi,\gamma)})$ is a geometric morphism of 1-dimensional total propagators, the map $\boldsymbol{\eta}^{(\pi,\gamma)}$ sends a given γ-frame $\mathcal{O}$, into a $\gamma_{\Gamma''_\gamma(\pi)}$-orthogonal set and $\boldsymbol{\eta}^{(\pi,\gamma)}$ is injective when restricted to γ-frames. Given a γ-frame $\mathcal{O}$, from the details in the proof of proposition 4.7, we see that $\mathrm{W}^*(\mathcal{O})$ is canonically represented in $L^2(\mathrm{W}^*(\mathcal{O})) \simeq \bigoplus_{p\in\mathcal{O}} \mathcal{H}_p^{\mathcal{E}|_\mathcal{O}}$, where (denoting by $|_\mathcal{O} : \Gamma''_\gamma(\pi) \to \mathrm{W}^*(\mathcal{O})$ the canonical restriction isomorphism) each of the orthogonal Hilbert subspaces $\mathcal{H}_p^{\mathcal{E}|_\mathcal{O}}$ coincides with the invariant irreducible $\boldsymbol{\eta}_p^{(\pi,\gamma)} \circ |_\mathcal{O}^{-1}$-GNS space of $\mathrm{W}^*(\mathcal{O})$ with GNS-vector $| \boldsymbol{\eta}_p^{(\pi,\gamma)} \rangle\langle \boldsymbol{\eta}_p^{(\pi,\gamma)} |$. From the partition of unity $\mathrm{Id}_{L^2(\mathrm{W}^*(\mathcal{O}))} = \sum_{p\in\mathcal{O}} | \boldsymbol{\eta}_p^{(\pi,\gamma)} \rangle\langle \boldsymbol{\eta}_p^{(\pi,\gamma)} |$ we have that $\boldsymbol{\eta}^{(\pi,\gamma)}(\mathcal{O})$ is a $\gamma_{\Gamma''_\gamma(\pi)}$-frame and hence $(\boldsymbol{\eta}^{(\pi,\gamma)}, \mathbf{\Omega}^{(\pi,\gamma)})$ is frame-preserving.

Given a morphism $(\mathcal{E}^1, \pi^1, \gamma^1, \mathcal{P}^1) \xrightarrow{(\xi,\Xi)} (\mathcal{E}^2, \pi^2, \gamma^2, \mathcal{P}^2)$ in 1-$\mathscr{W}$, we denote, with a slight abuse of notation:

$$
\begin{aligned}
(\mathcal{E}_\mathbb{F}, \pi_\mathbb{F}, \gamma_\mathbb{F}, \mathcal{P}_\mathbb{F}) : &= \Sigma \circ \mathbb{F}(\mathcal{E}, \pi, \gamma, \mathcal{P}) = \\
&= \left((\mathcal{E}_{\Gamma''_\gamma(\pi)})_{[\pi,\gamma]''}, (\pi_{\Gamma''_\gamma(\pi)})_{[\pi,\gamma]''}, (\gamma_{\Gamma''_\gamma(\pi)})_{[\pi,\gamma]''}, (\mathcal{P}_{\Gamma''_\gamma(\pi)})_{[\pi,\gamma]''}\right)
\end{aligned}
$$

and, for the morphism $(\mathcal{E}^1_\mathbb{F}, \pi^1_\mathbb{F}, \gamma^1_\mathbb{F}, \mathcal{P}^1_\mathbb{F}) \xrightarrow{(\xi^\mathbb{F},\Xi^\mathbb{F})} (\mathcal{E}^2_\mathbb{F}, \pi^2_\mathbb{F}, \gamma^2_\mathbb{F}, \mathcal{P}^2_\mathbb{F})$, we will use the notation $(\xi^\mathbb{F}, \Xi^\mathbb{F}) := \Sigma_{(\mathbb{F}_{(\xi,\Xi)})} = (\xi^{(\mathbb{F}_{(\xi,\Xi)})}, \Xi^{(\mathbb{F}_{(\xi,\Xi)})})$.

In order to show that $\mathfrak{E} : \Sigma \circ \mathbb{F} \to \mathrm{Id}_{1\text{-}\mathscr{W}}$ is a natural transformation, we have to prove the following $(\xi,\Xi) \circ (\boldsymbol{\eta}^{(\pi^1,\gamma^1)}, \mathbf{\Omega}^{(\pi^1,\gamma^1)}) = (\boldsymbol{\eta}^{(\pi^2,\gamma^2)}, \mathbf{\Omega}^{(\pi^2,\gamma^2)}) \circ (\xi^\mathbb{F}, \Xi^\mathbb{F})$. This means:

$$
\begin{aligned}
&\left(\boldsymbol{\eta}^{(\pi^1,\gamma^1)} \circ \xi\right) = \left(\xi^\mathbb{F} \circ \boldsymbol{\eta}^{(\pi^2,\gamma^2)}\right) : \quad \mathcal{P}^2 \xrightarrow{\delta} \mathcal{P}^1_\mathbb{F}, \\
&\left(\Xi \circ \mathbf{\Omega}^{(\pi^1,\gamma^1)} \circ \zeta^{\pi^1_\mathbb{F}}_{(\boldsymbol{\eta}^{(\pi^1,\gamma^1)},\ \xi)}\right) = \left(\mathbf{\Omega}^{(\pi^2,\gamma^2)} \circ \Xi^\mathbb{F} \circ \zeta^{\pi^1_\mathbb{F}}_{(\xi^\mathbb{F},\ \boldsymbol{\eta}^{(\pi^2,\gamma^2)})}\right) : \quad \delta^\bullet(\mathcal{E}^1_\mathbb{F}) \to \mathcal{E}^2.
\end{aligned}
$$

For the first equation above, for $q \in \mathcal{P}^2$ and $\sigma \in \Gamma''_{\gamma^1}(\pi^1)$, we have:

$$
\begin{aligned}
\boldsymbol{\eta}^{(\pi^1,\gamma^1)}_{\xi(q)}(\sigma) &= \zeta^{\pi^1}_{\xi(q)}(\sigma(\xi(q), \xi(q))) \\
(\xi^\mathbb{F} \circ \boldsymbol{\eta}^{(\pi^2,\gamma^2)})_q(\sigma) &= \boldsymbol{\eta}_q^{(\pi^2,\gamma^2)}(\mathbb{F}_{(\xi,\Xi)}(\sigma)) = \boldsymbol{\eta}_q^{(\pi^2,\gamma^2)}(\Xi \circ \xi^\bullet(\sigma) \circ (\mathrm{Id}_{\mathcal{P}^1})_\xi^{-1}) \\
&= (\zeta_q^{\pi^2} \circ \Xi_q)(\sigma(\xi(q), \xi(q))),
\end{aligned}
$$

from the Gel'fand-Mazur canonical isomorphisms $\zeta^{\pi^1}_{\xi(q)} = \zeta^{\pi^2}_q \circ \Xi_q$, the equality follows.

For the second equation, for all $p, q \in \mathcal{P}^2$ and $\sigma \in \Gamma''_{\gamma^1}(\pi^1)$, we have:

$$\begin{aligned}
&\Xi_{pq} \circ \boldsymbol{\Omega}^{(\pi^1,\gamma^1)}_{\xi(p)\xi(q)} \Big(\mid \boldsymbol{\eta}^{(\pi^1,\gamma^1)}_{\xi(p)} \rangle\langle \boldsymbol{\eta}^{(\pi^1,\gamma^1)}_{\xi(p)} \mid \varpi_{[\pi^1,\gamma^1]}[\sigma] \mid \boldsymbol{\eta}^{(\pi^1,\gamma^1)}_{\xi(q)} \rangle\langle \boldsymbol{\eta}^{(\pi^1,\gamma^1)}_{\xi(q)} \mid \Big) \\
&= \Xi_{pq}(\sigma(\xi(p), \xi(q))) \\
&= \boldsymbol{\Omega}^{(\pi^2,\gamma^2)}_{pq} \Big(\mid \boldsymbol{\eta}^{(\pi^2,\gamma^2)}_{p} \rangle\langle \boldsymbol{\eta}^{(\pi^2,\gamma^2)}_{p} \mid \varpi_{[\pi^2,\gamma^2]}[\mathbb{\Gamma}_{(\xi,\Xi)}(\sigma)] \mid \boldsymbol{\eta}^{(\pi^2,\gamma^2)}_{q} \rangle\langle \boldsymbol{\eta}^{(\pi^2,\gamma^2)}_{q} \mid \Big) \\
&= \boldsymbol{\Omega}^{(\pi^2,\gamma^2)}_{pq} \circ \Xi^{\mathbb{\Gamma}}_{\boldsymbol{\eta}^{(\pi^2,\gamma^2)}_p \boldsymbol{\eta}^{(\pi^2,\gamma^2)}_q} \Bigg(\mid \boldsymbol{\eta}^{(\pi^2,\gamma^2)}_{p} \circ \mathbb{\Gamma}_{(\xi,\Xi)} \rangle\langle \boldsymbol{\eta}^{(\pi^2,\gamma^2)}_{p} \circ \mathbb{\Gamma}_{(\xi,\Xi)} \mid \varpi_{[\pi^1,\gamma^1]}[\sigma] \\
&\qquad\qquad = \mid \boldsymbol{\eta}^{(\pi^2,\gamma^2)}_{q} \circ \mathbb{\Gamma}_{(\xi,\Xi)} \rangle\langle \boldsymbol{\eta}^{(\pi^2,\gamma^2)}_{q} \circ \mathbb{\Gamma}_{(\xi,\Xi)} \mid \Bigg) \\
&= \boldsymbol{\Omega}^{(\pi^2,\gamma^2)}_{pq} \circ \Xi^{\mathbb{\Gamma}}_{\boldsymbol{\eta}^{(\pi^2,\gamma^2)}_p \boldsymbol{\eta}^{(\pi^2,\gamma^2)}_q} \Big(\mid \boldsymbol{\eta}^{(\pi^1,\gamma^1)}_{\xi(p)} \rangle\langle \boldsymbol{\eta}^{(\pi^1,\gamma^1)}_{\xi(p)} \mid \varpi_{[\pi^1,\gamma^1]}[\sigma] \mid \boldsymbol{\eta}^{(\pi^1,\gamma^1)}_{\xi(q)} \rangle\langle \boldsymbol{\eta}^{(\pi^1,\gamma^1)}_{\xi(q)} \mid \Big),
\end{aligned}$$

hence, for all $p, q \in \mathcal{P}^2$, $\Xi_{pq} \circ \boldsymbol{\Omega}^{(\pi^1,\gamma^1)}_{\xi(p)\xi(q)} = \boldsymbol{\Omega}^{(\pi^2,\gamma^2)}_{pq} \circ \Xi^{\mathbb{\Gamma}}_{\boldsymbol{\eta}^{(\pi^2,\gamma^2)}_p \boldsymbol{\eta}^{(\pi^2,\gamma^2)}_q} : (\mathcal{E}^1_{\mathbb{\Gamma}})_{\delta p \delta(q)} \to \mathcal{E}^2_{pq}$. □

Definition 4.22. *A propagator (π, γ) in $\mathscr{W}$ is* ***algebraically saturated*** *if $\mathfrak{E}_{(\pi,\gamma)}$ is an isomorphism. The full subcategory of algebraically saturated one-dimensional propagators is denoted by $\overline{1\text{-}\mathscr{W}}$*

Theorem 4.23. *There is an adjunction $\mathbb{\Sigma} \dashv \mathbb{\Gamma}$ between the two covariant functors $\mathscr{R} \underset{\mathbb{\Gamma}}{\overset{\mathbb{\Sigma}}{\rightleftarrows}} 1\text{-}\mathscr{W}$.*

The algebraic Gel'fand transform $\mathfrak{G} : \mathrm{Id}_{\mathscr{R}} \to \mathbb{\Gamma} \circ \mathbb{\Sigma}$ is the unit of the adjunction. The co-unit is given by the algebraic evaluation transform $\mathfrak{E} : \mathbb{\Sigma} \circ \mathbb{\Gamma} \to \mathrm{Id}_{1\text{-}\mathscr{W}}$, which "embeds" every one-dimensional propagator (π, γ) into its algebraically saturated spectral amplitude propagator $\mathbb{\Sigma} \circ \mathbb{\Gamma}(\pi, \gamma)$.

The previous adjunction of functors restricts to an adjunction of covariant functors $\mathbb{\Sigma}^{FD} \dashv \mathbb{\Gamma}^{FD}$, $\mathscr{R}_{FD} \underset{\mathbb{\Gamma}^{FD}}{\overset{\mathbb{\Sigma}^{FD}}{\rightleftarrows}} 1\text{-}\mathscr{W}_{FD}$, with unit $\mathfrak{G}^{FD} : \mathrm{Id}_{\mathscr{R}} \to \mathbb{\Gamma}^{FD} \circ \mathbb{\Sigma}^{FD}$ and co-unit $\mathfrak{E}^{FD} : \mathbb{\Sigma}^{FD} \circ \mathbb{\Gamma}^{FD} \to \mathrm{Id}_{1\text{-}\mathscr{W}}$, between the full-subcategories $\mathscr{R}_{FD}$ of irrep-classes of finite-dimensional C-algebras, with irrep-classes-preserving isomorphisms, and the category $1\text{-}\mathscr{W}_{FD}$ of algebraic finite one-dimensional total propagators, with isomorphisms.*

Proof. We must show the following adjunction triangle identities (where $\circ_h$ and $\circ_v$ denote the compositions, over objects and over 1-arrows, of the natural transformations involved):

$$(\iota_{\mathbb{\Gamma}} \circ_h \mathfrak{E}) \circ_v (\mathfrak{G} \circ_h \iota_{\mathbb{\Gamma}}) = \iota_{\mathbb{\Gamma}}, \qquad (\mathfrak{E} \circ_h \mathbb{\Sigma}) \circ_v (\mathbb{\Sigma} \circ_h \mathfrak{G}) = \iota_{\mathbb{\Sigma}}.$$

For the first equation, we need to check that, for every one-dimensional propagator $(\mathcal{E}, \pi, \gamma, \mathcal{P})$ in 1-$\mathscr{W}$, $(\mathbb{\Gamma}_{\mathfrak{E}_{(\pi,\gamma)}}) \circ (\mathfrak{G}_{(\Gamma''_\gamma(\pi),[\pi,\gamma]'')}) = \mathrm{Id}_{(\Gamma''_\gamma(\pi),[\pi,\gamma]'')}$, i.e. $\mathbb{\Gamma}_{\mathfrak{E}_{(\pi,\gamma)}}(\widehat{\boldsymbol{\sigma}}) = \sigma$, for every γ-invariant W*-section $\sigma \in \Gamma''_\gamma(\pi)$:

$$\begin{aligned}\left(\mathbb{\Gamma}_{\mathfrak{E}_{(\pi,\gamma)}}(\widehat{\boldsymbol{\sigma}})\right)(p,q) :&= \left(\mathbb{\Gamma}_{(\boldsymbol{\eta}^{(\pi,\gamma)},\boldsymbol{\Omega}^{(\pi,\gamma)})}(\widehat{\boldsymbol{\sigma}})\right)(p,q)\\ &= \left(\boldsymbol{\Omega}^{(\pi,\gamma)} \circ (\boldsymbol{\eta}^{(\pi,\gamma)})^\bullet(\widehat{\boldsymbol{\sigma}}) \circ (\mathrm{Id}_{\mathcal{P}_{\Gamma_\gamma(\pi)}})^{-1}_{\boldsymbol{\eta}^{(\pi,\gamma)}}\right)(p,q)\\ &= \boldsymbol{\Omega}^{(\pi,\gamma)} \circ \widehat{\boldsymbol{\sigma}}\left(\boldsymbol{\eta}^{(\pi,\gamma)}(p), \boldsymbol{\eta}^{(\pi,\gamma)}(q)\right)\\ &= \boldsymbol{\Omega}^{(\pi,\gamma)}\left(\mid \boldsymbol{\eta}^{(\pi,\gamma)}(p)\,\rangle\langle\, \boldsymbol{\eta}^{(\pi,\gamma)}(p) \mid \varpi(\sigma)_o \mid \boldsymbol{\eta}^{(\pi,\gamma)}(q)\,\rangle\langle\, \boldsymbol{\eta}^{(\pi,\gamma)}(q) \mid\right)\\ &= \sigma(p,q),\end{aligned}$$

for all $p, q \in \mathcal{P}$, where $[\boldsymbol{\eta}^{(\pi,\gamma)}(p)] = o = [\boldsymbol{\eta}^{(\pi,\gamma)}(q)]$.

For the second equation, we need to check, for every unitary equivalence class of irreps of a unital C*-algebra $(\mathcal{A}, o)$ in $\mathscr{R}$, that $(\mathfrak{E}_{((\pi_\mathcal{A})_o,(\gamma_\mathcal{A})_o)}) \circ (\mathbb{\Sigma}_{\mathfrak{G}_{(\mathcal{A},o)}}) = \iota((\pi_\mathcal{A})_o, (\gamma_\mathcal{A})_o)$ and this means

$$(\boldsymbol{\eta}^{((\pi_\mathcal{A})_o,(\gamma_\mathcal{A})_o)}, \boldsymbol{\Omega}^{((\pi_\mathcal{A})_o,(\gamma_\mathcal{A})_o)}) \circ (\xi^{\mathfrak{G}_{(\mathcal{A},o)}}, \Xi^{\mathfrak{G}_{(\mathcal{A},o)}}) = (\mathrm{Id}_{(\mathcal{P}_\mathcal{A})_o}, \zeta_{(\mathcal{E}_\mathcal{A})_o}) \quad \text{and hence}$$

$$\xi^{\mathfrak{G}_{(\mathcal{A},o)}} \circ \boldsymbol{\eta}^{((\pi_\mathcal{A})_o,(\gamma_\mathcal{A})_o)} = \mathrm{Id}_{(\mathcal{P}_\mathcal{A})_o} \tag{1}$$

$$\boldsymbol{\Omega}^{((\pi_\mathcal{A})_o,(\gamma_\mathcal{A})_o)} \circ \left(\boldsymbol{\eta}^{((\pi_\mathcal{A})_o,(\gamma_\mathcal{A})_o)}\right)^\bullet (\Xi^{\mathfrak{G}_{(\mathcal{A},o)}}) \circ \zeta^{(\pi_\mathcal{A})_o}_{\xi^{\mathfrak{G}_{(\mathcal{A},o)}},\, \boldsymbol{\eta}^{((\pi_\mathcal{A})_o,(\gamma_\mathcal{A})_o)}} = \zeta_{(\mathcal{E}_\mathcal{A})_o}. \tag{2}$$

To obtain equation (1), we observe that, for every $\omega \in (\mathcal{P}_\mathcal{A})_o$ and $x \in \mathcal{A}$:

$$\begin{aligned}\xi^{\mathfrak{G}_{(\mathcal{A},o)}} \circ \boldsymbol{\eta}^{((\pi_\mathcal{A})_o,(\gamma_\mathcal{A})_o)}_\omega(x) &= \boldsymbol{\eta}^{((\pi_\mathcal{A})_o,(\gamma_\mathcal{A})_o)}_\omega(\mathfrak{G}_{(\mathcal{A},o)}(x)) = \zeta^{(\pi_\mathcal{A})_o}_\omega \circ \hat{\boldsymbol{x}}(\omega,\omega)\\ &= \zeta^{(\pi_\mathcal{A})_o}_\omega(\mid \omega\,\rangle\langle\, \omega \mid \varpi_o(x) \mid \omega\,\rangle\langle\, \omega \mid)\\ &= \zeta^{(\pi_\mathcal{A})_o}_\omega(\omega(x) \mid \omega\,\rangle\langle\, \omega \mid) = \omega(x).\end{aligned}$$

To obtain equation (2), we observe that, for every $\omega, \rho \in (\mathcal{P}_\mathcal{A})_o$ and for every $x \in \mathcal{A}$:

$$\begin{aligned}&\Xi^{\mathfrak{G}_{(\mathcal{A},o)}}_{\boldsymbol{\eta}_\omega\boldsymbol{\eta}_\rho}(\mid \omega\,\rangle\langle\, \omega \mid \varpi_o(x) \mid \rho\,\rangle\langle\, \rho \mid)\\ &= \Xi^{\mathfrak{G}_{(\mathcal{A},o)}}_{\boldsymbol{\eta}_\omega\boldsymbol{\eta}_\rho}\left(\mid \xi^{\mathfrak{G}_{(\mathcal{A},o)}}(\boldsymbol{\eta}_\omega)\,\rangle\langle\, \xi^{\mathfrak{G}_{(\mathcal{A},o)}}(\boldsymbol{\eta}_\omega) \mid \varpi_o(x) \mid \xi^{\mathfrak{G}_{(\mathcal{A},o)}}(\boldsymbol{\eta}_\rho)\,\rangle\langle\, \xi^{\mathfrak{G}_{(\mathcal{A},o)}}(\boldsymbol{\eta}_\rho) \mid\right)\\ &= \mid \boldsymbol{\eta}^{((\pi_\mathcal{A})_o,(\gamma_\mathcal{A})_o)}_\omega\,\rangle\langle\, \boldsymbol{\eta}^{((\pi_\mathcal{A})_o,(\gamma_\mathcal{A})_o)}_\omega \mid \varpi_{[(\pi_\mathcal{A})_o,(\gamma_\mathcal{A})_o]}(\hat{\boldsymbol{x}}) \mid \boldsymbol{\eta}^{((\pi_\mathcal{A})_o,(\gamma_\mathcal{A})_o)}_\rho\,\rangle\langle\, \boldsymbol{\eta}^{((\pi_\mathcal{A})_o,(\gamma_\mathcal{A})_o)}_\rho \mid,\end{aligned}$$

$$\begin{aligned}\boldsymbol{\Omega}^{((\pi_\mathcal{A})_o,(\gamma_\mathcal{A})_o)}_{\omega\rho}\Bigg(&\mid \boldsymbol{\eta}^{((\pi_\mathcal{A})_o,(\gamma_\mathcal{A})_o)}_\omega\,\rangle\langle\, \boldsymbol{\eta}^{((\pi_\mathcal{A})_o,(\gamma_\mathcal{A})_o)}_\omega \mid \varpi_{[(\pi_\mathcal{A})_o,(\gamma_\mathcal{A})_o]}(\hat{\boldsymbol{x}})\\ &\mid \boldsymbol{\eta}^{((\pi_\mathcal{A})_o,(\gamma_\mathcal{A})_o)}_\rho\,\rangle\langle\, \boldsymbol{\eta}^{((\pi_\mathcal{A})_o,(\gamma_\mathcal{A})_o)}_\rho \mid\Bigg)\\ &= \hat{\boldsymbol{x}}(\omega,\rho) = \mid \omega\,\rangle\langle\, \omega \mid \varpi_o(x) \mid \rho\,\rangle\langle\, \rho \mid,\end{aligned}$$

and hence, from the following commuting diagram,[39]

$$\begin{array}{ccccccc}
(\mathcal{E}_{\mathcal{A}})_o|_{\omega\rho} & \xleftarrow{\xi^{(\pi_{\mathcal{A}})_o}|_{\boldsymbol{\eta}_\omega\boldsymbol{\eta}_\rho}} & (\xi^\bullet((\mathcal{E}_{\mathcal{A}})_o)|_{\boldsymbol{\eta}_\omega\boldsymbol{\eta}_\rho} & \xrightarrow{\Xi^{\mathfrak{G}}|_{\boldsymbol{\eta}_\omega\boldsymbol{\eta}_\rho}} & \mathcal{E}_{\Sigma\circ\Gamma(\mathcal{A},o)}|_{\boldsymbol{\eta}_\omega\boldsymbol{\eta}_\rho} & & \\
\uparrow{\scriptstyle \zeta_{(\mathcal{E}_{\mathcal{A}})_o}|_{\omega\rho}} & & \uparrow{\scriptstyle \boldsymbol{\eta}^{\beta_1}|_{\omega\rho}} & & \uparrow{\scriptstyle \boldsymbol{\eta}^{\beta_2}|_{\omega\rho}} & & \\
(\mathrm{Id}_{(\mathcal{P}_{\mathcal{A}})_o})^\bullet((\mathcal{E}_{\mathcal{A}})_o)|_{\omega\rho} & \xrightarrow[\zeta^{(\pi_{\mathcal{A}})_o}_{\xi,\boldsymbol{\eta}}|_{\omega\rho}]{} & \boldsymbol{\eta}^\bullet(\xi^\bullet((\mathcal{E}_{\mathcal{A}})_o))|_{\omega\rho} & \xrightarrow[\boldsymbol{\eta}^\bullet(\Xi^{\mathfrak{G}})|_{\omega\rho}]{} & \boldsymbol{\eta}^\bullet(\mathcal{E}_{\Sigma\circ\Gamma(\mathcal{A},o)})|_{\omega\rho} & \xrightarrow[\boldsymbol{\Omega}|_{\omega\rho}]{} & (\mathcal{E}_{\mathcal{A}})_o|_{\omega\rho}
\end{array}$$

we finally obtain:

$$\boldsymbol{\Omega}^{((\pi_{\mathcal{A}})_o,(\gamma_{\mathcal{A}})_o)} \circ \left(\boldsymbol{\eta}^{((\pi_{\mathcal{A}})_o,(\gamma_{\mathcal{A}})_o)}\right)^\bullet (\Xi^{\mathfrak{G}_{(\mathcal{A},o)}}) \circ \zeta^{(\pi_{\mathcal{A}})_o}_{\xi^{\mathfrak{G}_{(\mathcal{A},o)}},\ \boldsymbol{\eta}^{((\pi_{\mathcal{A}})_o,(\gamma_{\mathcal{A}})_o)}} \circ (\zeta_{(\mathcal{E}_{\mathcal{A}})_o})^{-1} = \mathrm{Id}_{(\mathcal{E}_{\mathcal{A}})_o}. \qquad \square$$

Restricting the previous adjunction $\Sigma \dashv \Gamma$ to the full reflective subcategories $\overline{\mathscr{R}}$ and $\overline{1\text{-}\mathscr{W}}$ we obtain:[40]

Corollary 4.24. *There is an adjoint equivalence* $\overline{\mathscr{R}} \underset{\Gamma}{\overset{\Sigma}{\rightleftarrows}} \overline{1\text{-}\mathscr{W}}$ *,* $\Sigma \dashv \Gamma$*, with unit isomorphism* $\mathfrak{G} : \mathrm{Id}_{\overline{\mathscr{R}}} \to \Gamma \circ \Sigma$ *and co-unit isomorphism* $\mathfrak{E} : \Sigma \circ \Gamma \to \mathrm{Id}_{\overline{1\text{-}\mathscr{W}}}$*, between the reflective subcategory* $\overline{\mathscr{R}}$ *of algebraically saturated unitary equivalence classes of irreps of unital C*-algebras and the reflective subcategory* $\overline{1\text{-}\mathscr{W}}$ *of algebraically saturated one-dimensional total propagators.*

The previous adjoint equivalence restricts to an adjoint equivalence $\Sigma^{FD} \dashv \Gamma^{FD}$*,* $\mathscr{R}_{FD} \underset{\Gamma}{\overset{\Sigma}{\rightleftarrows}} \overline{1\text{-}\mathscr{W}}_{FD}$ *, with unit and co-unit respectively given by:*

$$\mathfrak{G}^{FD} : \mathrm{Id}_{\mathscr{R}} \to \Gamma^{FD} \circ \Sigma^{FD}, \quad \mathfrak{E}^{FD} : \Sigma^{FD} \circ \Gamma^{FD} \to \mathrm{Id}_{\overline{1\text{-}\mathscr{W}}},$$

between the full-sucategories $\mathscr{R}_{FD}$ *irrep-classes of finite-dimensional C*-algebras, with irrep-class-preserving isomorphisms, and the category* $\overline{1\text{-}\mathscr{W}}_{FD}$ *of algebraically saturated finite one-dimensional total propagators, with isomorphisms.*

Remark 4.25. Denote by $\mathcal{G}(\pi)$ the group of $*$-automorphisms of the 1-C*-category $\pi : \mathcal{E} \to \mathcal{P}$ of a (discrete) propagator $(\mathcal{E}, \pi, \gamma, \mathcal{P})$ and let us denote by $\nabla(\pi, \gamma)$ the set of all possible transition amplitudes γ' for this 1-C*-category $\pi : \mathcal{E} \to \mathcal{P}$, such that there exists an isomorphism of propagators $(\xi, \Xi) : (\mathcal{E}, \pi, \gamma, \mathcal{P}) \to (\mathcal{E}, \pi, \gamma', \mathcal{P})$ such that $(\xi, \Xi) \in \mathcal{G}(\pi)$ and hence $\gamma' = \Xi \circ \gamma \circ \xi^{-1}$. Notice that the global "gauge group" $\mathcal{G}(\pi)$ is acting canonically on the set $\nabla(\pi, \gamma)$ of admissible transition amplitudes (that here play the role of "γ-frame-connections") as $\gamma^1 \mapsto \Xi \circ \gamma^1 \circ \xi^{-1}$. The (discrete) propagator (π, γ) is algebraically saturated if and only if $\nabla(\pi, \gamma)$ is a torsor for the gauge group $\mathcal{G}(\pi)$ i.e. the previous action of $\mathcal{G}(\pi)$ on $\nabla(\pi, \gamma)$ is transitive and effective. Denoting

[39] Where $\boldsymbol{\eta}^{\beta^1} : \boldsymbol{\eta}^\bullet((\xi^\bullet((\mathcal{E}_{\mathcal{A}})_o)) \to \xi^\bullet((\mathcal{E}_{\mathcal{A}})_o) \to$ and $\boldsymbol{\eta}^{\beta^2} : \boldsymbol{\eta}^\bullet(\mathcal{E}_{\Sigma\circ\Gamma(\mathcal{A},o)}) \to \mathcal{E}_{\Sigma\circ\Gamma(\mathcal{A},o)}$ are the canonical isomorphisms in the definition of the respective $\boldsymbol{\eta}$-pull-backs.

[40] With a small abuse of notation we continue to denote functors and transformations in these restriction with the same symbols.

by $\mathscr{O}(\pi,\gamma)$ the set of γ-frames of the (discrete) propagator (π,γ), we have the principal bundle over $\mathscr{O}(\pi,\gamma)$ with $\mathcal{O}$-fiber $\mathcal{U}_{\mathcal{O}} := \{\sigma \in \Gamma(\pi|_{\mathcal{O}}) \mid \sigma^\star \odot \sigma = \gamma|_{\mathcal{O}} = \sigma \odot \sigma^\star\}$ and we consider the group of "unitary sections" of $\pi|_{\mathcal{O}} : \mathcal{E}|_{\mathcal{O}} \to \mathcal{O}$, for $\mathcal{O} \in \mathscr{O}(\pi,\gamma)$. The global gauge group $\mathcal{G}(\pi)$ is fiberwise acting transitively and effectively on $\mathcal{U}_{\mathcal{O}}$, since every $(\xi,\Xi) \in \mathcal{G}(\pi)$ uniquely determines a unitary $\Xi \circ \gamma|_{\mathcal{O}} \in \mathcal{U}_{\mathcal{O}}$. Denoting by $\nabla(\pi,\gamma)^{\mathscr{O}}$ the family of unitary pair-groupoids between γ-frames obtained by frame localization of any transition amplitude $\gamma' \in \nabla(\pi,\gamma)$, we have a gauge action of the global gauge group $\mathcal{G}(\pi)$ on the set of "connections" $\nabla(\pi,\gamma)^{\mathscr{O}}$ by conjugation: $\gamma'|_{\mathcal{O}} \mapsto (\Xi \circ \gamma_{\mathcal{O}_2}) \odot \gamma'_{\mathcal{O}_2\mathcal{O}_1} \odot (\Xi \circ \gamma_{\mathcal{O}_1})^{-1}$. The saturation condition is equivalent to requiring that $\nabla(\pi,\gamma)^{\mathscr{O}}$ is a torsor for such action of $\mathcal{G}(\pi)$. ⌟

We now start to introduce the fundamental spectral environment for our work: the category $\mathscr{E}_{FD}$ whose objects are *"bundles of finite total propagators of 1-dimensional C*-categories over a finite set"*.

Definition 4.26. *A* ***discrete non-commutative spaceoid*** $(\mathcal{E},\pi,\gamma,\mathcal{P},\chi,\mathcal{X})$ *consists of the following diagram*

$$\begin{array}{ccc} \mathcal{E} & \underset{\gamma}{\overset{\pi}{\rightleftarrows}} & \mathcal{P}\times_\chi \mathcal{P} \\ & {\scriptstyle\mu_1}\searrow \quad \swarrow{\scriptstyle\mu_0} & \\ & \mathcal{X} & \end{array}$$

with $\mu_1 = \mu_0 \circ \pi$ and $\mu_1 \circ \gamma = \mu_0$, where:

- *$\mathcal{X}$ is a set, $\mathcal{P} \xrightarrow{\chi} \mathcal{X}$ is a bundle, and $\{(p,q) \in \mathcal{P} \mid \chi(p) = \chi(q)\} =: \mathcal{P}\times_\chi \mathcal{P} \xrightarrow{\mu_0} \mathcal{X}$ is its fiberwise product bundle, hence $\mu_0(p,q) := \chi(p) = \chi(q)$, for $(p,q) \in \mathcal{P}\times_\chi\mathcal{P}$,*
- *for all $o \in \mathcal{X}$, the fiber restriction $\mathcal{E}_o \xrightarrow{\pi_o:=(t_o,s_o)} \mathcal{P}_o \times \mathcal{P}_o$ of the map π is a 1-dimensional C*-category with objects $\mathcal{P}_o$, hence $\mu := (\mu_1,\mu_0)$ is a bundle of 1-C*-categories,*
- *$\gamma_o \in$ 1-$\mathscr{W}(\pi_o)$ i.e. $\mathcal{E}_o \underset{\gamma_o}{\overset{\pi_o}{\rightleftarrows}} \mathcal{P}_o \times \mathcal{P}_o$ is a total $\mathcal{E}_o$-valued propagator on $\mathcal{P}_o$, for all $o \in \mathcal{X}$.*

A ***finite discrete non-commutative spaceoid*** *is a discrete spaceoid $(\mathcal{E},\pi,\gamma,\mathcal{P},\chi,\mathcal{X})$ with finite base space $\mathcal{X}$, whose fibers $(\pi_o,\gamma_o) \in \mathscr{W}_{FD}$ are finite propagators for all $o \in \mathcal{X}$.*

A ***morphism of discrete non-commutative spaceoids*** $\mu^1 \xrightarrow{(\lambda,\Lambda)} \mu^2$*, between two discrete non-commutative spaceoids $\mu^j := (\mathcal{E}^j,\pi^j,\gamma^j,\mathcal{P}^j,\chi^j,\mathcal{X}^j)$, for $j = 1,2$, is given by a pair (λ,Λ) where:*

- *$\lambda : \mathcal{X}^1 \to \mathcal{X}^2$ is a map between sets,*
- *$\Lambda : \lambda^\bullet(\mu^2) \to \mu^1$ is a fibrewise frame-preserving geometric morphism of propagators in the category 1-$\mathscr{W}$: namely, for all $o \in \mathcal{X}^1$, $\Lambda_o := (\Lambda^0_o,\Lambda^1_o)$ is a geometric morphism from the λ-pull-back of μ^2 to μ^1.*

We denote by $\mathscr{E}$ the category of such morphisms with composition and identity as in the case of $\mathscr{B}_{FD}$. The subcategory of fibrewise $*$-isomorphisms of finite spaceoids in $\mathscr{E}$ is denoted by $\mathscr{E}_{FD}$.

We proceed now to define a pair of adjoint covariant functors between $\mathscr{B}_{FD}$ and $\mathscr{E}_{FD}$. These are just obtained by "fiberwise application" of the previous adjunction $\mathbb{\Sigma}^{FD} \dashv \mathbb{\Gamma}^{FD}$ in theorem 4.23.

Definition 4.27. *The **fiber section functor** $\Gamma|^{FD} : \mathscr{E}_{FD} \to \mathscr{B}_{FD}$ associates to a finite discrete spaceoid $(\mathcal{E}, \pi, \gamma, \mathcal{P}, \chi, \mathcal{X})$ the bundle $\Gamma|^{FD}(\pi, \gamma, \chi)$ on $\mathcal{X}$ with o-fibers $\mathbb{\Gamma}^{FD}(\pi_o, \gamma_o) := (\Gamma''_{\gamma_o}(\pi_o), [\pi_o, \gamma_o]'')$ consisting of the primitive finite-dimensional C*-algebras $\Gamma''_{\gamma_o}(\pi_o)$, of γ_o-invariant C*-sections of the finite total $\mathcal{E}_o$-valued propagator $(\mathcal{E}_o, \pi_o, \gamma_o, \mathcal{P}_o)$ over $\mathcal{P}_o$ (and their unique irrep-classes $[\pi_o, \gamma_o]''$).*

To every morphism $(\mathcal{E}^1, \pi^1, \gamma^1, \mathcal{P}^1, \chi^1, \mathcal{X}^1) \xrightarrow{(\lambda,\Lambda)} (\mathcal{E}^2, \pi^2, \gamma^2, \mathcal{P}^2, \chi^2, \mathcal{X}^2)$ of finite discrete spaceoids in $\mathscr{E}_{FD}$, the fiber section functor associates the morphism of finite bundles of primitive finite-dimensional C-algebras given by:*

$$\Gamma|^{FD}(\pi^1, \gamma^1, \chi^1) \xrightarrow{\Gamma|^{FD}_{(\lambda,\Lambda)}} \Gamma|^{FD}(\pi^2, \gamma^2, \chi^2), \quad \Gamma|^{FD}_{(\lambda,\Lambda)} := (\lambda^\Gamma, \Lambda^\Gamma),$$

where $\lambda^\Gamma := \lambda$, and $\lambda^\bullet(\Gamma|^{FD}(\pi^2, \gamma^2, \chi^2)) \xrightarrow{\Lambda^\Gamma} \Gamma|^{FD}(\pi^1, \gamma^1, \chi^1)$ is fiberwise defined as $\Lambda^\Gamma_o := \mathbb{\Gamma}^{FD}_{\Lambda_o}$, the Λ^0_o-pull-back of $\gamma^2_{\lambda(o)}$-invariant sections

$$\Lambda^\Gamma_o : \sigma_{\lambda(o)} \mapsto \Lambda^1_o \circ (\Lambda^0_o)^\bullet(\sigma_{\lambda(o)}) \circ (\mathrm{Id}_{\mathcal{P}^2_{\lambda(o)}})^{-1}_{\Lambda^0_o},$$

for all $\sigma_{\lambda(o)} \in \Gamma_{\gamma^2_{\lambda(o)}}(\pi^2_{\lambda(o)})$, for $o \in \mathcal{X}^1$.

For all $o \in \mathcal{X}^1$, the map Λ^Γ_o is a unital $*$-isomorphism of primitive finite-dimensional C*-algebras. A direct computation ensures that $\Gamma|^{FD}_{(\lambda^1,\Lambda^1)\circ(\lambda^2,\Lambda^2)} = \Gamma|^{FD}_{(\lambda^1,\Lambda^1)} \circ \Gamma|^{FD}_{(\lambda^2,\Lambda^2)}$ and $\Gamma|^{FD}_{(\mathrm{Id}_\mathcal{X}, \eta_{(\mathcal{E},\pi,\gamma,\mathcal{P},\chi,\mathcal{X})})} = (\mathrm{Id}_\mathcal{X}, \eta_{\Gamma|^{FD}(\pi,\gamma,\chi)})$, providing the covariant functoriality of $\Gamma|^{FD}$.

Definition 4.28. *The **fiber spectrum functor** $\Sigma|^{FD} : \mathscr{B}_{FD} \to \mathscr{E}_{FD}$ associates to a bundle $(\mathcal{F}, \theta, \mathcal{X})$ in $\mathscr{B}^0_{FD}$ of primitive finite-dimensional C*-algebras on a finite set $\mathcal{X}$, its **spectral finite discrete spaceoid** $\Sigma|^{FD}(\theta) \in \mathscr{E}_{FD}$, over the finite set $\mathcal{X}$, whose o-fibers, for all $o \in \mathcal{X}$, are $\mathbb{\Sigma}^{FD}(\mathcal{F}_o, \kappa_\theta(o)) \in 1\text{-}\mathscr{W}_{FD}$, where κ_θ is the canonical bijection in remark 3.1.*

To every morphism $(\mathcal{F}^1, \theta^1, \mathcal{X}^1) \xrightarrow{(\lambda,\Lambda)} (\mathcal{F}^2, \theta^2, \mathcal{X}^2)$ in $\mathscr{B}_{FD}$, the fiber spectrum functor associates the morphism $\Sigma|^{FD}(\theta^1) \xrightarrow{\Sigma|^{FD}_{(\lambda,\Lambda)}} \Sigma|^{FD}(\theta^2)$ of finite spectral discrete spaceoids given by $\Sigma|^{FD}_{(\lambda,\Lambda)} := (\lambda^\Sigma, \Lambda^\Sigma)$, where $\lambda^\Sigma := \lambda$ and $\lambda^\bullet(\Sigma|^{FD}(\theta^2)) \xrightarrow{\Lambda^\Sigma} \Sigma|^{FD}(\theta^1)$ is fibrewise defined, for all $o \in \mathcal{X}^1$, as the isomorphism $\Lambda^\Sigma_o := \mathbb{\Sigma}^{FD}_{\Lambda_o} := (\xi^{\Lambda_o}, \Xi^{\Lambda_o})$ of finite propagators, where $\xi^{\Lambda_o} : \mathcal{P}_{\mathcal{F}^1_o} \to \mathcal{P}_{\mathcal{F}^2_{\lambda(o)}}$ is the Λ_o-pull-back of pure states under the $*$-isomorphisms $\Lambda_o : \mathcal{F}^2_{\lambda(o)} \to \mathcal{F}^1_o$, and Ξ^{Λ_o} is the isomorphism of 1-C-categories $| \omega \circ \Lambda_o \rangle\langle \omega \circ \Lambda_o | x | \rho \circ \Lambda_o \rangle\langle \rho \circ \Lambda_o | \mapsto | \omega \rangle\langle \omega | \Lambda_o(x) | \rho \rangle\langle \rho |$, for all $\omega, \rho \in \mathcal{P}_{\mathcal{F}^1_o}$.*

For all $o \in \mathcal{X}^1$, the map Λ^Σ_o is an isomorphism in 1-$\mathscr{W}_{FD}$.
A direct computation assures the covariant functoriality of $\Sigma|^{FD}$:

$$\Sigma|^{FD}_{(\lambda^1,\Lambda^1)\circ(\lambda^2,\Lambda^2)} = \Sigma|^{FD}_{(\lambda^1,\Lambda^1)} \circ \Sigma|^{FD}_{(\lambda^2,\Lambda^2)}, \quad \Sigma|^{FD}_{(\mathrm{Id}_\mathcal{X},\eta_{(\mathcal{F},\theta,\mathcal{X})})} = (\mathrm{Id}_\mathcal{X}, \eta_{\Sigma|^{FD}(\theta)}).$$

Definition 4.29. *The **fiber Gel'fand transform** $\mathfrak{G}|^{FD}$ associates to a bundle $(\mathcal{F},\theta,\mathcal{X})$ in $\mathscr{B}^0_{FD}$ the morphism*

$$\mathfrak{G}|^{FD}_\theta : \Gamma|^{FD} \circ \Sigma|^{FD}(\theta) \xrightarrow{(\upsilon^\theta,\Upsilon^\theta)} (\mathcal{F},\theta,\mathcal{X})$$

in $\mathscr{B}^1_{FD}$, with $\upsilon^\theta := \mathrm{Id}_\mathcal{X} : \mathcal{X} \to \mathcal{X}$, and Υ^θ defined fibrewise, for all $o \in \mathcal{X}$, by $\Upsilon^\theta_o := \mathfrak{G}_{(\mathcal{F}_o,\kappa_\theta(o))} : \mathcal{F}_o \to \Gamma|^{FD} \circ \Sigma|^{FD}(\theta)_o$.

*The **fiber evaluation transform** $\mathfrak{E}|^{FD} : \mathrm{Id}_{\mathscr{E}_{FD}}$ associates to finite discrete spaceoids $(\mathcal{E},\pi,\gamma,\mathcal{P},\chi,\mathcal{X})$ in $\mathscr{E}^0_{FD}$ the morphisms of discrete spaceoids*

$$\mathfrak{E}|^{FD}_{(\pi,\gamma,\chi)} : (\mathcal{E},\pi,\gamma,\mathcal{P},\chi,\mathcal{X}) \xrightarrow{(\eta^{(\pi,\gamma,\chi)},\Omega^{(\pi,\gamma,\chi)})} \Sigma|^{FD} \circ \Gamma|^{FD}(\pi,\gamma,\chi)$$

in $\mathscr{E}^1_{FD}$ where $\eta^{(\pi,\gamma,\chi)} := \mathrm{Id}_\mathcal{X} : \mathcal{X} \to \mathcal{X}$ and $\Omega^{(\pi,\gamma,\chi)}$ is fibrewise given by the geometric morphism of propagators $\Omega^{(\pi,\gamma,\chi)}_o := \mathfrak{E}_{(\pi_o,\gamma_o)} : (\mathcal{E}_o,\pi_o,\gamma_o,\mathcal{P}_o) \to \Sigma|^{FD}\circ\Gamma|^{FD}(\pi_o,\gamma_o,\mathcal{P}_o)$, for all $o \in \mathcal{X}$.

From the fiberwise definitions, we have that $\mathfrak{G}|^{FD} : \mathrm{Id}_{\mathscr{B}_{FD}} \to \Gamma|^{FD} \circ \Sigma|^{FD}$ is a natural isomorphism and that $\mathfrak{E}|^{FD} : \mathrm{Id}_{\mathscr{E}_{FD}} \to \Sigma|_{FD} \circ \Gamma|^{FD}$ is a natural transformation.

Theorem 4.30. *There is an adjunction $\Sigma|^{FD} \vdash \Gamma|^{FD}$ between the covariant functors $\mathscr{B}_{FD} \underset{\Gamma|^{FD}}{\overset{\Sigma|^{FD}}{\rightleftarrows}} \mathscr{E}_{FD}$, with unit $\mathfrak{E}|^{FD} : \mathrm{Id}_{\mathscr{E}^{FD}} \to \Sigma|^{FD} \circ \Gamma|^{FD}$ the fiber evaluation transform and with co-unit $\mathfrak{G}|^{FD} : \Gamma|^{FD} \circ \Sigma|^{FD} \to \mathrm{Id}_{\mathscr{B}_{FD}}$ the fiber Gel'fand transform isomorphism.*

Restricting the previous adjunction $\Sigma|^{FD} \vdash \Gamma|^{FD}$ to the reflective full subcategory $\overline{\mathscr{E}}_{FD}$, whose objects are the saturated algebraic finite non-commutative spaceoids, we obtain an adjoint equivalence.[41]

Proof. The result follows by a "fibrewise application" of the previous adjunction in theorem 4.23.[42] □

Our final duality is obtained by composing the adjoint duality in 3.2 with the adjoint equivalence in 4.30:

$$\dashv\ \underline{\Sigma}^{FD}\ \underline{\Gamma}^{FD}\ \vdash \qquad \mathscr{A}_{FD} \underset{\underline{\Gamma}^{FD}}{\overset{\underline{\Sigma}^{FD}}{\rightleftarrows}} \mathscr{B}_{FD} \underset{\Gamma|^{FD}}{\overset{\Sigma|^{FD}}{\rightleftarrows}} \overline{\mathscr{E}}_{FD} \qquad \Sigma|^{FD} \vdash \Gamma|^{FD}.$$

[41] Here, with some minor abuse of notation, we do not give alternative notation for the functors, units and co-units.

[42] One has to remember that the "direction" of morphisms in the categories $\mathscr{B}_{FD}$ and $\mathscr{E}_{Fd}$ is the "opposite" of the "direction" in $\mathscr{R}$ and 1-$\mathscr{W}$ and hence the direction of the Gel'fand and evaluation transforms are reversed.

Definition 4.31. *The contravariant* ***section functor*** Γ^{FD} *and the contravariant* ***spectrum functor*** Σ^{FD} *are defined by compositions of the previous contraviariant base and covariant fiber functors:*

$$\Gamma^{FD} := \underline{\Gamma}^{FD} \circ \Gamma|^{FD} : \mathscr{E}_{FD} \to \mathscr{A}_{FD}, \qquad \Sigma^{FD} := \Sigma|^{FD} \circ \underline{\Sigma}^{FD} : \mathscr{A}_{FD} \to \mathscr{E}_{FD}.$$

Considering the (big) strict 2-category of natural transformations between functors, with categories as objects, (denoting by $\circ_h$ the "horizontal" composition and $\circ_v$ the "vertical" compositions of natural transformations), we define non-commutative Gel'fand and evaluation transforms.

Definition 4.32. *The* ***non-commutative Gel'fand transform*** *is the natural transformation:*

$$\mathfrak{G}^{FD} := \left(\underline{\Gamma}^{FD} \circ_h \mathfrak{G}|^{FD} \circ_h \underline{\Sigma}^{FD}\right) \circ_v \underline{\mathfrak{G}}^{FD} : \mathrm{Id}_{\mathscr{A}_{FD}} \to \Gamma^{FD} \circ \Sigma^{FD}.$$

The ***non-commutative evaluation transform*** *is the natural transformation:*

$$\mathfrak{E}^{FD} := \left(\Sigma|^{FD} \circ_h \underline{\mathfrak{E}}^{FD} \circ_h \Gamma|^{FD}\right) \circ_v \mathfrak{E}|^{FD} : \mathrm{Id}_{\mathscr{E}_{FD}} \to \Sigma^{FD} \circ \Gamma^{FD}.$$

Theorem 4.33. *There is a right-right contravariant adjunction* $\dashv\ \Sigma^{FD}\ \Gamma^{FD} \vdash$ *between the pair of contravariant functors* $\mathscr{A}_{FD} \underset{\Gamma^{FD}=\underline{\Gamma}^{FD}\circ\,\Gamma|^{FD}}{\overset{\Sigma^{FD}=\Sigma|^{FD}\circ\,\underline{\Sigma}^{FD}}{\rightleftarrows}} \mathscr{E}_{FD}$ *with units* $\mathfrak{G}^{FD}$ *and* $\mathfrak{E}^{FD}$.

Restricting this right-right contravariant adjunction to the saturated full subcategory $\overline{\mathscr{E}}_{FD}$*, we obtain the discrete non-commutative Gel'fand-Naĭmark adjoint duality*

$$\mathscr{A}_{FD} \underset{\Gamma^{FD}=\underline{\Gamma}^{FD}\circ\,\Gamma|^{FD}}{\overset{\Sigma^{FD}=\Sigma|^{FD}\circ\,\underline{\Sigma}^{FD}}{\rightleftarrows}} \overline{\mathscr{E}}_{FD}\ .$$

Proof. The theorem is immediately obtained using the standard categorical composition of adjuctions.

By remark 3.4 and proposition 3.6, considering the dual category $\mathscr{B}^{\circ}_{FD}$, we have the covariant adjoint equivalence $({}^{\circ}\underline{\Sigma}^{FD} \dashv \underline{\Gamma}^{FD\circ})$ with unit $\underline{\mathfrak{G}}^{FD} : \mathrm{Id}_{\mathscr{A}_{FD}} \to \underline{\Gamma}^{FD\circ} \circ {}^{\circ}\underline{\Sigma}^{FD}$ the base Gel'fand transform isomorphism and co-unit $\underline{\mathfrak{E}}^{FD\circ} : {}^{\circ}\underline{\Sigma}^{FD} \circ \underline{\Gamma}^{FD\circ} \to \mathrm{Id}_{\mathscr{B}^{\circ}_{FD}}$ the dual base evaluation transform isomorphism.

By theorem 4.30 and by remark 3.4, considering both the dual categories $\mathscr{B}^{\circ}_{FD}$ and $\mathscr{E}^{\circ}_{FD}$, we have the covariant adjunction $({}^{\circ}\Sigma|^{FD\circ} \dashv {}^{\circ}\Gamma|^{FD\circ})$ with unit the dual fiber Gel'fand transform $\mathfrak{G}|^{FD\circ} : \mathrm{Id}_{\mathscr{B}^{\circ}_{FD}} \to {}^{\circ}\Gamma|^{FD\circ} \circ {}^{\circ}\Sigma|^{FD\circ}$ and co-unit the dual fiber evaluation transform $\mathfrak{E}|^{FD\circ} : {}^{\circ}\Sigma|^{FD\circ} \circ {}^{\circ}\Gamma|^{FD\circ} \to \mathrm{Id}_{\mathscr{E}^{\circ}_{FD}}$.

Composing the previous covariant adjunctions (see [27, proposition 4.4.4]), we obtain the covariant adjunction $({}^{\circ}\Sigma|^{FD\circ} \circ {}^{\circ}\underline{\Sigma}^{FD}) \dashv (\underline{\Gamma}^{FD\circ} \circ {}^{\circ}\Gamma|^{FD\circ})$ with the following unit and co-unit:[43]

[43]The symbol $\tilde{\circ}_v$ reminds that this vertical composition is performed "pointwise" composing in the dual category $\mathscr{E}^{\circ}_{FD}$.

$\left(\underline{\Gamma}^{FD\circ} \circ_h \mathfrak{G}|^{FD\circ} \circ_h {}^\circ\underline{\Sigma}^{FD}\right) \circ_v \underline{\mathfrak{G}}^{FD} : \mathrm{Id}_{\mathscr{A}_{FD}} \to (\underline{\Gamma}^{FD\circ} \circ {}^\circ\Gamma|^{FD\circ}) \circ ({}^\circ\Sigma|^{FD\circ} \circ {}^\circ\underline{\Sigma}^{FD})$,
$\mathfrak{E}|^{FD\circ} \tilde{\circ}_v \left({}^\circ\Sigma|^{FD\circ} \circ_h \underline{\mathfrak{E}}^{FD\circ} \circ_h {}^\circ\Gamma|^{FD\circ}\right) : ({}^\circ\Sigma|^{FD\circ} \circ {}^\circ\underline{\Sigma}^{FD}) \circ (\underline{\Gamma}^{FD\circ} \circ {}^\circ\Gamma|^{FD\circ}) \to \mathrm{Id}_{\mathscr{E}^\circ_{FD}}$.

Finally, by duality, passing back to the original categories using again remark 3.4, we obtain a contravariant right-right adjunction $\dashv \ \underline{\Sigma}^{FD} \ \underline{\Gamma}^{FD} \ \vdash$ with units respectively the non-commutative Gel'fand and evaluation transforms as defined in 4.32.

Upon restriction of the fiber adjoint equivalence to the saturated full subcategory $\overline{\mathscr{E}}_{FD}$, since all of the Gel'fand and evaluation transforms are now natural isomorphisms, we obtain the adjoint duality. □

5 Commutative and C*-categorical Dualities

In this section we clarify the connection between the (discrete) non-commutative Gel'fand-Naĭmark duality here developed, the usual commutative Gel'fand-Naĭmark duality for (finite) Abelian C*-algebras and the duality for full Abelian (finite-objects) C*-categories discussed in [3]. In particular we describe how the discrete non-commutative spaceoids in definition 4.26 are related to the usual (finite) compact Hausdorff Gel'fand spectra and to the (discrete) topological spaceoids introduced in previous works [3, 4].

Let $\mathscr{A}_{FC} \hookrightarrow \mathscr{A}_{FD}$ denote the full-faithful functorial "inclusion" of the category $\mathscr{A}_{FC}$ of unital $*$-homomorphisms of finite-dimensional commutative C*-algebras as a full subcategory of $\mathscr{A}_{FD}$. By theorem 2.2, a finite-dimensional Abelian C*-algebra $\mathcal{A}$ is of the form $\bigoplus_{n=1}^N \mathbb{C}$, for a certain non-zero natural $N \in \mathbb{N}_0$ and, from remark 2.3, $\mathcal{X}_\mathcal{A}$ is a finite discrete space that is canonically homeomorphic to the usual discrete compact Hausdorff Gel'fand spectrum $\mathrm{Sp}(\mathcal{A})$, via the map $\mathrm{Sp}(\mathcal{A}) \ni \omega \mapsto [\omega] \in \mathcal{X}_\mathcal{A}$. Denoting by $\mathscr{S}_{FC}$ the category of continuous maps between finite compact Hausdorff topological spaces, by $\Gamma^{FC} : \mathscr{S}_{FC} \to \mathscr{A}_{FC}$ the restriction of the usual "continuous-maps Gel'fand functor" $\mathscr{S}^0_{FC} \ni X \mapsto \Gamma_{FC}(X) := C(X;\mathbb{C}) \in \mathscr{A}^0_{FC}$ and denoting by $\Sigma^{FC} : \mathscr{A}_{FC} \to \mathscr{S}_{FC}$ the restriction of the usual "Gel'fand spectrum functor" $\mathscr{A}^0_{FC} \ni \mathcal{A} \mapsto \mathrm{Sp}(\mathcal{A}) \in \mathscr{S}^0_{FC}$ (both acting as "pull-backs" on morphisms), we have the following "restriction" of the usual commutative Gel'fand-Naĭmark right-right adjoint duality to the discrete case: ($\dashv \ \Sigma^{FC} \ \Gamma^{FC} \ \vdash$) with units given by the usual Gel'fand transform $\mathfrak{G}^{FC} : \mathrm{Id}_{\mathscr{A}_{FC}} \to \Gamma^{FC} \circ \Sigma^{FC}$ with $\mathfrak{G}^{FC}_\mathcal{A} : x \mapsto \hat{x} \in C(\mathrm{Sp}(X);\mathbb{C})$, for $x \in \mathcal{A} \in \mathscr{A}^0_{FC}$, and the usual evaluation transform $\mathfrak{E}^{FC} : \mathrm{Id}_{\mathscr{S}_{FC}} \to \Sigma^{FC} \circ \Gamma^{FC}$ with $\mathfrak{E}^{FC}_X : p \mapsto \mathrm{ev}_p \in \mathrm{Sp}(C(X;\mathbb{C}))$, for $p \in X \in \mathscr{S}^0_{FC}$.

The following proposition describes the precise relationship between the discrete version of the usual Gel'fand-Naĭmark adjoint duality and our discrete duality via discrete spaceoids. The proof is totally elementary, the only real difficulty being the exact specification of the structures (functors and natural transformations) involved.

Proposition 5.1. *Let $\Phi : \mathscr{S}_{FC} \to \mathscr{S}_{FD}$ be the functor that to every finite discrete space $X \in \mathscr{S}^0_{FC}$ associates the discrete spaceoid $(\mathcal{E}^X, \pi^X, \gamma^X, \mathcal{P}^X, \chi^X, \mathcal{X}^X)$ consisting of the bundle, over X, of trivial 1-dimensional propagators: $\mathcal{X}^X := X := \mathcal{P}^X$, $\chi^X := \mathrm{Id}_X$,*

$(\mathcal{E}^X, \pi^X, \mathcal{P}^X \times_{\chi^X} \mathcal{P}^X)$ the trivial $\mathbb{C}$-line bundle over $\mathcal{P}^X \times_{\chi^X} \mathcal{P}^X \simeq X$, $\gamma^X : \mathcal{P}^X \times_{\chi^X} \mathcal{P}^X \to \mathcal{E}^X$ the constant section $\gamma^X(p,p) := 1_{\mathbb{C}}$, for all $p \in X = \mathcal{P}^X$.[44]
The following are weakly commuting square diagrams between functors

$$\begin{array}{ccc} \mathcal{A}_{FD} & \underset{\Gamma^{FD}}{\overset{\Sigma^{FD}}{\rightleftarrows}} & \mathscr{S}_{FD} \\ \uparrow & & \uparrow \Phi \\ \mathcal{A}_{FC} & \underset{\Gamma^{FC}}{\overset{\Sigma^{FC}}{\rightleftarrows}} & \mathscr{S}_{FC} \end{array},$$

in detail: there is a natural isomorphism of functors $\Gamma^{FD} \circ \Phi \xrightarrow{\mathfrak{F}} \Gamma^{FC}$ given by the natural transformation $\mathscr{S}^0_{FC} \ni X \xrightarrow{\mathfrak{F}} \mathfrak{F}_X \in \mathscr{A}^1_{FD}$, where, for all $\sigma \in \Gamma^{FD}(\Phi(X))$, we define $\mathfrak{F}_X(\sigma) \in \Gamma^{FC}(X) := C(X;\mathbb{C})$ as the function $\mathfrak{F}_X(\sigma)_p = \zeta_p(\sigma_p)$, for all $p \in X$, where ζ_p denotes the unique Gel'fand-Mazur isomorphism between the 1-dimensional C-algebra $\mathcal{E}^X_{pp}$ and $\mathbb{C}$;*

there is a natural isomorphism of functors $\Sigma^{FD}|_{\mathscr{A}_{FC}} \xrightarrow{\mathfrak{T}} \Phi \circ \Sigma^{FC}$ given by the natural transformation $\mathscr{A}^0_{FC} \ni \mathcal{A} \xrightarrow{\mathfrak{T}} \mathfrak{T}_{\mathcal{A}} \in \mathscr{S}^1_{FD}$, where $\Sigma^{FD}(\mathcal{A}) \xrightarrow{\mathfrak{T}_{\mathcal{A}}} \Phi \circ \Sigma^{FC}(\mathcal{A})$ is the isomorphism of spaceoids $\mathfrak{T}_{\mathcal{A}} := (\xi^{\mathcal{A}}, \Xi^{\mathcal{A}})$ with $(\Xi^{\mathcal{A}})^0 := \mathrm{Id}_{\mathrm{Sp}(\mathcal{A})} : \mathcal{P}^{\mathcal{A}} \to \mathrm{Sp}(\mathcal{A})$, $\xi^{\mathcal{A}} : \mathcal{X}^{\mathcal{A}} \to \mathcal{X}^{\mathrm{Sp}(\mathcal{A})}$ is given by $\xi^{\mathcal{A}} : [\omega] \mapsto \omega \in \Phi(\mathrm{Sp}(\mathcal{A}))$, for all $\omega \in \mathcal{P}^{\mathcal{A}} = \mathrm{Sp}(\mathcal{A})$, and $(\Xi^{\mathcal{A}})^1|_{\omega\omega} := \zeta_\omega : (\mathcal{E}_{\mathcal{A}})_{\omega\omega} \to \mathcal{E}^{\mathrm{Sp}(\mathcal{A})} = \mathbb{C}$ is just the Gel'fand-Mazur isomorphism ζ_ω, for all $\omega \in \mathrm{Sp}(\mathcal{A})$.

There is a morphism of (right-right) adjoint dualities:[45]

$$(\dashv \Sigma^{FC}\ \Gamma^{FC} \vdash) \to (\dashv \Sigma^{FD}\ \Gamma^{FD} \vdash)$$
$$\mathfrak{F} \circ \mathfrak{G}^{FD}_{\mathcal{A}} = \mathfrak{G}^{FC}_{\mathcal{A}}, \quad \forall \mathcal{A} \in \mathscr{A}^0_{FC}, \qquad \mathfrak{T} \circ \mathfrak{E}^{FD}_{\Phi(X)} = \Phi \circ \mathfrak{E}^{FC}_X, \quad \forall X \in \mathscr{S}^0_{FC}.$$

The term "spaceoid" was originally introduced in a previous paper [3] providing spectral descriptions of commutative full C*-categories. Since then, it was already clear that the notions there described (with some suitable adjustments) were sufficiently powerful to support a spectral analysis for non-commutative C*-algebras. It is a necessary duty to explain how the discrete non-commutative spaceoids here defined are related to one of the three equivalent definitions of topological spaceoid as discussed in [4]).

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[44] The functor Φ associates to every map $f : X \to Y$ the morphism $(f, f, F) : \Phi(Y) \to \Phi(X)$ of trivial discrete spaceoids, with $f : \mathcal{X}^X \to \mathcal{X}^Y$, $f : \mathcal{P}^X \to \mathcal{P}^Y$ and where $F : f^\bullet(\mathcal{E}^Y) \to \mathcal{E}^X$ is the necessarily unique fibrewise linear isomorphism preserving the identity of the 1-dimensional $\mathbb{C}$-fibers $\mathcal{E}^Y_{f(p)f(p)}$ and $\mathcal{E}^X_{pp}$, for $p \in X$.

[45] By this we mean that the natural transformations of the adjunctions weakly-commute with the respective inclusion functors, modulo the same natural isomorphisms $\mathfrak{F}$, $\mathfrak{T}$ involved in the weak-commutation of the above diagrams of functors.

Definition 5.2. *A **C*-categorical spaceoid**[46] is a bundle $\mathcal{C} \xrightarrow{\mu} \mathcal{X}$, over a compact Hausdorff space $\mathcal{X}$ of full one-dimensional C*-categories $\mathcal{C}_o$, for $o \in \mathcal{X}$, with a set Δ such that $\mathcal{C}_o^0 = \Delta$, for all $o \in \mathcal{X}$.*

In the specific "discrete" situation studied in this paper, we will of course choose X to be a finite discrete topological space and Δ a finite set.

Proposition 5.3. *Every finite C*-categorical spaceoid $(\mathcal{C}, \mu, \mathcal{X})$ is a specific case of discrete non-commutative spaceoid, where, for every $o \in X$, we have the (generally non-saturated) finite propagator $(\mathcal{C}_o, \pi, \gamma, \Delta)$, with $\gamma_o : \Delta \times \Delta \to \mathcal{C}$ given by the "identity section" $\gamma_o(A, A) := 1_{(\mathcal{C}_o)_{AA}}$, for all $A \in \Delta$ and $\gamma_o(AB) = 0_{(\mathcal{C}_o)_{AB}}$, whenever $A \neq B$, for all $A, B \in \Delta$.*

In practice, we have just a bundle, over a finite set, of full one-dimensional C*-categories with the same finite set of objects Δ and the propagator γ is trivial, hence there is exactly only one frame for every point of X.

6 Preview of Topological/Uniform Duality

In this last section, we briefly look beyond the discrete non-commutative Gel'fand-Naĭmark duality, in the direction of the topological/uniform theory fully developed in the companion second paper [6], anticipating some of the typical technical difficulties that will be dealt with there.

In principle, a topological version of "base duality" as in section 3 already exists, since J.Dauns-K.-H.Hofmann theorem [10, 12] and J.Varela duality [30] can deal with arbitrary (unital) C*-algebras. Unfortunately, the type of Banach C*-bundles (with semi-continuous norm) appearing there as spectra, have fibers that in general are not primitive C*-algebras, making it quite difficult to perform the subsequent spectral analysis necessary for the "fiber duality" of section 4. This serious issue will force us to introduce, for such general case, a new topology on $\mathcal{X}_{\mathcal{A}}$ making it compact pre-regular (actually completely regular).

Whenever the structure space $\widehat{\mathcal{A}}$ (the set $\mathcal{X}_{\mathcal{A}}$ with the quotient topology induced by the weak*-topology on $\mathcal{P}_{\mathcal{A}}$) is compact Hausdorff, Varela duality and Dauns-Hofmann theorem reduce to a previous duality result by J.M.G.Fell [14, 15]: the spectra are just usual Banach C*-bundles (with continuous norm) with fibers that are always primitive C*-algebras and hence there is no problem to proceed to the second stage of "fiber duality" as in section 4, as soon as suitable topologies are imposed on the discrete spaceoids. The standard choice of topologies will be to equip:

- $\mathcal{P}_{\mathcal{A}}$ with its weak*-topology, as subspace of $\mathcal{A}^* := \mathcal{B}(\mathcal{A}; \mathbb{C})$, the Banach dual space of $\mathcal{A}$,

[46] This corresponds to the "first picture" for "topological spaceoids" in [4].

- $\mathcal{P}_{\mathcal{A}} \times_{\chi_{\mathcal{A}}} \mathcal{P}_{\mathcal{A}} \subset \mathcal{P}_{\mathcal{A}} \times \mathcal{P}_{\mathcal{A}}$ with the subspace topology of the product weak*-topology on $\mathcal{P}_{\mathcal{A}} \times \mathcal{P}_{\mathcal{A}}$,

- $\mathcal{E}_{\mathcal{A}}$ with the tubular topology induced by the family of Gel'fand transforms $\hat{x} : \mathcal{P}_{\mathcal{A}} \times_{\chi_{\mathcal{A}}} \mathcal{P}_{\mathcal{A}} \to \mathcal{E}_{\mathcal{A}}$.

Unfortunately, the space $\mathcal{P}_{\mathcal{A}}$ (and even $(\mathcal{P}_{\mathcal{A}})_o := \chi_{\mathcal{A}}^{-1}(o)$, for $o \in \mathcal{X}_{\mathcal{A}}$) with the weak*-topology is not always compact, a well-known problem that (at least in those approaches to spectral theory, reconstructing C*-algebras via continuous functions/sections on the space of pure states, see for example the works by R.Cirelli-A.Manià-L.Pizzocchero [8] and N.Landsman [21, 22]) requires the introduction of suitable uniformities. This will force us, in the general case, to work with "uniform spaceoids" defined as uniform bundles of "uniform propagators" (with γ uniformly continuous with values in a "uniform Fell line-bundle") instead of just bundles of 1-C*-category-valued propagators, as we did in section 4.[47]

Whenever $\widehat{\mathcal{A}}$ is (compact) Hausdorff and the C*-algebra $\mathcal{A}$ has only finite-dimensional irreducible representations, A.J.Lazar [23] has described a specific form of Dauns-Hofmann theorem where the spectra are "scaled" Banach bundles over $\widehat{\mathcal{A}}$, with fibers primitive finite-dimensional C*-algebras. Since in this case $(\mathcal{P}_{\mathcal{A}})_o$ is compact for all $o \in \mathcal{X}_{\mathcal{A}} = \widehat{\mathcal{A}}$, and since for every γ_o-frame $\mathcal{F}$ the restriction of the (uniform) propagator $(\mathcal{E}_{\mathcal{A}})|_{\mathcal{F}}$ is a 1-C*-category, the techniques already contained in section 4 immediately allow to obtain the correct fiber duality and a non-commutative Gel'fand-Naĭmark spectral analysis.

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