

ON (σ, τ) - LEFT JORDAN IDEALS AND GENERALIZED DERIVATIONS

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Abstract

Let R be a prime ring with characteristic not 2 and $\sigma, \tau, \alpha, \beta, \lambda, \mu, \gamma$ automorphisms of R . Let $h : R \rightarrow R$ be a nonzero right (resp. left)-generalized (α, β) - derivation associated with (α, β) - derivation d (resp. d_1). Let W, V be nonzero left (σ, τ) -Jordan ideals of R . The main object in this paper is to study the situations. (1) $h(W) = 0$, (2) $[b, W]_{\lambda, \mu} = 0$ or $[W, b]_{\lambda, \mu} = 0$, (3) $(b, W)_{\lambda, \mu} = 0$ or $(W, b)_{\lambda, \mu} = 0$, (4) $b[W, a]_{\lambda, \mu} = 0$ or $[W, a]_{\lambda, \mu} b = 0$ (5) $b(W, a)_{\lambda, \mu} = 0$ or $(W, a)_{\lambda, \mu} b = 0$, (6) $bW \subset C_{\lambda, \mu}(V)$ or $Wb \subset C_{\lambda, \mu}(V)$. (7) $(h(R), b)_{\lambda, \mu} a = 0$ or $a(h(R), b)_{\lambda, \mu} = 0$.

1 Introduction

Let R be a ring and σ, τ two mappings of R . For each $r, s \in R$ we set $[r, s] = rs - sr$, $(r, s) = rs + sr$, $[r, s]_{\sigma, \tau} = r\sigma(s) - \tau(s)r$ and $(r, s)_{\sigma, \tau} = r\sigma(s) + \tau(s)r$. Let U be an additive subgroup of R . If $(U, R) \subset U$ then U is called a Jordan ideal of R . The definition of (σ, τ) -Jordan ideal of R is introduced in [6] as follows: (i) U is called a right (σ, τ) -Jordan ideal of R if $(U, R)_{\sigma, \tau} \subset U$, (ii) U is called a left (σ, τ) -Jordan ideal if $(R, U)_{\sigma, \tau} \subset U$. (iii) U is called a (σ, τ) -Jordan ideal if U is both right and left (σ, τ) -Jordan ideal of R . Every Jordan ideal of R is a $(1, 1)$ -Jordan ideal of R , where $1 : R \rightarrow R$ is an identity map. The following example is given in [6]. If $R = \{ \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \mid x \text{ and } y \text{ are integers} \}$, $U = \{ \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} \mid x \text{ is integer} \}$, $\sigma \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$ and $\tau \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} x & -y \\ 0 & 0 \end{pmatrix}$ then U is (σ, τ) -right Jordan ideal but not a Jordan ideal of R .

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A derivation d is an additive mapping on R which satisfies $d(rs) = d(r)s + rd(s), \forall r, s \in R$. The notion of generalized derivation was introduced by Brešar [2] as follows. An additive mapping $h : R \rightarrow R$ will be called a generalized derivation if there exists a derivation d of R such that $h(xy) = h(x)y + xd(y), \forall x, y \in R$.

An additive mapping $d : R \rightarrow R$ is said to be a (σ, τ) -derivation if $d(rs) = d(r)\sigma(s) + \tau(r)d(s)$ for all $r, s \in R$. Every derivation $d : R \rightarrow R$ is a $(1, 1)$ -derivation. Chang [3] gave the following definition. Let R be a ring, σ and τ automorphisms of R and $d : R \rightarrow R$ a (σ, τ) -derivation. An additive mapping $h : R \rightarrow R$ is said to be a right generalized (σ, τ) -derivation of R associated with d if $h(xy) = h(x)\sigma(y) + \tau(x)d(y)$, for all $x, y \in R$ and h is said to be a left generalized (σ, τ) -derivation of R associated with d if $h(xy) = d(x)\sigma(y) + \tau(x)h(y)$, for all $x, y \in R$. h is said to be a generalized (σ, τ) -derivation of R associated with d if it is both a left and right generalized (σ, τ) -derivation of R associated with d . Besides every (σ, τ) -derivation $d : R \rightarrow R$ is a generalized (σ, τ) -derivation associated with d and every derivation $d : R \rightarrow R$ is a generalized $(1, 1)$ -derivation associated with d . A generalized $(1, 1)$ -derivation is simply called a generalized derivation. It is clear that the definition of generalized derivation which is given in [2] is a right generalized derivation associated with derivation d according to Chang's definition.

The mapping $h(r) = (a, r)_{\sigma, \tau}, \forall r \in R$ is a left-generalized (σ, τ) -derivation with (σ, τ) -derivation $d_1(r) = [a, r]_{\sigma, \tau}, \forall r \in R$ and right-generalized (σ, τ) -derivation with (σ, τ) -derivation $d(r) = -[a, r]_{\sigma, \tau}, \forall r \in R$. Every (σ, τ) -derivation $d : R \rightarrow R$ is a generalized (σ, τ) -derivation with d .

Throughout the paper, R will be a prime ring with center Z , characteristic not 2 and $\sigma, \tau, \alpha, \beta, \lambda, \mu, \gamma$ automorphisms of R . We set $C_{\sigma, \tau}(R) = \{c \in R \mid c\sigma(r) = \tau(r)c, \forall r \in R\}$, and shall use the following relations frequently:

$$\begin{aligned} [rs, t]_{\sigma, \tau} &= r[s, t]_{\sigma, \tau} + [r, \tau(t)]s = r[s, \sigma(t)] + [r, t]_{\sigma, \tau}s \\ [r, st]_{\sigma, \tau} &= \tau(s)[r, t]_{\sigma, \tau} + [r, s]_{\sigma, \tau}\sigma(t) \\ (rs, t)_{\sigma, \tau} &= r(s, t)_{\sigma, \tau} - [r, \tau(t)]s = r[s, \sigma(t)] + (r, t)_{\sigma, \tau}s. \\ (r, st)_{\sigma, \tau} &= \tau(s)(r, t)_{\sigma, \tau} + [r, s]_{\sigma, \tau}\sigma(t) = -\tau(s)[r, t]_{\sigma, \tau} + (r, s)_{\sigma, \tau}\sigma(t) \end{aligned}$$

2 Results

Lemma 1. [1, Lemma 3] *Let U be a nonzero ideal of R and $d : R \rightarrow R$ a (σ, τ) -derivation. If $a \in R$ such that $ad(U) = 0$ (or $d(U)a = 0$) then $a = 0$ or $d = 0$.*

Lemma 2. *Let I be a nonzero ideal of R and $a, b \in R$.*

- (i) If $b, ba \in C_{\lambda, \mu}(I)$ or $(b, ab \in C_{\lambda, \mu}(I))$ then $b = 0$ or $a \in Z$.
- (ii) If $b\gamma(I, a)_{\alpha, \beta} = 0$ or $\gamma(I, a)_{\alpha, \beta}b = 0$ then $b = 0$ or $a \in Z$.

Proof. (i) If $b, ba \in C_{\lambda, \mu}(I)$ then we have, for all $x \in I$

$$0 = [ba, x]_{\lambda, \mu} = b[a, \lambda(x)] + [b, x]_{\lambda, \mu}a = b[a, \lambda(x)]$$

and so $b[a, \lambda(x)] = 0, \forall x \in I$. Replacing x by $xr, r \in R$ then we get $b\lambda(I)[a, R] = 0$. Since $\lambda(I)$ is a nonzero ideal of R , then we have $b = 0$ or $a \in Z$.

If $b, ab \in C_{\lambda, \mu}(I)$ then considering as above and using the relation

$$0 = [ab, x]_{\lambda, \mu} = a[b, x]_{\lambda, \mu} + [a, \mu(x)]b = [a, \mu(x)]b, \forall x \in I$$

we obtain the result.

(ii) If $b\gamma(I, a)_{\alpha, \beta} = 0$ then, for all $x \in I, r \in R$

$$0 = b\gamma(xr, a)_{\alpha, \beta} = b\gamma(x)\gamma[r, \alpha(a)] + b\gamma(x, a)_{\alpha, \beta}\gamma(r) = b\gamma(x)\gamma[r, \alpha(a)].$$

That is $b\gamma(I)\gamma[R, \alpha(a)] = 0$. Since $\gamma(I)$ is a nonzero ideal, then we obtain that $b = 0$ or $a \in Z$ by the last relation

If $\gamma(I, a)_{\alpha, \beta}b = 0$ then, for all $x \in I, r \in R$

$$0 = \gamma(rx, a)_{\alpha, \beta}b = \gamma(r)\gamma(x, a)_{\alpha, \beta}b - \gamma[r, \beta(a)]\gamma(x)b = -\gamma[r, \beta(a)]\gamma(x)b$$

which gives $\gamma[R, \beta(a)]\gamma(I)b = 0$. Considering as above we get the required result. \square

Corollary 1. [7, Lemma 4] *Let b and ab be in the center of a prime ring R . If b is not zero, then a is in Z , the center of R .*

Theorem 1. *Let I, J be nonzero ideals of R and $a, b \in R$. Let W be a left (σ, τ) -Jordan ideal of R .*

- (i) If $b\gamma(I, a)_{\alpha, \beta} \subset C_{\lambda, \mu}(J)$ or $\gamma(I, a)_{\alpha, \beta}b \subset C_{\lambda, \mu}(J)$ then $b = 0$ or $a \in Z$.
- (ii) If $b\gamma(W) \subset C_{\lambda, \mu}(J)$ or $\gamma(W)b \subset C_{\lambda, \mu}(J)$ then $b = 0$ or $W \subset Z$.

Proof. (i) If $b\gamma(I, a)_{\alpha, \beta} \subset C_{\lambda, \mu}(J)$ then we have, for all $x \in I$

$$C_{\lambda, \mu}(J) \ni b\gamma(x\alpha(a), a)_{\alpha, \beta} = b\gamma(x)\gamma[\alpha(a), \alpha(a)] + b\gamma(x, a)_{\alpha, \beta}\gamma\alpha(a) = b\gamma(x, a)_{\alpha, \beta}\gamma\alpha(a).$$

Then

$$b\gamma(I, a)_{\alpha, \beta}\gamma\alpha(a) \subset C_{\lambda, \mu}(J). \quad (2.1)$$

If we use Lemma2 (i) in (2.1) then we get

$$b\gamma(I, a)_{\alpha, \beta} = 0 \text{ or } a \in Z.$$

If $b\gamma(I, a)_{\alpha, \beta} = 0$ then we have $b = 0$ or $a \in Z$ by Lemma2 (ii).

If $\gamma(I, a)_{\alpha, \beta}b \subset C_{\lambda, \mu}(J)$ then we have, for all $x \in I$

$$C_{\lambda, \mu}(J) \ni \gamma(\beta(a)x, a)_{\alpha, \beta}b = \gamma\beta(a)\gamma(x, a)_{\alpha, \beta}b - \gamma[\beta(a), \beta(a)]\gamma(x)b = \gamma\beta(a)\gamma(x, a)_{\alpha, \beta}b.$$

That is

$$\gamma\beta(a)\gamma(I, a)_{\alpha, \beta}b \in C_{\lambda, \mu}(J). \quad (2.2)$$

If we use Lemma2 (i) then (2.2) gives that

$$\gamma(I, a)_{\alpha, \beta}b = 0 \text{ or } a \in Z.$$

If $\gamma(I, a)_{\alpha, \beta}b = 0$ then using Lemma2 (ii) we obtain the required result.

(ii) If $b\gamma(W) \in C_{\lambda, \mu}(J)$ or $\gamma(W)b \in C_{\lambda, \mu}(J)$ then we have $b\gamma(R, W)_{\sigma, \tau} \in C_{\lambda, \mu}(J)$ or $\gamma(R, W)_{\sigma, \tau}b \in C_{\lambda, \mu}(J)$. Thus $b = 0$ or $W \subset Z$ by (i). \square

Corollary 2. [4, Theorem 2.2] *Let W be a left (σ, τ) -Jordan ideal of R . If $aW = 0$ (or $Wa = 0$) and $a \in R$, then $a = 0$ or $W \subset Z$.*

Lemma 3. *Let $h : R \rightarrow R$ be a nonzero right generalized (α, β) -derivation associated with a nonzero (α, β) -derivation $d : R \rightarrow R$ and I a nonzero ideal of R . If a is a noncentral element of R such that $h\lambda(I, a)_{\sigma, \tau} = 0$ or $(h\lambda(I), a)_{\alpha\lambda\sigma, \beta\lambda\tau} = 0$ then $d\lambda\sigma(a) = 0$.*

Proof. If $h\lambda(I, a)_{\sigma, \tau} = 0$ then we get, for all $x \in I$

$$\begin{aligned} 0 &= h\lambda(x\sigma(a), a)_{\sigma, \tau} = h\{\lambda(x)\lambda[\sigma(a), \sigma(a)] + \lambda(x, a)_{\sigma, \tau}\lambda\sigma(a)\} \\ &= h\{\lambda(x, a)_{\sigma, \tau}\lambda\sigma(a)\} = h\lambda(x, a)_{\sigma, \tau}\alpha\lambda\sigma(a) + \beta\lambda(x, a)_{\sigma, \tau}d\lambda\sigma(a) \\ &= \beta\lambda(x, a)_{\sigma, \tau}d\lambda\sigma(a). \end{aligned}$$

That is

$$\beta\lambda(I, a)_{\sigma, \tau}d\lambda\sigma(a) = 0. \quad (2.3)$$

Since $a \notin Z$ then (2.3) gives that $d\lambda\sigma(a) = 0$ by Lemma2 (ii).

If $(h\lambda(I), a)_{\alpha\lambda\sigma, \beta\lambda\tau} = 0$ then we have, for all $x \in I$

$$\begin{aligned} 0 &= (h\lambda(x\sigma(a)), a)_{\alpha\lambda\sigma, \beta\lambda\tau} = (h\lambda(x)\alpha\lambda\sigma(a) + \beta\lambda(x)d\lambda\sigma(a), a)_{\alpha\lambda\sigma, \beta\lambda\tau} \\ &= h\lambda(x)[\alpha\lambda\sigma(a), \alpha\lambda\sigma(a)] + (h\lambda(x), a)_{\alpha\lambda\sigma, \beta\lambda\tau}\alpha\lambda\sigma(a) \\ &+ \beta\lambda(x)(d\lambda\sigma(a), a)_{\alpha\lambda\sigma, \beta\lambda\tau} - [\beta\lambda(x), \beta\lambda\tau(a)]d\lambda\sigma(a) \\ &= \beta\lambda(x)(d\lambda\sigma(a), a)_{\alpha\lambda\sigma, \beta\lambda\tau} - [\beta\lambda(x), \beta\lambda\tau(a)]d\lambda\sigma(a) \end{aligned}$$

and so

$$\beta\lambda(x)(k, a)_{\alpha\lambda\sigma, \beta\lambda\tau} - [\beta\lambda(x), \beta\lambda\tau(a)]k = 0, \forall x \in I \quad (2.4)$$

where $k = d\lambda\sigma(a)$. Replacing x by rx , $r \in R$ in (2.4) and using (2.4) we get, for all $r \in R, x \in I$

$$\begin{aligned} 0 &= \beta\lambda(r)\beta\lambda(x)(k, a)_{\alpha\lambda\sigma, \beta\lambda\tau} - \beta\lambda(r)[\beta\lambda(x), \beta\lambda\tau(a)]k - [\beta\lambda(r), \beta\lambda\tau(a)]\beta\lambda(x)k \\ &= -[\beta\lambda(r), \beta\lambda\tau(a)]\beta\lambda(x)k \end{aligned}$$

so $[R, \beta\lambda\tau(a)]\beta\lambda(I)k = 0$. Since $\beta\lambda(I)$ is a nonzero ideal of R and $a \notin Z$, then we obtain that $d\lambda\sigma(a) = 0$ by the last relation. \square

Lemma 4. *Let I be a nonzero ideal of R and $h : R \longrightarrow R$ a nonzero left-generalized (α, β) - derivation associated with a nonzero (α, β) - derivation $d_1 : R \longrightarrow R$. If a is a noncentral element of R such that $h\lambda(I, a)_{\sigma, \tau} = 0$ or $(h\lambda(I), a)_{\alpha\lambda\sigma, \beta\lambda\tau} = 0$ then $d_1\lambda\tau(a) = 0$.*

Proof. If $h\lambda(I, a)_{\sigma, \tau} = 0$ then we have, for all $x \in I$

$$\begin{aligned} 0 &= h\lambda(\tau(a)x, a)_{\sigma, \tau} = h\{\lambda\tau(a)\lambda(x, a)_{\sigma, \tau} - \lambda[\tau(a), \tau(a)]\lambda(x)\} = h\{\lambda\tau(a)\lambda(x, a)_{\sigma, \tau}\} \\ &= d_1\lambda\tau(a)\alpha\lambda(x, a)_{\sigma, \tau} + \beta\lambda\tau(a)h\lambda(x, a)_{\sigma, \tau} = d_1\lambda\tau(a)\alpha\lambda(x, a)_{\sigma, \tau}. \end{aligned}$$

That is

$$d_1\lambda\tau(a)\alpha\lambda(I, a)_{\sigma, \tau} = 0. \quad (2.5)$$

Since a is noncentral, using 2 (ii) and (2.5) we get $d_1\lambda\tau(a) = 0$.

Similarly, if $(h\lambda(I), a)_{\alpha\lambda\sigma, \beta\lambda\tau} = 0$ then we get, for all $x \in I$

$$\begin{aligned} 0 &= (h\lambda(\tau(a)x), a)_{\alpha\lambda\sigma, \beta\lambda\tau} = (d_1\lambda\tau(a)\alpha\lambda(x) + \beta\lambda\tau(a)h\lambda(x), a)_{\alpha\lambda\sigma, \beta\lambda\tau} \\ &= d_1\lambda\tau(a)[\alpha\lambda(x), \alpha\lambda\sigma(a)] + (d_1\lambda\tau(a), a)_{\alpha\lambda\sigma, \beta\lambda\tau}\alpha\lambda(x) \\ &\quad + \beta\lambda\tau(a)(h\lambda(x), a)_{\alpha\lambda\sigma, \beta\lambda\tau} - [\beta\lambda\tau(a), \beta\lambda\tau(a)]h\lambda(x) \\ &= d_1\lambda\tau(a)[\alpha\lambda(x), \alpha\lambda\sigma(a)] + (d_1\lambda\tau(a), a)_{\alpha\lambda\sigma, \beta\lambda\tau}\alpha\lambda(x). \end{aligned}$$

That is

$$k[\alpha\lambda(x), \alpha\lambda\sigma(a)] + (k, a)_{\alpha\lambda\sigma, \beta\lambda\tau}\alpha\lambda(x) = 0, \forall x \in I \quad (2.6)$$

where $k = d_1\lambda\tau(a)$. Replacing x by $xr, r \in R$ in (2.6) we get, for all $x \in I, r \in R$

$$\begin{aligned} 0 &= k\alpha\lambda(x)[\alpha\lambda(r), \alpha\lambda\sigma(a)] + k[\alpha\lambda(x), \alpha\lambda\sigma(a)]\alpha\lambda(r) + (k, a)_{\alpha\lambda\sigma, \beta\lambda\tau}\alpha\lambda(x)\alpha\lambda(r) \\ &= k\alpha\lambda(x)[\alpha\lambda(r), \alpha\lambda\sigma(a)] \end{aligned}$$

so $k\alpha\lambda(I)[R, \alpha\lambda\sigma(a)] = 0$. The last relation gives that $d_1\lambda\tau(a) = 0$ or $[R, \alpha\lambda\sigma(a)] = 0$. Since $a \notin Z$ then we have $d_1\lambda\tau(a) = 0$. \square

Theorem 2. *Let $h : R \longrightarrow R$ be a nonzero right-generalized (α, β) - derivation associated with (α, β) - derivation $d : R \longrightarrow R$ and left-generalized (α, β) - derivation associated with (α, β) - derivation $d_1 : R \longrightarrow R$. Let a be a noncentral element of R and I a nonzero ideal of R . Then $h\lambda(I, a)_{\sigma, \tau} = 0$ if and only if $(h\lambda(I), a)_{\alpha\lambda\sigma, \beta\lambda\tau} = 0$.*

Proof. If $(h\lambda(I), a)_{\alpha\lambda\sigma, \beta\lambda\tau} = 0$ or $h\lambda(I, a)_{\sigma, \tau} = 0$ then $d\lambda\sigma(a) = 0$ and $d_1\lambda\tau(a) = 0$ by Lemma 3 and Lemma 4. Using these results we get, for all $x \in I$

$$\begin{aligned} h\lambda(x, a)_{\sigma, \tau} = 0 &\Leftrightarrow h\lambda(x\sigma(a) + \tau(a)x) = 0 \\ &\Leftrightarrow h\lambda(x)\alpha\lambda\sigma(a) + \beta\lambda(x)d\lambda\sigma(a) + d_1\lambda\tau(a)\alpha\lambda(x) + \beta\lambda\tau(a)h\lambda(x) = 0 \\ &\Leftrightarrow h\lambda(x)\alpha\lambda\sigma(a) + \beta\lambda\tau(a)h\lambda(x) = 0 \\ &\Leftrightarrow (h\lambda(x), a)_{\alpha\lambda\sigma, \beta\lambda\tau} = 0. \end{aligned}$$

That is $(h\lambda(I), a)_{\alpha\lambda\sigma, \beta\lambda\tau} = 0$ if and only if $h\lambda(I, a)_{\sigma, \tau} = 0$. \square

Corollary 3. [5, Theorem 7] *Let R be a prime ring of characteristic different from two, $d : R \rightarrow R$ a nonzero derivation and $a \in R$. Then $(d(R), a) = 0$ if and only if $d(R, a) = 0$*

Theorem 3. *Let $d : R \rightarrow R$ be a nonzero (α, β) -derivation and I a nonzero ideal of R . If $b \in R$ such that $d\lambda(I, b)_{\sigma, \tau} = 0$ then $\sigma(b) - \tau(b) \in Z$.*

Proof. If $b \in Z$ then we have $\sigma(b) - \tau(b) \in Z$. Hence let $b \notin Z$. Since d is an (α, β) -derivation then d is a right (and left)-generalized (α, β) -derivation associated with d .

If $d\lambda(I, b)_{\sigma, \tau} = 0$ then $(d\lambda(I), b)_{\alpha\lambda\sigma, \beta\lambda\tau} = 0$ by Theorem 2. Using this relation, we get, for all $x, y \in I$

$$\begin{aligned} 0 &= (d\lambda(x(y, b)_{\sigma, \tau}), b)_{\alpha\lambda\sigma, \beta\lambda\tau} \\ &= (d\lambda(x)\alpha\lambda(y, b)_{\sigma, \tau} + \beta\lambda(x)d\lambda(y, b)_{\sigma, \tau}, b)_{\alpha\lambda\sigma, \beta\lambda\tau} = (d\lambda(x)\alpha\lambda(y, b)_{\sigma, \tau}, b)_{\alpha\lambda\sigma, \beta\lambda\tau} \\ &= d\lambda(x)[\alpha\lambda(y, b)_{\sigma, \tau}, \alpha\lambda\sigma(b)] + (d\lambda(x), b)_{\alpha\lambda\sigma, \beta\lambda\tau}\alpha\lambda(y, b)_{\sigma, \tau} \\ &= d\lambda(x)[\alpha\lambda(y, b)_{\sigma, \tau}, \alpha\lambda\sigma(b)] \end{aligned}$$

which gives that $d\lambda(I)[\alpha\lambda(y, b)_{\sigma, \tau}, \alpha\lambda\sigma(b)] = 0, \forall y \in I$. Since $\lambda(I)$ is a nonzero ideal of R and $d \neq 0$ then using Lemma 1 we obtain that

$$[(y, b)_{\sigma, \tau}, \sigma(b)] = 0, \forall y \in I. \quad (2.7)$$

Replacing y by $\tau(b)y$ in (2.7) we have, for all $y \in I$

$$\begin{aligned} 0 &= [(\tau(b)y, b)_{\sigma, \tau}, \sigma(b)] = [\tau(b)(y, b)_{\sigma, \tau} - [\tau(b), \tau(b)]y, \sigma(b)] \\ &= [\tau(b)(y, b)_{\sigma, \tau}, \sigma(b)] = \tau(b)[(y, b)_{\sigma, \tau}, \sigma(b)] + [\tau(b), \sigma(b)](y, b)_{\sigma, \tau} \\ &= [\tau(b), \sigma(b)](y, b)_{\sigma, \tau}. \end{aligned}$$

That is $[\tau(b), \sigma(b)](I, b)_{\sigma, \tau} = 0$. Since $b \notin Z$, then using Lemma 2 (ii) we get $[\tau(b), \sigma(b)] = 0$ and so

$$\sigma(b)\tau(b) = \tau(b)\sigma(b). \quad (2.8)$$

If we consider (2.7) and (2.8) we have, for all $y \in I$

$$\begin{aligned} 0 &= [(y, b)_{\sigma, \tau}, \sigma(b)] = y\sigma(b)\sigma(b) + \tau(b)y\sigma(b) - \sigma(b)y\sigma(b) - \sigma(b)\tau(b)y \\ &= y\sigma(b)\sigma(b) + \tau(b)y\sigma(b) - \sigma(b)y\sigma(b) - \tau(b)\sigma(b)y \\ &= [y, \sigma(b)]\sigma(b) + \tau(b)[y, \sigma(b)] = ([y, \sigma(b)], b)_{\sigma, \tau}. \end{aligned}$$

That is

$$([y, \sigma(b)], b)_{\sigma, \tau} = 0, \forall y \in I. \quad (2.9)$$

Replacing y by yz , $z \in I$ in (2.9) we get, for all $y, z \in I$

$$\begin{aligned} 0 &= (y[z, \sigma(b)] + [y, \sigma(b)]z, b)_{\sigma, \tau} = (y[z, \sigma(b)], b)_{\sigma, \tau} + ([y, \sigma(b)]z, b)_{\sigma, \tau} \\ &= y([z, \sigma(b)], b)_{\sigma, \tau} - [y, \tau(b)][z, \sigma(b)] + [y, \sigma(b)][z, \sigma(b)] + ([y, \sigma(b)], b)_{\sigma, \tau}z \\ &= -[y, \tau(b)][z, \sigma(b)] + [y, \sigma(b)][z, \sigma(b)] = [y, \sigma(b) - \tau(b)][z, \sigma(b)] \end{aligned}$$

which gives that

$$[y, \sigma(b) - \tau(b)][I, \sigma(b)] = 0, \forall y \in I. \quad (2.10)$$

Replacing y by ry , $r \in R$ in (2.10) we have $[R, \sigma(b) - \tau(b)]I[I, \sigma(b)] = 0$. Since b is noncentral then we obtain that $\sigma(b) - \tau(b) \in Z(R)$ by the last relation. \square

Corollary 4. *Let $d : R \rightarrow R$ be a nonzero (α, β) -derivation and W a nonzero left (σ, τ) -Jordan ideal of R . If $d\gamma(W) = 0$ then $\sigma(v) - \tau(v) \in Z, \forall v \in W$.*

Proof. If $d\gamma(W) = 0$ then $d\gamma(R, v)_{\sigma, \tau} = 0, \forall v \in W$. This gives that $\sigma(v) - \tau(v) \in Z, \forall v \in W$ by Theorem 3. \square

Theorem 4. *Let W be a nonzero left (σ, τ) -Jordan ideal of R . Let $h : R \rightarrow R$ be a nonzero left-generalized (α, β) -derivation associated with a nonzero (α, β) -derivation $d_1 : R \rightarrow R$ of R and $b \in R$.*

- (i) If $h\lambda(W) = 0$ then $\sigma(v) - \tau(v) \in Z, \forall v \in W$.
- (ii) If $[W, b]_{\lambda, \mu} = 0$ then $b \in Z$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$.
- (iii) If $[b, W]_{\lambda, \mu} = 0$ then $b \in C_{\lambda, \mu}(R)$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$.

Proof. (i) If $h\lambda(W) = 0$ then $h\lambda(R, v)_{\sigma, \tau} = 0, \forall v \in W$. This means that, for any $v \in W$,

$$v \in Z \text{ or } d_1\lambda\tau(v) = 0$$

by Lemma 4. Let $K = \{v \in W \mid v \in Z\}$ and $L = \{v \in W \mid d_1\lambda\tau(v) = 0\}$. Then K and L are two additive subgroups of W such that $W = K \cup L$. Since a group cannot be the union of proper subgroups, according to Brauer's Trick either $W = K$ or $W = L$. That is

$$W \subset Z \text{ or } d_1\lambda\tau(W) = 0.$$

It is clear that, if $W \subset Z$ then $\sigma(v) - \tau(v) \in Z, \forall v \in W$. On the other hand, if $d_1\lambda\tau(W) = 0$ then we have $\sigma(v) - \tau(v) \in Z, \forall v \in W$ by Corollary 4.

(ii) The mapping $g(r) = [r, b]_{\lambda, \mu}, \forall r \in R$ is a left-generalized derivation associated with derivation $d_2(r) = [r, \mu(b)], \forall r \in R$. If $g = 0$ then $d_2 = 0$ and so $b \in Z$ is obtained.

Let $d_2 \neq 0$. If $[W, b]_{\lambda, \mu} = 0$ then $g(W) = 0$. Using (i) we have $\sigma(v) - \tau(v) \in Z, \forall v \in W$. Finally, we obtain that $b \in Z$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$.

(iii) The mapping $d_3(r) = [b, r]_{\lambda, \mu}, \forall r \in R$ is a (λ, μ) -derivation and so left-generalized derivation associated with d_3 . If $d_3 = 0$ then $b \in C_{\lambda, \mu}(R)$. Let $d_3 \neq 0$.

If $[b, W]_{\lambda, \mu} = 0$ then $d_3(W) = 0$. This means that $\sigma(v) - \tau(v) \in Z, \forall v \in W$ by Corollary 4. Finally, we obtain that $b \in C_{\lambda, \mu}(R)$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$ for any cases. \square

Theorem 5. *Let W be a nonzero left (σ, τ) -Jordan ideal of R and $b \in R$.*

- (i) If $(b, W)_{\lambda, \mu} = 0$ then $b \in C_{\lambda, \mu}(R)$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$.
- (ii) If $(W, b)_{\lambda, \mu} = 0$ then $b \in Z$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$.

Proof. (i) The mapping $h(r) = (b, r)_{\lambda, \mu}, \forall r \in R$ is a left-generalized (λ, μ) -derivation associated with (λ, μ) -derivation $d(r) = [b, r]_{\lambda, \mu}, \forall r \in R$. If $h = 0$ then $d = 0$ and so $b \in C_{\lambda, \mu}(R)$ is obtained. Let $d \neq 0$. If $(b, W)_{\lambda, \mu} = 0$ then we have $h(W) = 0$. Using Theorem 4 (i) we obtain that $\sigma(v) - \tau(v) \in Z, \forall v \in W$. Finally we obtain that $b \in C_{\lambda, \mu}$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$ for any cases.

(ii) Similarly, the mapping $g(r) = (r, b)_{\lambda, \mu}, \forall r \in R$ is a left-generalized derivation associated with derivation $d_1(r) = -[r, \mu(b)], \forall r \in R$. If $g = 0$ then $d_1 = 0$ and so $b \in Z$ is obtained. Let $d_1 \neq 0$. If $(W, b)_{\lambda, \mu} = 0$ then $g(W) = 0$. This gives that $\sigma(v) - \tau(v) \in Z, \forall v \in W$ by Theorem 4 (i). Finally, we obtain that $b \in Z$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$ for any cases. \square

Lemma 5. *Let $h : R \rightarrow R$ be a nonzero right-generalized (α, β) -derivation associated with a nonzero (α, β) -derivation $d : R \rightarrow R$. If $a, b \in R$ such that $a[h(R), b]_{\lambda, \mu} = 0$ then $a[a, \mu(b)] = 0$ or $d\alpha^{-1}\lambda(b)$.*

Proof. If $a[h(R), b]_{\lambda, \mu} = 0$ then we get, for all $r \in R$

$$\begin{aligned} 0 &= a[h(r\alpha^{-1}\lambda(b)), b]_{\lambda, \mu} = a[h(r)\lambda(b) + \beta(r)d\alpha^{-1}\lambda(b), b]_{\lambda, \mu} \\ &= ah(r)[\lambda(b), \lambda(b)] + a[h(r), b]_{\lambda, \mu}\lambda(b) + a\beta(r)[d\alpha^{-1}\lambda(b), b]_{\lambda, \mu} \\ &\quad + a[\beta(r), \mu(b)]d\alpha^{-1}\lambda(b) \\ &= a\beta(r)[d\alpha^{-1}\lambda(b), b]_{\lambda, \mu} + a[\beta(r), \mu(b)]d\alpha^{-1}\lambda(b). \end{aligned}$$

That is

$$a\beta(r)[k, b]_{\lambda, \mu} + a[\beta(r), \mu(b)]k = 0, \forall r \in R. \quad (2.11)$$

where $k = d\alpha^{-1}\lambda(b)$. Replacing r by $\beta^{-1}(a)r$ in (2.11) and using (2.11) we have, for all $r \in R$

$$\begin{aligned} 0 &= aa\beta(r)[k, b]_{\lambda, \mu} + a[a\beta(r), \mu(b)]k \\ &= aa\beta(r)[k, b]_{\lambda, \mu} + aa[\beta(r), \mu(b)]k + a[a, \mu(b)]\beta(r)k \\ &= a[a, \mu(b)]\beta(r)k \end{aligned}$$

which gives $a[a, \mu(b)]Rk = 0$. Using that primeness of R we get $a[a, \mu(b)] = 0$ or $d\alpha^{-1}\lambda(b) = 0$. \square

Lemma 6. *Let $h : R \rightarrow R$ be a nonzero left-generalized (α, β) -derivation associated with a nonzero (α, β) -derivation $d_1 : R \rightarrow R$. If $a, b \in R$ such that $[h(R), b]_{\lambda, \mu} a = 0$ then $d_1 \beta^{-1} \mu(b) = 0$ or $[a, \lambda(b)]a = 0$.*

Proof. If $[h(R), b]_{\lambda, \mu} a = 0$ then we get, for all $r \in R$

$$\begin{aligned} 0 &= [h(\beta^{-1} \mu(b)r), b]_{\lambda, \mu} a = [d_1 \beta^{-1} \mu(b) \alpha(r) + \mu(b)h(r), b]_{\lambda, \mu} a \\ &= d_1 \beta^{-1} \mu(b) [\alpha(r), \lambda(b)]a + [d_1 \beta^{-1} \mu(b), b]_{\lambda, \mu} \alpha(r) a \\ &\quad + \mu(b) [h(r), b]_{\lambda, \mu} a + [\mu(b), \mu(b)] h(r) a \\ &= d_1 \beta^{-1} \mu(b) [\alpha(r), \lambda(b)]a + [d_1 \beta^{-1} \mu(b), b]_{\lambda, \mu} \alpha(r) a \end{aligned}$$

which gives that

$$k[\alpha(r), \lambda(b)]a + [k, b]_{\lambda, \mu} \alpha(r) a = 0, \forall r \in R \quad (2.12)$$

where $k = d_1 \beta^{-1} \mu(b)$. Replacing r by $r\alpha^{-1}(a)$ in (2.12) and using (2.12) we have, for all $r \in R$

$$\begin{aligned} 0 &= k[\alpha(r)a, \lambda(b)]a + [k, b]_{\lambda, \mu} \alpha(r) aa \\ &= k\alpha(r)[a, \lambda(b)]a + k[\alpha(r), \lambda(b)]aa + [k, b]_{\lambda, \mu} \alpha(r) aa \\ &= k\alpha(r)[a, \lambda(b)]a. \end{aligned}$$

That is $kR[a, \lambda(b)]a = 0$. Since R is a prime ring, then the last relation gives that $d_1 \beta^{-1} \mu(b) = 0$ or $[a, \lambda(b)]a = 0$. \square

Theorem 6. *Let W be a nonzero left (σ, τ) -Jordan ideal of R and $a, b \in R$.*

- (i) If $a[W, b]_{\lambda, \mu} = 0$ then $a[a, \mu(b)] = 0$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$.
- (ii) If $[W, b]_{\lambda, \mu} a = 0$ then $[a, \lambda(b)]a = 0$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$.

Proof. Let us consider a nonzero element v of W . The mapping defined by $h(r) = (r, v)_{\sigma, \tau}, \forall r \in R$ is a left-generalized derivation associated with derivation $d_1(r) = -[r, \tau(v)], \forall r \in R$ and right-generalized derivation associated with derivation $d(r) = [r, \sigma(v)], \forall r \in R$. If $h = 0$ then $d = 0 = d_1$ and so $v \in Z$ is obtained. Let $d \neq 0$ and $d_1 \neq 0$.

(i) If $a[W, b]_{\lambda, \mu} = 0$ then we have $a[(R, v)_{\sigma, \tau}, b]_{\lambda, \mu} = 0$ and so $a[h(R), b]_{\lambda, \mu} = 0$. Since h is a right-generalized derivation associated with d , then using Lemma 5 we get $d\lambda(b) = 0$ or $a[a, \mu(b)] = 0$. That is

$$[\lambda(b), \sigma(v)] = 0 \text{ or } a[a, \mu(b)] = 0.$$

On the other hand, if $v = 0$ then $[\lambda(b), \sigma(v)] = 0$. Hence, considering the same argument for all $v \in W$ we have

$$[\sigma^{-1} \lambda(b), W] = 0 \text{ or } a[a, \mu(b)] = 0.$$

If $[\sigma^{-1}\lambda(b), W] = 0$ then $b \in Z$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$ by Theorem4(iii). Finally we obtained that $a[a, \mu(b)] = 0$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$ for any cases.

(ii) If $[W, b]_{\lambda, \mu} a = 0$ then we have $[(R, v)_{\sigma, \tau}, b]_{\lambda, \mu} a = 0$ and so $[h(R), b]_{\lambda, \mu} a = 0$. Since h is a left-generalized derivation associated with d_1 then, using Lemma 6, we get $d_1\mu(b) = 0$ or $[a, \lambda(b)]a = 0$. That is $[\mu(b), \tau(v)] = 0$ or $[a, \lambda(b)]a = 0$. Considering as in the proof of (i) we get

$$[\tau^{-1}\mu(b), W] = 0 \text{ or } [a, \lambda(b)]a = 0.$$

If $[\tau^{-1}\mu(b), W] = 0$ then we have $b \in Z$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$ by Theorem4(iii). If $b \in Z$ then $[a, \lambda(b)]a = 0$.

Finally, we obtained that $[a, \lambda(b)]a = 0$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$ for any cases. \square

Lemma 7. *Let $h : R \rightarrow R$ be a nonzero right-generalized (α, β) -derivation associated with a nonzero (α, β) -derivation $d : R \rightarrow R$ and $a, b \in R$. If $a(h(R), b)_{\lambda, \mu} = 0$ then $d\alpha^{-1}\lambda(b) = 0$ or $a[a, \mu(b)] = 0$.*

Proof. If $a(h(R), b)_{\lambda, \mu} = 0$ then we get, for all $r \in R$

$$\begin{aligned} 0 &= a(h(r\alpha^{-1}\lambda(b)), b)_{\lambda, \mu} = a(h(r)\lambda(b) + \beta(r)d\alpha^{-1}\lambda(b), b)_{\lambda, \mu} \\ &= ah(r)[\lambda(b), \lambda(b)] + a(h(r), b)_{\lambda, \mu}\lambda(b) + a\beta(r)(d\alpha^{-1}\lambda(b), b)_{\lambda, \mu} \\ &\quad - a[\beta(r), \mu(b)]d\alpha^{-1}\lambda(b) \end{aligned}$$

which gives that

$$a\beta(r)(k, b)_{\lambda, \mu} - a[\beta(r), \mu(b)]k = 0, \forall r \in R. \quad (2.13)$$

where $k = d\alpha^{-1}\lambda(b)$. Replacing r by $\beta^{-1}(a)r$ in (2.13) we have, for all $r \in R$

$$\begin{aligned} 0 &= aa\beta(r)(k, b)_{\lambda, \mu} - a[a\beta(r), \mu(b)]k \\ &= aa\beta(r)(k, b)_{\lambda, \mu} - aa[\beta(r), \mu(b)]k - a[a, \mu(b)]\beta(r)k \\ &= -a[a, \mu(b)]\beta(r)k. \end{aligned}$$

That is $a[a, \mu(b)]Rd\alpha^{-1}\lambda(b) = 0$. Since R is a prime ring, we obtain that $d\alpha^{-1}\lambda(b) = 0$ or $a[a, \mu(b)] = 0$. \square

Lemma 8. *Let $h : R \rightarrow R$ be a nonzero left-generalized (α, β) -derivation associated with a nonzero (α, β) -derivation $d_1 : R \rightarrow R$ and $a, b \in R$. If $(h(R), b)_{\lambda, \mu} a = 0$ then $d_1\beta^{-1}\mu(b) = 0$ or $[a, \lambda(b)]a = 0$.*

Proof. If $(h(R), b)_{\lambda, \mu} a = 0$ then we get, for all $r \in R$

$$\begin{aligned} 0 &= (h(\beta^{-1}\mu(b)r), b)_{\lambda, \mu} a = (d_1\beta^{-1}\mu(b)\alpha(r) + \mu(b)h(r), b)_{\lambda, \mu} a \\ &= d_1\beta^{-1}\mu(b)[\alpha(r), \lambda(b)]a + (d_1\beta^{-1}\mu(b), b)_{\lambda, \mu}\alpha(r)a \\ &\quad + \mu(b)(h(r), b)_{\lambda, \mu} a - [\mu(b), \mu(b)]h(r)a \\ &= d_1\beta^{-1}\mu(b)[\alpha(r), \lambda(b)]a + (d_1\beta^{-1}\mu(b), b)_{\lambda, \mu}\alpha(r)a \end{aligned}$$

which gives that

$$k[\alpha(r), \lambda(b)]a + (k, b)_{\lambda, \mu}\alpha(r)a = 0, \forall r \in R. \quad (2.14)$$

where $k = d_1\beta^{-1}\mu(b)$. If we take $r\alpha^{-1}(a)$ instead of r in (2.14), we get, for all $r \in R$

$$\begin{aligned} 0 &= k[\alpha(r)a, \lambda(b)]a + (k, b)_{\lambda, \mu}\alpha(r)aa \\ &= k\alpha(r)[a, \lambda(b)]a + k[\alpha(r), \lambda(b)]aa + (k, b)_{\lambda, \mu}\alpha(r)aa \\ &= k\alpha(r)[a, \lambda(b)]a. \end{aligned}$$

That is $kR[a, \lambda(b)]a = 0$. This gives that $d_1\beta^{-1}\mu(b) = 0$ or $[a, \lambda(b)]a = 0$ in prime rings. \square

Theorem 7. Let W be a nonzero left (σ, τ) -Jordan ideal of R and $a, b \in R$.

- (i) If $a(W, b)_{\lambda, \mu} = 0$ then $a[a, \mu(b)] = 0$ or $\sigma(v) - \tau(v) \in Z, \forall v \in V$.
- (ii) If $(W, b)_{\lambda, \mu} a = 0$ then $[a, \lambda(b)]a = 0$ or $\sigma(v) - \tau(v) \in Z, \forall v \in V$.

Proof. Let us consider a nonzero element v of W . The mapping defined by $h(r) = (r, v)_{\sigma, \tau}, \forall r \in R$ is a left-generalized derivation associated with derivation $d_1(r) = -[r, \tau(v)], \forall r \in R$ and right-generalized derivation associated with derivation $d(r) = [r, \sigma(v)], \forall r \in R$. If $h = 0$ then $d = 0 = d_1$ and so $v \in Z$ is obtained. Let $d \neq 0$ and $d_1 \neq 0$.

(i) If $a(W, b)_{\lambda, \mu} = 0$ then we have $a((R, v)_{\sigma, \tau}, b)_{\lambda, \mu} = 0$. That is $a(h(R), b)_{\lambda, \mu} = 0$. Since h is a right-generalized derivation with d then using Lemma 7 we obtain that $d\lambda(b) = 0$ or $a[a, \mu(b)] = 0$. That is

$$[\lambda(b), \sigma(v)] = 0 \text{ or } a[a, \mu(b)] = 0.$$

On the other hand, if $v = 0$ then $[\lambda(b), \sigma(v)] = 0$. Hence, considering the same argument for all $v \in W$ we have

$$[b, W] = 0 \text{ or } a[a, \mu(b)] = 0.$$

If $[b, W] = 0$ then we have $b \in Z$ or $\sigma(v) - \tau(v) \in Z, \forall v \in W$ by Theorem 4 (iii). If $b \in Z$ then $a[a, \mu(b)] = 0$.

Finally, we obtained that $a[a, \mu(b)] = 0$ or $\sigma(v) - \tau(v) \in Z, \forall v \in V$ for any cases.

(ii) If $(W, b)_{\lambda, \mu} a = 0$ then $(h(R), b)_{\lambda, \mu} a$ is obtained. Since h is a left-generalized derivation associated with d_1 then using Lemma 8 and considering as in the proof of (i) we get $[a, \lambda(b)]a = 0$ or $\sigma(v) - \tau(v) \in Z, \forall v \in V$. \square

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