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THE INTERSECTION PROBLEMS OF REAL PARAMETRIC CURVES AND SURFACES BY MEANS OF MATRIX BASED IMPLICIT REPRESENTATIONS: A NEW APPROACH

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Abstract

Evaluating the intersection of two real rational parameterized algebraic surfaces is an important problem in solid modeling. In [9, 10, 22], we have already developed an approach based on generalized matrix representations of parameterized curves and surfaces in order to represent the intersection points or curves as the generalized eigenvalues of a matrix or the zero set of a matrix determinant. These computations based on complicate techniques from linear algebra such as QR-Decomposition, ΔW-Decomposition to obtain square matrices that hold the necessary properties. In this paper, we propose an new method to obtain intersection represented matrices that are square without complicate computing.

1 Introduction

In geometric modeling, parameterized algebraic curves and surfaces are used intensively. To manipulate them, it is useful to have an implicit representation, in addition to their given parametric representation. Indeed, a parametric representation is for instance well adapted for visualization purposes whereas an implicit representation allows significant improvements in the computation

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of intersections. Nevertheless, implicit representations are known to be very hard to compute in general. Matrix-based implicit representations of plane curves and surfaces already appeared several times in the literature (see e.g. [4, 11, 26]). However, all these approaches aimed at building a non-singular matrix whose determinant is an implicit polynomial equation. The case of plane curves is well understood: it is always possible to build such a non-singular matrix, in particular by means of the moving lines method introduced by Sederberg [26]. The case of surfaces is much more involved because of their rich geometry and the occurrence of base points (the points where the parameterization is not well defined). Thus, in order to find a non-singular matrix whose determinant is an implicit polynomial equation, one has to consider some very particular classes of parameterizations (see e.g. [3, 8]). In [9, 10, 22], we show that matrix-based implicit representations can be built for (almost all) parameterized al- gebraic curves, including space curves, and surfaces if the requirement of getting a non-singular matrix is deleted. Indeed, the matrices we will introduce are in general singular matrices, but they still represent the curve or surface: the vanishing of a determinant will be replaced by a drop-of-rank property. Our approach is hence to keep these matrices as implicit representations on their own and to develop their study and use. Added and combined to the parametric representations, we believe that these implicit matrix representations can be a powerful tool. The goal of this paper is continued to overcome this difficulty by developing a simple method for computing an implicit representation of a parameterized curve or surface in the form of a matrix, then present new way to solve the intersection problems.

The paper is organized as follows. In Section 2, we review the construction of curve and surface matrix representations. In Section 3, we present some techniques in linear algebra such as linearization of a polynomial matrix and rank of real matrix. In Section 4, we present some algorithms to solve the intersection problems and characterizes the singular pointson the curve by rank of a representation matrix in Section 5.

2 Implicit Matrix Representations of Parameterized Curves and Surfaces

In this section, we recall the construction in a general framework a family of matrices given by parameterization ϕ from [3, 4, 8, 9, 23].

2.1 Matrix based implicit representations of parameterized surfaces

Given a parametric rational surface, we briefly recall from [3, 8] how to build a matrix that *represents* this surface in a sense that we will make explicit. Let a parameterization

$$\begin{array}{rcl} \mathbb{P}^2_{\mathbb{R}} & \xrightarrow{\phi} & \mathbb{P}^3_{\mathbb{R}} \qquad (1) \\ (s:t:u) & \mapsto & (f_1:f_2:f_3:f_4)(s,t,u) \qquad (2) \end{array}$$

of a surface $\mathbf{S}$ such that $f_1, f_2, f_3, f_4 \in \mathbb{R}[s,t,u]$ are homogeneous polynomials of the same degree $d \geq 1$ and $\gcd(f_1, \ldots, f_4) \in \mathbb{R} \setminus \{0\}$. Denote by x, y, z, w the homogeneous coordinates of the projective space $\mathbb{P}^3_{\mathbb{R}}$. Notice that s, t, u are the homogeneous parameters of the surface $\mathbf{S}$ and that an affine parameterization of $\mathbf{S}$ can be obtained by "inverting" one of these parameters; for instance, setting $s' = s/u$ and $t' = t/u$ we get the following affine parameterization of $\mathbf{S}$:

$$\begin{array}{rcl} \mathbb{R}^2 & \xrightarrow{\phi} & \mathbb{R}^3 \\ (s', t') & \mapsto & \left(\dfrac{f_1(s',t',1)}{f_4(s',t',1)}, \dfrac{f_2(s',t',1)}{f_4(s',t',1)}, \dfrac{f_3(s',t',1)}{f_4(s',t',1)} \right) \end{array}$$

The implicit equation of $\mathbf{S}$ is a homogeneous polynomial of smallest degree $S(x,y,z,w) \in \mathbb{R}[x,y,z,w]$ such that $S(f_1, f_2, f_3, f_4) = 0$ (observe that it is defined up to multiplication by a nonzero element in $\mathbb{R}$). It is well known that the quantity $\deg(\mathbf{S})\deg(\phi)$ is equal to d^2 minus the number of common roots of f_1, f_2, f_3, f_4 in $\mathbb{P}^2_{\mathbb{R}}$, that are called *base points* of the parameterization ϕ, counted with suitable multiplicities. The notation $\deg(\mathbf{S})$ stands for the degree of the surface $\mathbf{S}$, which is nothing but the degree of the implicit equation of $\mathbf{S}$ and $\deg(\phi)$ is equal to the number of pre-images of a general point on $\mathbf{S}$ by the parameterization ϕ.

For every non-negative integer ν, we build a matrix $\mathtt{M}(\phi)_\nu$ as follows. Consider the set $\mathcal{L}(\phi)_\nu$ of polynomials of the form

$$a_1(s,t,u)x + a_2(s,t,u)y + a_3(s,t,u)z + a_4(s,t,u)w$$

such that

- $a_i(s,t,u) \in \mathbb{R}[s,t,u]$ is homogeneous of degree ν for $i = 1, \ldots, 4$,
- $\sum_{i=1}^4 a_i(s,t,u) f_i(s,t,u) \equiv 0$ in $\mathbb{R}[s,t,u]$.

The set $\mathcal{L}(\phi)_\nu$ has a natural structure of $\mathbb{R}$-vector space of finite dimension because each polynomial $a_i(s,t,u)$ is homogeneous of degree ν and that the set of homogeneous polynomials of degree ν in the variables s, t, u is an $\mathbb{R}$-vector space of dimension $\binom{\nu+2}{2}$ with canonical basis the set of monomials $\{s^\nu, s^{\nu-1}t, \cdots, u^\nu\}$. So, denote by $L^{(1)}, \ldots, L^{(n_\nu)}$ a basis of the $\mathbb{R}$-vector space $\mathcal{L}(\phi)_\nu$; it can be computed by solving a single linear system whose indeterminates are the coefficients of the polynomials $a_i(s,t,u)$, $i = 1, 2, 3, 4$. The matrix $\mathtt{M}(\phi)_\nu$ is then by definition the matrix of coefficients of $L^{(1)}, \ldots, L^{(n_\nu)}$

as homogeneous polynomials of degree ν in the variables s, t, u. In other words, we have the equality of matrices:

$$\begin{bmatrix} s^\nu & s^{\nu-1}t & \cdots & u^\nu \end{bmatrix} \mathtt{M}(\phi)_\nu = \begin{bmatrix} L^{(1)} & L^{(2)} & \cdots & L^{(n_\nu)} \end{bmatrix}.$$

Notice that we have chosen for simplicity the monomial basis for the $\mathbb{R}$-vector space of homogeneous polynomials of degree ν in s, t, u. However, any other choice, for instance the Bernstein basis, can be made without affecting the result.

For every integer $\nu \geq 2d-2$, the matrix $\mathtt{M}(\phi)_\nu$ is said to be a *representation matrix* of ϕ because it satisfies the following properties under the assumption that the base points of ϕ, if any, form *locally a complete intersection*, which means that at each base point, the ideal of polynomials (f_1, f_2, f_3, f_4) can be generated by two equations (see [8, Definition 4.8] for more details):

- The entries of $\mathtt{M}(\phi)_\nu$ are linear forms in $\mathbb{R}[x, y, z, w]$.
- The matrix $\mathtt{M}(\phi)_\nu$ has $\binom{\nu+2}{2}$ rows (which is nothing but the dimension of the $\mathbb{R}$- vector space of homogeneous polynomials of degree ν in three variables, here s, t, u) and possesses at least as many columns as rows.
- The rank of $\mathtt{M}(\phi)_\nu$ is $\binom{\nu+2}{2}$(the rank of $\mathtt{M}(\phi)_\nu$ measures the independency of the columns (and the rows) as linear combinations with coefficients in $\mathbb{R}$).
- When specializing $\mathtt{M}(\phi)_\nu$ at a given point $P \in \mathbb{P}^3_{\mathbb{R}}$, its rank drops if and only if P belongs to $\mathbf{S}$.
- The greatest common divisor of the $\binom{\nu+2}{2}$-minors of $\mathtt{M}(\phi)_\nu$ is equal to the implicit equation of $\mathbf{S}$ raised to the power $\deg(\phi)$.

From a computational point of view, the matrix $\mathtt{M}(\phi)_\nu$ with the smallest possible value of ν has to be chosen. It is rarely a square matrix. Also, notice that the last property given above is never used for computations; our aim is to keep the matrix $\mathtt{M}(\phi)_\nu$ as an implicit representation of $\mathbf{S}$ in place of its implicit equation.

Example 1. The Steiner surface $\mathbf{S}$ of degree 2 parameterized by

$$\phi_1 : \mathbb{P}^2 \to \mathbb{P}^3_{\mathbb{R}} : (s : t : u) \mapsto (s^2 + t^2 + u^2 : tu : st : su)$$

admits the matrix representation

$$M(x,y,z,w) := \begin{pmatrix} -x & 0 & -y & 0 & -y & y & 0 & z & 0 \\ y & -y & 0 & w & 0 & -x & -y & 0 & 0 \\ 0 & 0 & w & 0 & 0 & 0 & z & 0 & -x \\ w & 0 & 0 & -y & 0 & z & 0 & -y & y \\ 0 & w & 0 & 0 & 0 & z & 0 & 0 & y \\ w & 0 & 0 & 0 & z & 0 & 0 & 0 & y \end{pmatrix}.$$

Example 2. Let $\mathbf{S}$ be the rational surface of degree 3 that is parametrized by

$$\phi : \mathbb{P}^2 \to \mathbb{P}^3_{\mathbb{R}} : (s : t : u) \mapsto (f_1 : f_2 : f_3 : f_4)$$

where

$$f_1 = s^3 + t^2u, f_2 = s^2t + t^2u, f_3 = s^3 + t^3, f_4 = s^2u + t^2u.$$

Then, a matrix representation of $\mathbf{S}$ is

$$\begin{pmatrix} 0 & 0 & 0 & w-y & 0 & 0 & z-x \\ w & 0 & 0 & x & w-y & 0 & 0 \\ x-y-z & 0 & 0 & -z & 0 & w-y & 0 \\ 0 & w & 0 & 0 & x & 0 & -y \\ 0 & x-y-z & w & 0 & -z & x & y+z-x \\ 0 & 0 & x-y-z & 0 & 0 & -z & 0 \end{pmatrix}.$$

2.2 Matrix-based implicit representations of parameterized curves in space

Let f_0, f_1, f_2, f_3 be homogeneous polynomials in $\mathbb{R}[s,t]$ of the same degree $d \geq 1$ such that their greatest common divisor is a non-zero constant in $\mathbb{R}$. Consider the regular map of a parametric space curve

$$\begin{array}{ccc} \mathbb{P}^1_{\mathbb{R}} & \xrightarrow{\phi} & \mathbb{P}^3_{\mathbb{R}} \\ (s:t) & \mapsto & (f_0 : f_1 : f_2 : f_3)(s,t). \end{array}$$

Consider the set of syzygies of $\mathbf{f} := (f_0, f_1, f_2, f_3)$, that is to say the set

$$\mathrm{Syz}(\mathbf{f}) = \left\{ (g_0(s,t), \ldots, g_3(s,t)) \ : \ \sum_{i=0}^{3} g_i(s,t) f_i(s,t) = 0 \right\} \subset \bigoplus_{i=0}^{3} \mathbb{R}[s,t].$$

From a classical structure theorem of commutative algebra called the Hilbert-Burch Theorem (see for instance [14, §20.4]), $\mathrm{Syz}(\mathbf{f})$ is known to be a *free* and graded $\mathbb{R}[s,t]$-module of rank 3. Moreover, there exists non-negative integers μ_1, μ_2, μ_3 and 3 vectors of polynomials

$$(u_{i,0}(s,t), u_{i,1}(s,t), u_{i,2}(s,t), u_{i,3}(s,t)) \in \mathrm{Syz}(\mathbf{f}) \subset \mathbb{R}[s,t]^4, \quad i = 1,2,3, \quad (3)$$

such that

- for every $i \in \{1,2,3\}$, $j \in \{0,1,2,3\}$, $u_{i,j}(s,t)$ is a homogeneous polynomial in $\mathbb{R}[s,t]$ of degree $\mu_i \geq 0$,
- three vectors in (3) form an $\mathbb{R}[s,t]$-basis of $\mathrm{Syz}(\mathbf{f})$,

- $\mu_1 + \mu_2 + \mu_3 = d$ where $d = \deg f_i$.
- For every $j \in \{0, \ldots, 3\}$, the determinant of the matrix obtained by deleting the column $(u_{i,j})_{i=1,2,3}$ from the matrix

$$M(s,t) := \begin{pmatrix} u_{1,0}(s,t) & u_{1,1}(s,t) & u_{1,2}(s,t) & u_{1,3}(s,t) \\ u_{2,0}(s,t) & u_{2,1}(s,t) & u_{2,2}(s,t) & u_{2,3}(s,t) \\ u_{3,0}(s,t) & u_{3,1}(s,t) & u_{3,2}(s,t) & u_{3,3}(s,t) \end{pmatrix} \quad (4)$$

  is equal to $(-1)^j c\, f_j(s,t) \in \mathbb{R}[s,t]$ where $c \in \mathbb{R} \setminus \{0\}$.

A collection of vectors as in (3) that satisfy the above properties is called a μ-basis of the parameterization ϕ. It is important to notice that a μ-basis is far from unique, but the collection of integers (μ_1, μ_2, μ_3) is unique if we order it. Therefore, in the sequel we will always assume that a μ-basis is ordered so that $0 \leq \mu_1 \leq \mu_2 \leq \mu_3$.

For every integer $i = 1, 2, 3$ and every integer $\nu \in \mathbb{N}$, consider the matrix $\mathrm{Sylv}_\nu(u_i)$ that satisfies to the identity

$$\left[s^\nu \ s^{\nu-1}t \ \cdots \ st^{\nu-1} \ t^\nu \right] \times \mathrm{Sylv}_\nu(u_i) = \left[s^{\nu-\mu_i} u_i \ s^{\nu-\mu_i-1} t u_i \ \cdots \ st^{\nu-\mu_i-1} u_i \ t^{\nu-\mu_i} u_i \right].$$

It is a $(\nu+1) \times (\nu - \mu_i + 1)$-matrix which usually appears as a building block in well-known Sylvester matrices. It follows that the matrix

$$\mathrm{Sylv}_\nu(u_1, u_2, u_3) := \left(\begin{array}{c|c|c} \mathrm{Sylv}_\nu(u_1) & \mathrm{Sylv}_\nu(u_2) & \mathrm{Sylv}_\nu(u_3) \end{array} \right).$$

It has $\nu + 1$ rows and $3(\nu+1) - d$ columns. Its entries are *linear forms* in $\mathbb{R}[x, y, z, w]$; in particular, it can be evaluated at any point $(x : y : z : w) \in \mathbb{P}^3_{\mathbb{R}}$ and yielding a matrix with coefficients in $\mathbb{R}$.

In [9], we proved that for all $\nu \geq \mu_3 + \mu_2 - 1$ the matrix $\mathtt{M}(\phi)_\nu := \mathrm{Sylv}_\nu(u_1, u_2, u_3)$ is a *matrix-based representation* of the curve $\mathcal{C}$, i.e.,

(i) $\mathtt{M}(\phi)_\nu$ is generically full rank, that is to say generically of rank $\nu + 1$,

(ii) the rank of $\mathtt{M}(\phi)_\nu$ drops exactly on the curve $\mathcal{C}$.

Of course, in practice the most useful matrix is the smallest one, that is to say $\mathtt{M}(\phi)_{\mu_3+\mu_2-1}$.

Example 3. Suppose that the parameterization ϕ is given by

$$\begin{aligned} f_0(s,t) &= 3s^4t^2 - 9s^3t^3 - 3s^2t^4 + 12st^5 + 6t^6, \\ f_1(s,t) &= -3s^6 + 18s^5t - 27s^4t^2 - 12s^3t^3 + 33s^2t^4 + 6st^5 - 6t^6, \\ f_2(s,t) &= s^6 - 6s^5t + 13s^4t^2 - 16s^3t^3 + 9s^2t^4 + 14st^5 - 6t^6, \\ f_3(s,t) &= -2s^4t^2 + 8s^3t^3 - 14s^2t^4 + 20st^5 - 6t^6. \end{aligned}$$

A μ-basis for $\mathcal{C}$ is

$$\begin{aligned} p &= (s^2 - 3st + t^2)x + t^2 y, \\ q &= (s^2 - st + 3t^2)y + (3s^2 - 3st - 3t^2)z, \\ r &= 2t^2 z + (s^2 - 2st - 2t^2)w. \end{aligned}$$

From $\deg(p) = \deg(q) = \deg(r) = 2$, we have $\mu_3 + \mu_2 - 1 = 3$ and hence a matrix representation of $\mathcal{C}$ is given by

$$\mathtt{M}(\phi)_3 = \begin{pmatrix} x+y & 0 & 3y-3z & 0 & 2z-2w & 0 \\ -3x & x+y & -y-3z & 3y-3z & -2w & 2z-2w \\ x & -3x & y+3z & -y-3z & w & -2w \\ 0 & x & 0 & y+3z & 0 & w \end{pmatrix}.$$

3 Linearization of a polynomial matrix in the monomial basis

3.1 Linearization of a polynomial matrix

Let A and B be two matrices of size $m \times n$ with entries in $\mathbb{R}$. We will call a *generalized eigenvalue* of A and B a value in the set

$$\lambda(A, B) := \{t \in \mathbb{R} : \operatorname{rank}(A - tB) < \min\{m, n\}\}.$$

In the case $m = n$, the matrices A and B have n generalized eigenvalues if and only if $\operatorname{rank}(B) = n$. If $\operatorname{rank}(B) < n$, then $\lambda(A, B)$ can be finite, empty or infinite. Moreover, if B is invertible then $\lambda(A, B) = \lambda(AB^{-1}, I) = \lambda(AB^{-1})$, which is the ordinary spectrum of AB^{-1}.

Suppose given an $m \times n$-matrix $M(t) = (a_{i,j}(t))$ with polynomial entries $a_{i,j}(t) \in \mathbb{R}[t]$. It can be equivalently written as a polynomial in t with coefficients $m \times n$-matrices with entries in $\mathbb{R}$: if $d = \max_{i,j}\{\deg(a_{i,j}(t))\}$ then

$$M(t) = M_d t^d + M_{d-1} t^{d-1} + \ldots + M_0$$

where $M_i \in \mathbb{R}^{m \times n}$.

The generalized companion matrices A, B of the matrix $M(t)$ are the matrices with entries in $\mathbb{R}$ of size $((d-1)m + n) \times dm$ that are given by

$$A = \begin{pmatrix} 0 & I_m & \ldots & \ldots & 0 \\ 0 & 0 & I_m & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & I_m \\ M_0^t & M_1^t & \ldots & \ldots & M_{d-1}^t \end{pmatrix}, B = \begin{pmatrix} I_m & 0 & \ldots & \ldots & 0 \\ 0 & I_m & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & I_m & 0 \\ 0 & 0 & \ldots & 0 & -M_d^t \end{pmatrix}$$

where I_m stands for the identity matrix of size m and M_i^t stands for the transpose of the matrix M_i. These companion matrices allow to *linearize* the polynomial matrix $M(t)$ in the sense that there exists two unimodular matrices $E(t)$ et $F(t)$, i.e., invertible matrices with non-vanishing determinant independent of t, with entries in $\mathbb{R}[t]$ and of size dm and $(d-1)m+n$ respectively, such that

$$E(t)\,(A-tB)\,F(t)=\left(\begin{array}{c|c} {}^{t}M(t) & 0 \\ \hline 0 & I_{m(d-1)} \end{array}\right). \qquad (5)$$

Then, we have

$$\operatorname{rank} M(t) \text{ drops} \Leftrightarrow \operatorname{rank}(A-tB) \text{ drops}.$$

3.2 Linearization of a bivariate polynomial matrix

Suppose given an $m \times n$-matrix $M(s,t)=(a_{i,j}(s,t))$ with polynomial entries $a_{i,j}(s,t) \in \mathbb{R}[s,t]$. It can be equivalently written as a polynomial in s whose coefficients are $m \times n$-matrices with entries in $\mathbb{R}[t]$. If $d = \max_{i,j}\{\deg_s(a_{i,j}(s,t))\}$ then set

$$M(s,t)=M_d(t)s^d+M_{d-1}(t)s^{d-1}+\ldots+M_0(t)$$

where $M_i(t) \in \mathbb{R}[t]^{m\times n}$ for all $i=0,\ldots,d$.

Definition 4. The generalized companion matrices $A(t), B(t)$ of the matrix $M(s,t)$ are the matrices with coefficients in $\mathbb{R}[t]$ of size $((d-1)m+n)\times dm$ that are given by

$$A(t)=\begin{pmatrix} 0 & I_m & \ldots & \ldots & 0 \\ 0 & 0 & I_m & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & \ldots & I_m \\ M_0^t(t) & M_1^t(t) & \ldots & \ldots & M_{d-1}^t(t) \end{pmatrix}, \quad B(t)=\begin{pmatrix} I_m & 0 & \ldots & \ldots & 0 \\ 0 & I_m & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & I_m & 0 \\ 0 & 0 & \ldots & \ldots & -M_d^t(t) \end{pmatrix}$$

where I_m stands for the identity matrix of size m and $M_i^t(t)$ stands for the transpose of the matrix $M_i(t)$.

These companion matrices allows to *linearize* the polynomial matrix $M(s,t)$ in the sense that there exists two unimodular matrices $E(s,t)$ et $F(s,t)$ with coefficients in $\mathbb{C}[s,t]$ and of size dm and $(d-1)m+n$ respectively, such that

$$E(s,t)\,(A(t)-sB(t))\,F(s,t)=\left(\begin{array}{c|c} {}^{t}M(s,t) & 0 \\ \hline 0 & I_{m(d-1)} \end{array}\right). \qquad (6)$$

It is important to notice that (6) implies that the computation of the spectrum of the polynomial matrix $M(s,t)$ can be reduced to the computation of the spectrum of the polynomial matrix $A(t)-sB(t)$ which has the advantage to be linear in the variable s.

We also refer the reader to [10, 15, 17, 22] for more details.

3.3 Rank of a real matrix

Given a matrix $A \in \mathbb{R}^{m\times n}$, denote by $\mathfrak{N}(A) := \{x \in \mathbb{R}^n \mid Ax = 0\}$ a null space of A and by A^t a transpose matrix of the matrix A.

Lemma 5. *With above notation, we have* $\mathfrak{N}(A^tA) = \mathfrak{N}(A)$.

Proof. Assume that $x \in \mathfrak{N}(A)$, we have $Ax = 0$. This implies that $A^tAx = 0$. Thus $x \in \mathfrak{N}(A^tA)$. Now, we get $x \in \mathfrak{N}(A^tA) \Rightarrow A^tAx = 0$. Thus,

$$0 =< A^tAx, x >=< Ax, Ax >= \|Ax\|_2^2$$

where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^m$. So that, $Ax = 0$. This implies $x \in \mathfrak{N}(A)$. □

Since $\mathfrak{N}(A^tA) = \mathfrak{N}(A)$, we obtain easily $\mathfrak{N}(AA^t) = \mathfrak{N}(A^t)$.

Proposition 6. *With above notation, we have*

$$\operatorname{rank}(A^tA) = \operatorname{rank}(A) = \operatorname{rank}(A^t) = \operatorname{rank}(AA^t).$$

Proof. From $\dim(\mathfrak{N}(A)) + \operatorname{rank}(A) = n$, we deduce that

$$\operatorname{rank}(A) = n - \dim(\mathfrak{N}(A)).$$

Otherwise, we have also $\operatorname{rank}(A^tA) = n - \dim(\mathfrak{N}(A^tA))$. Thus, by Lemma 5 we have $\operatorname{rank}(A^tA) = \operatorname{rank}(A)$. Similarly, $\operatorname{rank}(A^t) = \operatorname{rank}(AA^t)$ and the well known result is $\operatorname{rank}(A) = \operatorname{rank}(A^t)$, thus we obtain

$$\operatorname{rank}(A^tA) = \operatorname{rank}(A) = \operatorname{rank}(A^t) = \operatorname{rank}(AA^t).$$

□

Remark 7. Matrix A^tA is the square matrix and Proposition 6 is false if the entries of matrix A are complex number. For instance, if $A = [i, 1]$, where $i^2 = -1$, then $AA^t = 0$, thus $\operatorname{rank}(A) = 1 \neq 0 = \operatorname{rank}(A^tA)$.

4 The intersection problems of parameterized curves and surfaces

4.1 Curve/Curve intersection

Suppose given two rational curves, say $\mathcal{C}_1$ parameterized by

$$\mathbb{P}^1_{\mathbb{R}} \xrightarrow{\phi_1} \mathbb{P}^3_{\mathbb{R}} : (s:t) \mapsto (f_0(s,t) : f_1(s,t) : f_2(s,t) : f_3(s,t)) \qquad (7)$$

and $\mathcal{C}_2$ parameterized by the regular map

$$\mathbb{P}^1_{\mathbb{R}} \xrightarrow{\phi_2} \mathbb{P}^3_{\mathbb{R}} : (s:t) \mapsto (g_0(s,t) : g_1(s,t) : g_2(s,t) : g_3(s,t)). \qquad (8)$$

Let $\mathtt{M}(\phi_1)_\nu$ be a representation matrix of $\mathcal{C}_1$ for a suitable integer ν. The substitution in $\mathtt{M}(\phi_1)_\nu$ of the variables x, y, z, w by the homogeneous parameterization of $\mathcal{C}_2$ yields the matrix

$$\mathtt{M}(\phi_1)_\nu(s,t) := \mathtt{M}(\phi_1)_\nu(g_0(s,t),\dots,g_3(s,t)).$$

As a consequence of the properties of a representation matrix, we have the following property: Let $(s_0 : t_0) \in \mathbb{P}^1_{\mathbb{R}}$. Then $\operatorname{rank} \mathtt{M}(\phi_1)_\nu(s_0,t_0) < \nu+1$ if and only if the point $\phi_2(s_0,t_0)$ belongs to the intersection locus $\mathcal{C}_1 \cap \mathcal{C}_2$. The set $\mathcal{C}_1 \cap \mathcal{C}_2$ is in correspondence with the points of $\mathbb{P}^1_{\mathbb{R}}$ where the rank of $\mathtt{M}(\phi_1)_\nu(s,t)$ drops with $(s,t) \in \mathbb{R}^2$. By setting $t = 1$, the determination of the values of $s \in \mathbb{R}$ such that the rank of $\mathtt{M}(\phi_1)_\nu(s,1)$ drops. Now, we put $\mathtt{M}_\nu(s) = \mathtt{M}(\phi_1)_\nu(s,1)\mathtt{M}(\phi_1)^t_\nu(s,1)$ then $\mathtt{M}_\nu(s)$ is a regular matrix of size $(\nu+1)\times(\nu+1)$.

Since Proposition 6, we have the following theorem:

Theorem 8. *With above notation, we have*

$$\{s \in \mathbb{R} \mid \operatorname{rank}(\mathtt{M}(\phi_1)_\nu(s,1)) \ \textit{drops}\} = \{s \in \mathbb{R} \mid \operatorname{rank}(\mathtt{M}_\nu(s)) \ \textit{drops}\}.$$

So, to find the real points of $\mathcal{C}_1 \cap \mathcal{C}_2$, we only find the real values of s such that the rank of $\mathtt{M}_\nu(s)$ drops. Notice that $\mathtt{M}(\phi_1)_\nu(s,1)$ is not regular matrix but $\mathtt{M}_\nu(s)$ is the regular matrix.

Now, we get the following Algorithm:

Algorithm 1: Intersection of two parametric curves

Input: Two parametric curves $\mathcal{C}_1$ and $\mathcal{C}_2$ given by (7) and (8).
Output: The real intersection points of $\mathcal{C}_1$ and $\mathcal{C}_2$.
1. Compute the matrix representation $\mathtt{M}(\phi)_\nu(\phi_1)(x,y,z,w)$ *of* $\mathcal{C}_1$ *for a suitable* ν*.*
2. Replace x, y, z, w *by the parameterization of* $\mathcal{C}_2$ *in the matrix* $\mathtt{M}(\phi)_\nu(\phi_1)$ *to get the matrix* $\mathtt{M}(\phi)_\nu(\phi_1)(s)$ $(t = 1)$*.*
3. Compute $\mathtt{M}_\nu(s) = \mathtt{M}(\phi_1)_\nu(s)\mathtt{M}(\phi_1)^t_\nu(s)$ *and compute the generalized companion matrices* A *and* B *of* $\mathtt{M}(\phi)_\nu(s)$*.*
4. Compute the real generalized eigenvalues of (A,B)*.*
5. For each real eigenvalue s_0*,* $\phi_2(s_0:1)$ *is an real intersection point.*

Remark that this algorithm returns all the points in $\mathcal{C}_1 \cap \mathcal{C}_2$ except possibly the point $\phi(1:0)$. This point can be treated independently.

Example 9 ([9, Example 25]). Let $\mathcal{C}_1$ be the rational space curve given by the parameterization

$$\begin{aligned}
f_0(s,t) &= 3s^4t^2 - 9s^3t^3 - 3s^2t^4 + 12st^5 + 6t^6,\\
f_1(s,t) &= -3s^6 + 18s^5t - 27s^4t^2 - 12s^3t^3 + 33s^2t^4 + 6st^5 - 6t^6,\\
f_2(s,t) &= s^6 - 6s^5t + 13s^4t^2 - 16s^3t^3 + 9s^2t^4 + 14st^5 - 6t^6,\\
f_3(s,t) &= -2s^4t^2 + 8s^3t^3 - 14s^2t^4 + 20st^5 - 6t^6.
\end{aligned}$$

We want to compute the real intersection of $\mathcal{C}_1$ with the twisted cubic $\mathcal{C}_2$ which is parameterized by

$$g_0(s,t) = s^3, g_1(s,t) = s^2t, g_2(s,t) = st^2, g_3(s,t) = t^3.$$

From Example 3, we have a representation matrix of $\mathcal{C}_1$:

$$\mathtt{M}(\phi)_3 = \begin{pmatrix} x+y & 0 & 3y-3z & 0 & 2z-2w & 0 \\ -3x & x+y & -y-3z & 3y-3z & -2w & 2z-2w \\ x & -3x & y+3z & -y-3z & w & -2w \\ 0 & x & 0 & y+3z & 0 & w \end{pmatrix}.$$

A real point P at finite distance belongs to the intersection locus of $\mathcal{C}_1$ and $\mathcal{C}_2$ if and only if $P = (1 : t : t^2 : t^3)$ and t is one of the real generalized eigenvalues of the matrix $\mathtt{M}(t) := \mathtt{M}(\phi)_3(1,t)\big(\mathtt{M}(\phi)_3(1,t)\big)^t = (a_{ij})$ of size 4×4 where

$$\begin{aligned}
a_{11} &= 1 + 2t + 10t^2 - 18t^3 + 13t^4 - 8t^5 + 4t^6, \\
a_{12} &= -3 - 3t - 3t^2 - 6t^3 + 9t^4 - 4t^5 + 4t^6 \\
a_{13} &= 1 + t + 3t^2 + 6t^3 - 9t^4 + 2t^5 - 2t^6, \\
a_{14} &= 0 \\
a_{21} &= -3 - 3t - 3t^2 - 6t^3 + 9t^4 - 4t^5 + 4t^6 \\
a_{22} &= 10 + 2t + 11t^2 - 12t^3 + 22t^4 + 8t^6 - 8t^5 \\
a_{23} &= -6 - 3t - 4t^2 - 12t^3 + 2t^6 - 4t^5 \\
a_{24} &= 1 + t + 3t^2 + 6t^3 - 9t^4 + 2t^5 - 2t^6 \\
a_{31} &= 1 + t + 3t^2 + 6t^3 - 9t^4 + 2t^5 - 2t^6 \\
a_{32} &= -6 - 3t - 4t^2 - 12t^3 + 2t^6 - 4t^5 \\
a_{33} &= 10 + 2t^2 + 12t^3 + 18t^4 + 5t^6 \\
a_{34} &= -3 - t^2 - 6t^3 - 9t^4 - 2t^6 \\
a_{41} &= 0 \\
a_{42} &= 1 + t + 3t^2 + 6t^3 - 9t^4 + 2t^5 - 2t^6 \\
a_{43} &= -3 - t^2 - 6t^3 - 9t^4 - 2t^6 \\
a_{44} &= 1 + t^2 + 6t^3 + 9t^4 + t^6
\end{aligned}$$

We see that $\mathtt{M}(t)$ is the regular polynomial matrix, thus the computation yields a real eigenvalue $t = 0$, and thus $\mathcal{C}_1$ intersect $\mathcal{C}_2$ at the only real point $P = (1 : 0 : 0 : 0)$.

4.2 Curve/Surface intersection

Suppose given a parametric surface **S** represented by a homogeneous and irreducible implicit equation $S(x,y,z,w) = 0$ in $\mathbb{P}^3_{\mathbb{R}}$ and a rational space curve $\mathcal{C}$

represented by a parameterization

$$\phi : \mathbb{P}^1_{\mathbb{R}} \to \mathbb{P}^3_{\mathbb{R}} : (s:t) \mapsto (f_0(s,t) : f_1(s,t) : f_2(s,t) : f_3(s,t)) \qquad (9)$$

A standard problem in nonlinear computational geometry is to determine the set $\mathcal{C} \cap \mathbf{S} \subset \mathbb{P}^3_{\mathbb{R}}$, especially when it is finite. One way to proceed, is to compute the roots of the homogeneous polynomial

$$S(f_0(s,t), f_1(s,t), f_2(s,t), f_3(s,t)) \qquad (10)$$

because they are in correspondence with $\mathcal{C} \cap \mathbf{S}$ through the regular map ϕ. Observe that (10) is identically zero if and only if $\mathcal{C} \cap \mathbf{S}$ is infinite, equivalently $\mathcal{C} \subset \mathbf{S}$ (for $\mathcal{C}$ is irreducible).

Assume that $\mathtt{M}(x,y,z,w)$ is a matrix representation of the surface $\mathbf{S}$, meaning a representation of the polynomial $S(x,y,z,w)$. By replacing the variables x,y,z,w by the homogeneous polynomials $f_0(s,t), f_1(s,t), f_2(s,t), f_3(s,t)$ respectively, we get the matrix

$$\mathtt{M}(s,t) = \mathtt{M}(f_0(s,t), f_1(s,t), f_2(s,t), f_3(s,t)).$$

Therefore, we have the following easy property: for every point $(s_0 : t_0) \in \mathbb{P}^1_{\mathbb{R}}$ the rank of the matrix $\mathtt{M}(s_0,t_0)$ drops if and only if the point $(f_0(s_0,t_0) : f_1(s_0,t_0) : f_2(s_0,t_0) : f_3(s_0,t_0))$ belongs to the intersection locus $\mathcal{C} \cap \mathbf{S}$.

It follows that points in $\mathcal{C} \cap \mathbf{S}$ associated to points $(s:t)$ such that $s \neq 0$, are in correspondence with the set of values $t \in \mathbb{R}$ such that $M(1,t)$ drops of rank strictly less than its row and column dimensions, i.e., the set of generalized eigenvalues of $\mathtt{M}(1,t)$. Similarly the Algorithm in Section 4.1, we present an algorithm from linear algebra which allows us to obtain the intersection points in $\mathbb{P}^1_{\mathbb{R}}$ of $\mathcal{C} \cap \mathbf{S}$.

Algorithm 2: Intersection of parametric curves and surfaces

Input: A parametric surface $\mathbf{S}$ and a parametric curve $\mathcal{C}$ given by (1) and (9).

Output: The intersection points of $\mathbf{S}$ and $\mathcal{C}$.

1. Compute the matrix representation $\mathtt{M}_S(x,y,z,w)$ *of* $\mathbf{S}$.

2. Substitute x,y,z,w *by the parameterization of* $\mathcal{C}$ *in the matrix* $\mathtt{M}_S(x,y,z,w)$ *to get the matrix* $\mathtt{M}_S(t)$ $(s=1)$.

3. Compute $\mathtt{M}(t) = \mathtt{M}_S(t)\mathtt{M}^t_S(t)$ *and compute the generalized companion matrices* A *and* B *of* $\mathtt{M}(t)$.

4. Compute the real generalized eigenvalues of (A,B).

5. For each real eigenvalue t_0, $\phi(1:t_0)$ *is an real intersection point.*

4.3 Surface/Surface intersection

Suppose given two distinct parametric surfaces $\mathbf{S}_1$ and $\mathbf{S}_2$. A standard problem in nonlinear computational geometry is to determine the set $\mathbf{S}_1 \cap \mathbf{S}_2$ which is a curve in $\mathbb{P}^3_{\mathbb{R}}$. As we explained above, one can build a representation matrix of $\mathbf{S}_1$ that we will denote by $M(x, y, z, w)$. Let

$$\phi : \mathbb{P}^2_{\mathbb{R}} \to \mathbb{P}^3_{\mathbb{R}} : (s : t : u) \mapsto (a(s,t,u) : b(s,t,u) : c(s,t,u) : d(s,t,u))$$

be a parameterization of $\mathbf{S}_2$ where $a(s,t,u), b(s,t,u), c(s,t,u), d(s,t,u)$ are homogeneous polynomials of the same degree and without common factor in $\mathbb{R}[s,t,u]$. By substituting in the matrix $M(x,y,z,w)$ the variables x, y, z, w by the homogeneous polynomials $a(s,t,u), b(s,t,u), c(s,t,u)$, $d(s,t,u)$ respectively, we get the matrix

$$M(s,t,u) := M(\phi(s:t:u)) = M(a(s,t,u), b(s,t,u), c(s,t,u), d(s,t,u)).$$

From the properties of the representation matrix $M(x,y,z,w)$, we know that $M(s,t,u)$ has maximal rank ρ (where ρ is the number of rows of M). Moreover, for every point $(s_0 : t_0 : u_0) \in \mathbb{P}^2_{\mathbb{R}}$ we have

$$\operatorname{rank}(M(s_0,t_0,u_0)) < \rho \text{ if and only if } \begin{cases} \phi(s_0 : t_0 : u_0) \in \mathbf{S}_1 \cap \mathbf{S}_2 \text{ or} \\ (s_0 : t_0 : u_0) \text{ is a base point of } \phi. \end{cases} \tag{11}$$

The equivalence (11) shows that the spectrum of the matrix $M(s,t,u)$, that is to say the set

$$\left\{(s_0 : t_0 : u_0) \in \mathbb{P}^2_{\mathbb{R}} \text{ such that } \operatorname{rank} M(s_0,t_0,u_0) < \rho\right\},$$

yields the intersection locus $\mathbf{S}_1 \cap \mathbf{S}_2$ plus the base points of the parameterization ϕ of $\mathbf{S}_2$.

Theorem 10 ([10]). *The spectrum of the matrix* $\mathtt{M}(s,t,u)$ *is an algebraic curve in* $\mathbb{P}^2$*, which means that it is equal to the zero locus of a homogeneous polynomial in* $\mathbb{R}[s,t,u]$*. In particular, there are no isolated points in the spectrum of* $\mathtt{M}(s,t,u)$*.*

In [10], we extend the approach of Canny and Manocha [23] concerning surface/surface intersection to the significantly larger class of parameterizations introduced in Section 2. As a consequence if we use matrix representations to deal with the surface/surface intersection problem, we will at some point end up with a pencil of bivariate and non-square matrices that represents the intersection curve (after dehomogenization). Therefore, in order to be able to handle this intersection curve, for instance to determine its exact topology, it is necessary to extract a pencil of bivariate and square matrices that yields a matrix representation of the intersection curve as a matrix determinant. For that purpose, we developed an algorithm (called ΔW-Decomposition) based on the remarkable work of V. N. Kublanovskaya [19, 20]. We build two companion matrices $A(t)$ and $B(t)$ which allow us to *linearize* the polynomial matrix

$M(s,t,1)$ such that the spectrum of the matrix $M(s,t,1)$ coincides the spectrum of the matrix $A(t)-sB(t)$. Then, we provide an algorithm that extracts a square matrix whose determinant represents the intersection locus $\mathbf{S}_1 \cap \mathbf{S}_2$. A pencil of polynomial matrices $A(t)-sB(t)$ is equivalent to a pencil of the following form

$$P(t)(A(t)-sB(t))Q(t) = \begin{pmatrix} M_{1,1}(s,t) & 0 & 0 \\ M_{2,1}(s,t) & M_{2,2}(s,t) & 0 \\ M_{3,1}(s,t) & M_{3,2}(s,t) & M_{3,3}(s,t) \end{pmatrix} \qquad (12)$$

where $P(t), Q(t)$ are unimodular matrices and the pencil $M_{2,2}(s,t)$ is a regular pencil that corresponds to the intersection locus $\mathbf{S}_1 \cap \mathbf{S}_2$.

However, this work seem to be quite complicated. Thus, by applying Proposition 6, we can transform non square matrix $M(s,t,1)$ into the square matrix $\mathtt{M}(s,t) = M(s,t,1)M^t(s,t,1)$ where in $\mathbb{R}^2$ the spectrum of the matrix $M(s,t,1)$ coincides the spectrum of the square matrix $\mathtt{M}(s,t)$ whose determinant represents the intersection locus $\mathbf{S}_1 \cap \mathbf{S}_2$. Obviously, the way that we obtain the square matrix $\mathtt{M}(s,t)$ is much easier than the way to obtain $M_{2,2}(s,t)$ by applying ΔW-Decomposition. However, the inconvenient of this approach is that $\deg_{s,t} \det(\mathtt{M}(s,t))$ can be more two times greater than $\deg_{s,t} M_{2,2}(s,t)$ and $\det(\mathtt{M}(s,t))$ can contain some extract factors.

Now, we present the following algorithm and an illustrative example.

Algorithm 3: Matrix representation of an intersection curve

Input: Two parametric algebraic surfaces S_1 and S_2 such that the parameterization of S_1 has local complete intersection base points.

Output: The intersection curve $S_1 \cap S_2$ represented as a matrix determinant.

1. Compute a matrix representation of S_1, say $M(x,y,z,w)$.

2. Replace x,y,z,w by the parameterization of S_2 in the matrix $M(x,y,z,w)$ to get a matrix $M(s,t)$ (set $u=1$).

3. Return the regular matrix $\mathtt{M}(s,t) = M(s,t)M^t(s,t)$.

We recall our example in [10] for comparison of two methods.

Example 11 ([10, Example 5.3]). Given the Steiner surface S_1 parameterized by

$$\phi_1 : \mathbb{P}^2 \to \mathbb{P}^3_{\mathbb{R}} : (s:t:u) \mapsto (s^2+t^2+u^2 : tu : st : su)$$

which admits the matrix representation

$$M(x,y,z,w) := \begin{pmatrix} -x & 0 & -y & 0 & -y & y & 0 & z & 0 \\ y & -y & 0 & w & 0 & -x & -y & 0 & 0 \\ 0 & 0 & w & 0 & 0 & 0 & z & 0 & -x \\ w & 0 & 0 & -y & 0 & z & 0 & -y & y \\ 0 & w & 0 & 0 & 0 & z & 0 & 0 & y \\ w & 0 & 0 & 0 & z & 0 & 0 & 0 & y \end{pmatrix},$$

and the cubic surface S_2 parameterized by

$$\phi_2 : \mathbb{P}^2 \to \mathbb{P}^3_{\mathbb{R}} : (s:t:u) \mapsto (s^3+t^3 : stu : su^2+tu^2 : u^3).$$

To determine the intersection between $\mathbf{S}_1$ and $\mathbf{S}_2$, we will compute the spectrum of the polynomial matrix

$$M(s,t,u) = \begin{pmatrix} -s^3-t^3 & 0 & -stu & 0 & -stu & stu & 0 & su^2+tu^2 & 0 \\ stu & -stu & 0 & u^3 & 0 & -s^3-t^3 & -stu & 0 & 0 \\ 0 & 0 & u^3 & 0 & 0 & 0 & su^2+tu^2 & 0 & -s^3-t^3 \\ u^3 & 0 & 0 & -stu & 0 & su^2+tu^2 & 0 & -stu & stu \\ 0 & u^3 & 0 & 0 & 0 & su^2+tu^2 & 0 & 0 & stu \\ u^3 & 0 & 0 & 0 & su^2+tu^2 & 0 & 0 & 0 & stu \end{pmatrix}.$$

By dehomogenizing with respect to the variable u, we consider

$$M(s,t) = \begin{pmatrix} -s^3-t^3 & 0 & -st & 0 & -st & st & 0 & s+t & 0 \\ st & -st & 0 & 1 & 0 & -s^3-t^3 & -st & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & s+t & 0 & -s^3-t^3 \\ 1 & 0 & 0 & -st & 0 & s+t & 0 & -st & st \\ 0 & 1 & 0 & 0 & 0 & s+t & 0 & 0 & st \\ 1 & 0 & 0 & 0 & s+t & 0 & 0 & 0 & st \end{pmatrix}.$$

Thus, we obtain the regular matrix $\mathtt{M}(s,t) = M(s,t)M^t(s,t)$ and determinant of this matrix is

$$F(s,t) = 2(s^5t - s^2 - 2st - s^2t^2 - t^2 - 2s^3t^3 + t^5s)^2(s^6 + 2s^3t^3 + 2s^2t^2 + s^2 + 2st + t^6 + t^2 + 1)^2.$$

From $s^6 + 2s^3t^3 + 2s^2t^2 + s^2 + 2st + t^6 + t^2 + 1 > 0$ for all $(s,t) \in \mathbb{R}^2$, so it yields a real plane curve of degree 6 whose implicit equation is $t^2 + 2st + s^2t^2 + 2s^3t^3 - st^5 + s^2 - ts^5$. In this example, we see that the extract factor $s^6 + 2s^3t^3 + 2s^2t^2 + s^2 + 2st + t^6 + t^2 + 1$ is appeared in the implicit equation of intersection locus $\mathbf{S}_1 \cap \mathbf{S}_2$.

5 Rank of a representation matrix at a singular point

Let P be a point on $\mathcal{C}$. There exists at least one point $(s_1 : t_1) \in \mathbb{P}^1_{\mathbb{C}}$ such that $P = \phi(s_1 : t_1)$. Now, let $\mathcal{H}$ be a plane in $\mathbb{P}^3_{\mathbb{C}}$ passing through P, not containing

$\mathcal{C}$ and denote by $H(x, y, z, w)$ an equation (a linear form in $\mathbb{C}[x, y, z, w]$) of $\mathcal{H}$. We have the following degree d homogeneous polynomial in $\mathbb{C}[s, t]$

$$H(f_0(s,t), f_1(s,t), f_2(s,t), f_3(s,t)) = \prod_{i=1}^{d}(t_i s - s_i t) \tag{13}$$

where the points $(s_i : t_i) \in \mathbb{P}^1_{\mathbb{C}}$, $i = 1, \ldots, d$, are not necessarily distinct. We define the intersection multiplicity of $\mathcal{C}$ with $\mathcal{H}$ at the point P, denoted $i_P(\mathcal{C}, \mathcal{H})$, as the number of points $(s_i : t_i)_{i=1,\ldots,d}$ such that $\phi(s_i : t_i) = P$.

The *multiplicity* $m_P(\mathcal{C})$ of the point P on $\mathcal{C}$ is defined as the minimum of the intersection multiplicities $i_P(\mathcal{C}, \mathcal{H})$ where $\mathcal{H}$ runs over all the planes not containing $\mathcal{C}$ and passing through the point $P \in \mathcal{C}$. This minimum is reached for a sufficiently generic $\mathcal{H}$.

Suppose given a representation matrix $\mathtt{M}(\phi)_\nu$ of the curve $\mathcal{C}$ which is built from the μ-basis p, q, r of degree $\mu_1 \leq \mu_2 \leq \mu_3$. Its entries are linear forms in $\mathbb{R}[x, y, z, w]$ so that it makes sense to evaluate $\mathtt{M}(\phi)_\nu$ at a point P in $\mathbb{P}^3_{\mathbb{R}}$ to get a matrix $\mathtt{M}(\phi)_\nu(P)$ with entries in $\mathbb{R}$. In [9], we prove the following property: Given a point P in $\mathbb{P}^3_{\mathbb{C}}$, for every integer $\nu \geq \mu_2 + \mu_3 - 1$ we have

$$\operatorname{rank} \mathtt{M}(\phi)_\nu(P) = \nu + 1 - m_P(\mathcal{C}),$$

or equivalently $\operatorname{corank} \mathtt{M}(\phi)_\nu(P) = m_P(\mathcal{C})$. Here, we remark that if we put $\mathtt{M}_\nu = \mathtt{M}(\phi)_\nu \mathtt{M}(\phi)_\nu^t$ then $\mathtt{M}_\nu$ is a square matrix. Similarly, the results in [9] is stated the following:

Theorem 12. *Given a point P in $\mathbb{P}^3_{\mathbb{R}}$, for every integer $\nu \geq \mu_2 + \mu_3 - 1$ we have*

$$\operatorname{rank} \mathtt{M}_\nu(P) = \nu + 1 - m_P(\mathcal{C}),$$

This result provides a stratification of the points in $\mathbb{P}^3_{\mathbb{R}}$ with respect to the curve $\mathcal{C}$. Indeed, we have that

- if P is such that $\operatorname{rank} \mathtt{M}_\nu(P) = \nu + 1$ then $P \notin \mathcal{C}$,
- if P is such that $\operatorname{rank} \mathtt{M}_\nu(P) = \nu$ then P is a regular point (i.e. of multiplicity 1) on $\mathcal{C}$,
- if P is such that $\operatorname{rank} \mathtt{M}_\nu(P) = \nu - 1$ then P is singular point of multiplicity 2 on $\mathcal{C}$,
- and so on.

Moreover, if P is a singular point on $\mathcal{C}$ then necessarily

$$2 \leq m_P(\mathcal{C}) \leq \mu_2 \quad \text{or} \quad m_P(\mathcal{C}) = \mu_3.$$

Now, we return an illustrative example.

Example 13 ([9, Example 18]). Let $\mathcal{C}$ be the rational space curve parameterized by

$$\phi : \mathbb{P}^1_{\mathbb{R}} \to \mathbb{P}^3_{\mathbb{R}} : (s:t) \mapsto (s^5 : s^3t^2 : s^2t^3 : t^5).$$

A μ-basis for $\mathcal{C}$ is given by

$$\begin{aligned} p &= ty - sz, \\ q &= t^2x - s^2y, \\ r &= t^2z - s^2w. \end{aligned}$$

From $\deg(q) = \deg(r) = 2$, we can choose $\nu = 3$, then a matrix representation of $\mathcal{C}$ is given by

$$\mathtt{M}(\phi)_3 = \begin{pmatrix} y & 0 & 0 & x & 0 & z & 0 \\ -z & y & 0 & 0 & x & 0 & z \\ x & -z & y & -y & 0 & -w & 0 \\ 0 & 0 & -z & 0 & -y & 0 & -w \end{pmatrix}.$$

Substituting $x = s^5, y = s^3t^2, z = s^2t^3, w = t^5$, we obtain

$$\mathtt{M}(\phi)_3(s,t) = \begin{pmatrix} s^3t^2 & 0 & 0 & s^5 & 0 & s^2t^3 & 0 \\ -s^2t^3 & s^3t^2 & 0 & 0 & s^5 & 0 & s^2t^3 \\ 0 & -s^2t^3 & s^3t^2 & -s^3t^2 & 0 & -t^5 & 0 \\ 0 & 0 & -s^2t^3 & 0 & -s^3t^2 & 0 & -t^5 \end{pmatrix}.$$

and

$$M_3(s,t) = \mathtt{M}(\phi)_3(s,t)\mathtt{M}(\phi)_3(s,t)^t =$$

$$\begin{pmatrix} s^6t^4 + s^{10} + s^4t^6 & -s^5t^5 & -s^8t^2 - s^2t^8 & 0 \\ -s^5t^5 & 2s^4t^6 + s^6t^4 + s^{10} & -s^5t^5 & -s^8t^2 - s^2t^8 \\ -s^8t^2 - s^2t^8 & -s^5t^5 & s^4t^6 + 2s^6t^4 + t^{10} & -s^5t^5 \\ 0 & -s^8t^2 - s^2t^8 & -s^5t^5 & s^4t^6 + s^6t^4 + t^{10} \end{pmatrix}.$$

Now, the Smith form of $M_3(s,1)$ and $M_3(1,t)$ are respectively

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & s^{20} + s^{18} + 3s^{16} + 4s^{14} + 3s^{12} + 4s^{10} + 3s^8 + s^6 + s^4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & t^{20} + t^{18} + 3t^{16} + 4t^{14} + 3t^{12} + 4t^{10} + 3t^8 + t^6 + t^4 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, the singular factors (see [9]) of $\mathcal{C}$ are $d_4(s,t) = 1, d_3(s,t) = 1$, $d_2(s,t) = (s^{20} + s^{18} + 3s^{16} + 4s^{14} + 3s^{12} + 4s^{10} + 3s^8 + s^6 + s^4)(t^{20} + t^{18} + 3t^{16} + 4t^{14} + 3t^{12} + 4t^{10} + 3t^8 + t^6 + t^4)$.

The equation $d_2(s,t) = 0$ has only two real roots $s = 0$ or $t = 0$. Thus, $\mathcal{C}$ has only two singular points of multiplicity 2, the points $A = (0:0:0:1)$ and $B = (1:0:0:0)$ that correspond to the parameters $(0:1)$ and $(1:0)$ respectively.

6 Conclusion and future work

This paper presents an useful tool for solving intersection problems by the implicit representations of parametric curves or parametric surfaces. Its main interested of this tool is particularly to easily transform intersection problems into numerical linear algebra problems which can be solved using powerful and robust algorithms, such as the singular value decomposition and the computation of generalized eigenvalues or eigenvectors.

In future work, we plant to study their numerical stability and robustness of this approach and also compare the numerical stability and the robustness of the existing methods in particular situation.

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