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THE SECOND COHOMOLOGY GROUP OF ELEMENTARY QUADRATIC LIE SUPERALGEBRAS

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Abstract

By definition, a quadratic Lie superalgebra is a Lie superalgebra endowed with a non-degenerate supersymmetric bilinear form which satisfies the even and invariant properties. In this paper we give a new way of description of the cohomology group of quadratic Lie superalgebras by applying the super-Poisson bracket on the super exterior algebra. Moreover, we calculate all of the second cohomology group of elementary quadratic Lie superalgebras which have been classified in [5].

Introduction

As far as we know, the Killing form of a Lie superalgebra is supersymmetric, invariant and even. In some special cases, it also satisfies the non-degeneracy. Those lead to study of Lie superalgebras endowed with a supersymmetric, invariant, even and non-degenerate bilinear form. Such Lie superalgebras are called quadratic Lie superalgebras.

A concerned problem is to describe the cohomology of Lie superalgebras, which is an important tool in mathematics and theoretical physics. A classical

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example of a constant such that D. B. Fuchs and D. A. Leites in [6] calculated the cohomology groups of the classical Lie superalgebras with trivial coefficients, Y. C. Su and R. B. Zhang in [13] computed explicitly the first and second cohomology groups of the classical Lie superalgebras $\mathfrak{sl}_{m|n}$ and $\mathfrak{osp}_{2|2n}$ with coefficients in the finite-dimensional irreducible modules and the Kac modules, W. Bai and W. Liu in [2] described the cohomology groups of Heisenberg Lie superalgebras.

The article is divided into three sections: The first one is devoted to recall some basic concepts and results of Lie superalgebras, cohomology of Lie superalgebras and quadratic Lie superalgebras. The second one gives a new way to describe the cohomology group of quadratic Lie superalgebras by using the super $\mathbb{Z}\times\mathbb{Z}_2-$ Poisson bracket in the super-exterior algebra. The last section computes the second cohomology group of all elementary quadratic Lie superalgebras classified in [5].

All vector spaces considered in throughout the paper are finite-dimensional complex vector spaces.

1 Cohomology of Lie Superalgebras and Quadratic Lie Superalgebras

In this section, we recall some preliminary concepts and basic results which will be used later. For details we refer the reader to the paper [6] of D.B.Fuchs, D.A.Leites and the paper [11] of G. Pinczon, R. Ushirobira.

1.1 Lie Superalgebras and Cohomology

Definition 1.1.1. *A Lie superalgebra* $\mathfrak{g}$ is a $\mathbb{Z}_2-$graded vector space $\mathfrak{g}=\mathfrak{g}_{\overline{0}}\oplus\mathfrak{g}_{\overline{1}}$ endowed with a Lie super bracket [.,.] that satisfies the following conditions:

(i) The Lie super bracket [.,.] is bilinear and $[\mathfrak{g}_x,\mathfrak{g}_y]\subset\mathfrak{g}_{x+y}$ (*grading*);

(ii) $[X,Y]=-(-1)^{xy}[Y,X]$ (*skew-supersymmetry*);

(iii) $(-1)^{zx}[[X,Y],Z]+(-1)^{xy}[[Y,Z],X]+(-1)^{yz}[[Z,X],Y]=0$ (*super Jacobi identity*)

for all $x,y,z\in\mathbb{Z}_2$, $X\in\mathfrak{g}_x, Y\in\mathfrak{g}_y, Z\in\mathfrak{g}_z$.

Definition 1.1.2. Let $\mathfrak{g}=\mathfrak{g}_{\overline{0}}\oplus\mathfrak{g}_{\overline{1}}$ be a Lie superalgebra. Denote by $Alt(\mathfrak{g}_{\overline{0}},\mathbb{C})$ the algebra of alternating multilinear forms on $\mathfrak{g}_{\overline{0}}$ and by $Sym(\mathfrak{g}_{\overline{1}},\mathbb{C})$ the algebra of symmetric multilinear forms on $\mathfrak{g}_{\overline{1}}$. We define a $\mathbb{Z}\times\mathbb{Z}_2$-gradation on $Alt(\mathfrak{g}_{\overline{0}},\mathbb{C})$ and on $Sym(\mathfrak{g}_{\overline{1}},\mathbb{C})$ by

$$Alt^{(i,\overline{0})}(\mathfrak{g}_{\overline{0}},\mathbb{C})=Alt^{i}(\mathfrak{g}_{\overline{0}},\mathbb{C}), Alt^{(i,\overline{1})}(\mathfrak{g}_{\overline{0}},\mathbb{C})=\{0\}$$

and

$$Sym^{(i,\overline{i})}(\mathfrak{g}_{\overline{1}},\mathbb{C}) = Sym^{i}(\mathfrak{g}_{\overline{1}},\mathbb{C}), Sym^{(i,\overline{j})}(\mathfrak{g}_{\overline{1}},\mathbb{C}) = \{0\}$$

where $i,j \in \mathbb{Z}$; $\overline{i},\overline{j} \in \mathbb{Z}_2$ are respectively the residue classes modulo 2 of i,j and $\overline{i} \neq \overline{j}$. The super-exterior algebra of $\mathfrak{g}$ is

$$C(\mathfrak{g},\mathbb{C}) = Alt(\mathfrak{g}_{\overline{0}},\mathbb{C}) \otimes Sym(\mathfrak{g}_{\overline{1}},\mathbb{C})$$

endowed with the super-exterior product on $C(\mathfrak{g},\mathbb{C})$ defined by

$$(\Omega \otimes F) \wedge (\Omega' \otimes F') = (-1)^{f\omega'} (\Omega \wedge \Omega') \otimes FF',$$

for all

$$\Omega \in Alt(\mathfrak{g}_{\overline{0}},\mathbb{C}), \Omega' \in Alt^{\omega'}(\mathfrak{g}_{\overline{0}},\mathbb{C}), F \in Sym^{f}(\mathfrak{g}_{\overline{1}},\mathbb{C}), F' \in Sym(\mathfrak{g}_{\overline{1}},\mathbb{C}).$$

Remark that $C(\mathfrak{g},\mathbb{C})$ is a $\mathbb{Z}\times\mathbb{Z}_2-$graded algebra. More precisely, in terms of $\mathbb{Z}-$gradation, one has

$$C^{n}(\mathfrak{g},\mathbb{C}) = \overset{n}{\underset{m=0}{\oplus}} \left(Alt^{m}(\mathfrak{g}_{\overline{0}},\mathbb{C}) \otimes Sym^{n-m}(\mathfrak{g}_{\overline{1}},\mathbb{C})\right), C^{0}(\mathfrak{g},\mathbb{C}) = \mathbb{C},$$

and in terms of $\mathbb{Z}_2-$gradation,

$$C_{\overline{0}}(\mathfrak{g},\mathbb{C}) = Alt(\mathfrak{g}_{\overline{0}},\mathbb{C}) \otimes \left(\underset{j\geq 0}{\oplus} Sym^{2j}(\mathfrak{g}_{\overline{1}},\mathbb{C})\right)$$

$$\text{and } C_{\overline{1}}(\mathfrak{g},\mathbb{C}) = Alt(\mathfrak{g}_{\overline{0}},\mathbb{C}) \otimes \left(\underset{j\geq 0}{\oplus} Symm^{2j+1}(\mathfrak{g}_{\overline{1}},\mathbb{C})\right).$$

Definition 1.1.3. Denote by $End(C(\mathfrak{g},\mathbb{C}))$ the space of endomorphisms on $C(\mathfrak{g},\mathbb{C})$. A homogeneous endomorphism $D \in End(C(\mathfrak{g},\mathbb{C}))$ of degree (n,d) is called *a superderivation* of $C(\mathfrak{g},\mathbb{C})$ if it satisfies the following condition:

$$D(A \wedge A') = D(A) \wedge A' + (-1)^{na+db} A \wedge D(A)$$

for all $A \in C^{(a,b)}(\mathfrak{g},\mathbb{C})$ and $A' \in C(\mathfrak{g},\mathbb{C})$.

Denote by $Der^{n}_{d}(C(\mathfrak{g},\mathbb{C}))$ the space of superderivations of degree (n,d) of $C(\mathfrak{g},\mathbb{C})$. Then we have a $\mathbb{Z}\times\mathbb{Z}_2-$gradation of the space of superderivations of $C(\mathfrak{g},\mathbb{C})$:

$$Der(C(\mathfrak{g},\mathbb{C})) = \underset{(n,d)\in\mathbb{Z}\times\mathbb{Z}_2}{\oplus} Der^{n}_{d}(C(\mathfrak{g},\mathbb{C})).$$

Example 1.1.4. Let $X \in \mathfrak{g}_x$ be a homogeneous element in $\mathfrak{g}$ of degree x and define the endomorphism i_X of $C(\mathfrak{g},\mathbb{C})$ by

$$i_X(A)\,(X_1,\ldots,X_{a-1}) = (-1)^{xb} A\,(X, X_1,\ldots,X_{a-1})$$

for all $A \in C^{(a,b)}(\mathfrak{g}, \mathbb{C}); X_1, \ldots, X_{a-1} \in \mathfrak{g}$. Then

$$i_X(A \wedge A') = i_X(A) \wedge A' + (-1)^{-a+xb} A \wedge i_X(A)$$

for all $A \in C^{(a,b)}(\mathfrak{g}, \mathbb{C})$, $A' \in C(\mathfrak{g}, \mathbb{C})$. It means that i_X is a superderivation of degree $(-1, x)$.

Given $k \geq 0$, the differential operator $\delta_k : C^k(\mathfrak{g}, \mathbb{C}) \to C^{k+1}(\mathfrak{g}, \mathbb{C})$ is a superderivation of degree $\left(1, \overline{0}\right)$ defined by

$$\delta_k \omega \left(X_0, \ldots, X_k\right) = \sum_{r<s} (-1)^{s+x_s(x_{r+1}+\ldots+x_{s-1})} (X_0, \ldots, X_{r-1}, \left[X_r, X_s\right], X_{r+1}, \ldots, \widehat{X_s}, \ldots, X_k)$$

for all $\omega \in C^k(\mathfrak{g}, \mathbb{C})$, $X_0 \in \mathfrak{g}_{x_0}, \ldots, X_k \in \mathfrak{g}_{x_k}$, where the sign $\widehat{X_s}$ indicates that the element X_s is omitted. It is easy to check that $\delta^2 = \delta \circ \delta = 0$. By convention, $\delta_0 = 0$.

An element $\omega \in C^k(\mathfrak{g}, \mathbb{C})$ is called a k-*cocycle* if $\delta_k \omega = 0$ or a k-*coboundary* if there exists $\varphi \in C^{k-1}(\mathfrak{g}, \mathbb{C})$ such that $\omega = \delta_{k-1}\varphi$.

We denote by $Z^k(\mathfrak{g}, \mathbb{C})$ the set of all k-cocycles and by $B^k(\mathfrak{g}, \mathbb{C})$ the set of all k-coboundaries. That is

$$Z^k(\mathfrak{g}, \mathbb{C}) = Ker\delta_k,\ B^k(\mathfrak{g}, \mathbb{C}) = Im\delta_{k-1}$$

. Clearly, $B^k(\mathfrak{g}, \mathbb{C}) \subset Z^k(\mathfrak{g}, \mathbb{C})$. The quotient space $Z^k(\mathfrak{g}, \mathbb{C})/B^k(\mathfrak{g}, \mathbb{C})$ is denoted by $H^k(\mathfrak{g}, \mathbb{C})$ and called the k *-cohomology groups* of $\mathfrak{g}$ with trivial coefficients.

Definition 1.1.5. The dimension of the k-cohomology group $H^k(\mathfrak{g}, \mathbb{C})$ is called the k*-th Betti number* of $\mathfrak{g}$ and denoted by $b_k(\mathfrak{g})$.

Example 1.1.6. (see [2]) Let the Heisenberg Lie superalgebra

$$\mathfrak{h}_{2n+1,m} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}} = \mathbb{C}\{Z, X_1, \ldots, X_n, X_{n+1}, \ldots, X_{2n}\} \oplus \mathbb{C}\{Y_1, \ldots, Y_m\}$$

with non-zero super brackets

$$[X_i, X_{n+i}] = Z, [Y_j, Y_j] = Z, \forall i = \overline{1, n}, j = \overline{1, m}.$$

It is easy to compute that $\delta X_i^* = \delta Y_j^* = 0$, for all $i = \overline{1, 2n}, j = \overline{1, m}$ and

$$\delta Z^* = \sum_{i=1}^{n} X_{n+i}^* \wedge X_i^* - \frac{1}{2} \sum_{j=1}^{m} Y_j^* Y_j^*.$$

For the second cohomology group, we have

$$\delta \left(Z^* \wedge \omega\right) = \delta Z^* \wedge \omega - Z^* \wedge \delta\omega = 0 \Leftrightarrow \omega = 0.$$

Then

$$Z^2(\mathfrak{h}_{2n+1,m},\mathbb{C}) = \\ = \left\{X_i^*\wedge X_j^*, X_i^*\otimes Y_k^*, Y_k^*Y_l^* : i,j=\overline{1,2n}, i\neq j, k,l=\overline{1,m}\right\}$$

and

$$\dim Z^2(\mathfrak{h}_{2n+1,m},\mathbb{C}) = \binom{2n}{2}+2n.m+m+\binom{m}{2} = 2n^2-n+2nm+\frac{m^2+m}{2}.$$

That means $b_2(\mathfrak{h}_{2n+1,m}) = 2n^2-n+2nm+\dfrac{m^2+m}{2}-1.$

1.2 Quadratic Lie Superalgebras

Definition 1.2.1. Let $\mathfrak{g}=\mathfrak{g}_{\overline{0}}\oplus\mathfrak{g}_{\overline{1}}$ be a Lie superalgebra. Assume that B is a bilinear form defined on $\mathfrak{g}$ such that it satisfies the following properties:

(i) $B(X,Y)=(-1)^{xy}B(Y,X)$, $\forall X\in\mathfrak{g}_x, Y\in\mathfrak{g}_y$ (*supersymmetric*);

(ii) $B([X,Y],Z)=B(X,[Y,Z])$ for all $X,Y,Z\in\mathfrak{g}$ (*invariant*);

(iii) $B(X,Y)=0$, $\forall Y\in\mathfrak{g}$ implies $X=0$ (*non-degenerate*).

The pair $(\mathfrak{g},B)$ is called *a quadratic Lie superalgebra* if B is even, that is

$$B(X,Y)=0;\ \forall X\in\mathfrak{g}_{\overline{0}}, Y\in\mathfrak{g}_{\overline{1}}.$$

In this case, it is easy to check that $\left(\mathfrak{g}_{\overline{0}}, B\left|_{\mathfrak{g}_{\overline{0}}}\times\mathfrak{g}_{\overline{0}}\right.\right)$ is a quadratic Lie algebra and $\left(\mathfrak{g}_{\overline{1}}, B\left|_{\mathfrak{g}_{\overline{1}}}\times\mathfrak{g}_{\overline{1}}\right.\right)$ is a $\mathfrak{g}_{\overline{0}}$−module endowed with a symplectic structure.

Let $(\mathfrak{g},B),(\mathfrak{g}',B')$ be two quadratic Lie superalgebras. We say $(\mathfrak{g},B)$ and $(\mathfrak{g}',B')$ *isometrically isomorphic* (or i-isomorphic, for short) if there exists a Lie superalgebra isomorphism A from $\mathfrak{g}$ onto $\mathfrak{g}'$ satisfying

$$B'(A(X),A(Y))=B(X,Y),\forall X,Y\in\mathfrak{g}.$$

Then A is called an *i-isomorphism*. We write $(\mathfrak{g},B)\overset{i}{\cong}(\mathfrak{g}',B')$.

Definition 1.2.2. Let $(\mathfrak{g},B)$ be a quadratic Lie superalgebra and $\Im$ be a graded ideal of $\mathfrak{g}$.

(i) $\Im$ is called *non-degenerate* if the restriction of B to $\Im\times\Im$ is non-degenerate. Otherwise, we say $\Im$ *degenerate*.

(ii) $(\mathfrak{g},B)$ is called *irreducible* if $\mathfrak{g}$ does not have any non-degenerate graded ideal excepting $\{0\}$ and $\Im$.

(iii) A non-degenerate ideal $\Im$ is called irreducible if $\Im$ does not have any non-degenerate graded ideal excepting $\{0\}$ and $\Im$.

(iv) Ideal $\Im$ is called *totally isotropic* if $B(\Im, \Im) = \{0\}$.

The following proposition reduces the study of quadratic Lie superalgebras to non-degenerate graded ideals.

Proposition 1.2.3 (see [1]). *Let $(\mathfrak{g}, B)$ be a quadratic Lie superalgebra and $\Im$ be a graded ideal of $\mathfrak{g}$. Then $\Im^\perp$ is also a graded ideal of $\mathfrak{g}$. In addition, if $\Im$ is non-degenerate then so is $\Im^\perp$, $[\Im, \Im^\perp] = \{0\}$ and $\Im \cap \Im^\perp = \{0\}$. In this case, we denote $\mathfrak{g} = \Im \overset{\perp}{\oplus} \Im^\perp$.* □

2 The Second Cohomology Group of Elementary Quadratic Lie Superalgebras

In this section, we will compute the second cohomology group of all elementary quadratic Lie superalgebras classified in [5]. Firstly, we recall the concept of the super $\mathbb{Z} \times \mathbb{Z}_2-$ Poisson bracket on the super-exterior algebra of a quadratic Lie superalgebra, which is used to give a new way of description of cohomology.

2.1 The Super $\mathbb{Z} \times \mathbb{Z}_2-$Poisson Bracket on The Super-exterior Algebra

Let $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ be a $\mathbb{Z}_2-$graded vector space equipped with a non-degenerate even supersymmetric bilinear form B. In this case, $\mathfrak{g}_{\overline{1}}$ is a symplectic vector space. Hence, the dimension of $\mathfrak{g}_{\overline{1}}$ must be even and $\mathfrak{g}$ is aslo called a *quadratic* $\mathbb{Z}_2-$graded vector space. Now we recall the definition of the Poisson bracket on $Sym(\mathfrak{g}_{\overline{1}})$ and the super-Poisson bracket on $Alt(\mathfrak{g}_{\overline{0}}, \mathbb{C})$ which are used later.

Let $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n\}$ be a Darboux basis of $\mathfrak{g}_{\overline{1}}$, i.e. we have

$$B(X_i, X_j) = B(Y_i, Y_j) = 0, B(X_i, Y_j) = \delta_{ij},$$

for all $i, j = \overline{1, n}$. Let $\{p_1, \ldots, p_n, q_1, \ldots, q_n\}$ be its dual basis. Then the algebra $Sym(\mathfrak{g}_{\overline{1}}, \mathbb{C})$ regarded as the polynomial algebra

$$\mathbb{C}[p_1, \ldots, p_n, q_1, \ldots, q_n]$$

is equipped with the Poisson bracket as follows:

$$\{F, G\} = \sum_{i=1}^{n} \left(\frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q_i} - \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} \right), \forall F, G \in Sym(\mathfrak{g}_{\overline{1}}, \mathbb{C}).$$

For any $X \in \mathfrak{g}_{\overline{0}}$, let i_X be the derivation of $Alt(\mathfrak{g}_{\overline{0}}, \mathbb{C})$ defined by

$$\iota_X(\Omega)(Z_1, \dots, Z_k) = \Omega(X, Z_1, \dots, Z_k)$$

for all $\Omega \in Alt^{k+1}(\mathfrak{g}_{\overline{0}}, \mathbb{C}), X, Z_1, \dots, Z_k \in \mathfrak{g}_{\overline{0}}, k \geq 0$ and $\iota_X(1) = 0$.

Let $\{Z_1, \dots, Z_m\}$ be a fixed orthonormal basis of $\mathfrak{g}_{\overline{0}}$. The super-Poisson bracket on $Alt(\mathfrak{g}_{\overline{0}}, \mathbb{C})$ is defined by (see [11]):

$$\{\Omega, \Omega'\} = (-1)^{k+1} \sum_{j=1}^{m} \iota_{Z_j}(\Omega) \wedge \iota_{Z_j}(\Omega'), \forall \Omega \in Alt^k(\mathfrak{g}_{\overline{0}}, \mathbb{C}), \Omega' \in Alt(\mathfrak{g}_{\overline{0}}, \mathbb{C}).$$

The super $\mathbb{Z} \times \mathbb{Z}_2-$Poisson bracket on $C(\mathfrak{g}, \mathbb{C})$ is given by:

$$\{\Omega \otimes F, \Omega' \otimes G\} = (-1)^{f\omega'} \left(\{\Omega, \Omega'\} \otimes FG + \Omega \wedge \Omega' \otimes \{F, G\}\right)$$

for any $\Omega \in Alt(\mathfrak{g}_{\overline{0}}, \mathbb{C}), \Omega' \in Alt^{\omega'}(\mathfrak{g}_{\overline{0}}, \mathbb{C}), F \in Sym^f(\mathfrak{g}_{\overline{1}}, \mathbb{C})$, and $G \in Sym(\mathfrak{g}_{\overline{1}}, \mathbb{C})$.

Proposition 2.1.1 (see [5], [11]). *The algebra $C(\mathfrak{g}, \mathbb{C})$ is a graded Lie algebra with the super $\mathbb{Z} \times \mathbb{Z}_2-$Poisson bracket. More precisely, for all $A \in C^{(a,b)}(\mathfrak{g}, \mathbb{C}), A' \in C^{(a',b')}(\mathfrak{g}, \mathbb{C})$ and $A'' \in C(\mathfrak{g}, \mathbb{C})$, one has*

(i) $\{A', A\} = -(-1)^{aa'+bb'}\{A, A'\}$,

(ii) $\{\{A, A'\}, A''\} = \{A, \{A', A''\}\} - (-1)^{aa'+bb'}\{A', \{A, A''\}\}$.

Furthermore, $\{A, A' \wedge A''\} = \{A, A'\} \wedge A'' + (-1)^{aa'+bb'} A' \wedge \{A, A''\}$. □

Now, we choose an arbitrary basis $\left\{X_{\overline{0}}^1, \dots, X_{\overline{0}}^m\right\}$ of $\mathfrak{g}_{\overline{0}}$. Its dual basis is denoted by $\{\alpha_1, \dots, \alpha_m\}$. Let $\left\{Y_{\overline{0}}^1, \dots, Y_{\overline{0}}^m\right\}$ be the basis of $\mathfrak{g}_{\overline{0}}$ defined by $\alpha_i = B(Y_{\overline{0}}^i, \centerdot)$. That means

$$B(Y_{\overline{0}}^i, X_{\overline{0}}^j) = \delta_{ij}; \forall i, j = 1, \dots, n.$$

Set $\left\{X_{\overline{1}}^1, \dots, X_{\overline{1}}^n, Y_{\overline{1}}^1, \dots, Y_{\overline{1}}^n\right\}$ be a Darboux basis of $\mathfrak{g}_{\overline{1}}$. Then the super $\mathbb{Z} \times \mathbb{Z}_2-$Poisson bracket on $C(\mathfrak{g}, \mathbb{C})$ is also given by

$$\{A, A'\} = (-1)^{\omega+f+1} \sum_{i,j=1}^{m} B(Y_{\overline{0}}^i, Y_{\overline{0}}^j).\iota_{X_{\overline{0}}^i}(A) \wedge \iota_{X_{\overline{0}}^j}(A')$$
$$+ (-1)^{\omega} \sum_{k=1}^{n} \left(\iota_{X_{\overline{1}}^k}(A) \wedge \iota_{Y_{\overline{1}}^k}(A') - \iota_{Y_{\overline{1}}^k}(A) \wedge \iota_{X_{\overline{1}}^k}(A')\right),$$

for all $A \in Alt^{\omega}(\mathfrak{g}_{\overline{0}}, \mathbb{C}) \otimes Sym^f(\mathfrak{g}_{\overline{1}}, \mathbb{C}), A' \in C(\mathfrak{g}, \mathbb{C})$ (see [5]).

Remark 2.1.2. In [5], the authors introduced a useful tool. That was the 3-form I defined on any quadratic Lie superalgebra $(\mathfrak{g}, B)$ as follows

$$I\,(X,Y,Z) = B([X,Y],Z), \forall X,Y,Z \in \mathfrak{g}.$$

This 3-form is called the *3-form associated* to $\mathfrak{g}$. It is easy to prove that I is the homogeneous element of degree $(3,\overline{0})$ in the $\mathbb{Z}\times\mathbb{Z}_2-$graded algebra $C(\mathfrak{g},\mathbb{C}) = Alt(\mathfrak{g}_{\overline{0}},\mathbb{C}) \otimes Sym(\mathfrak{g}_{\overline{1}},\mathbb{C})$, $\{I,I\} = 0$ and $\delta = -\{I,.\}$ (see [5], Proposition 1.11). Using this proposition, the cohomology group $H^k(\mathfrak{g},\mathbb{C})$ can be computed through the super $\mathbb{Z}\times\mathbb{Z}_2-$Poisson bracket.

2.2 The Elementary Quadratic Lie Superalgebras

The main result of the paper is the description of the second cohomology group of elementary quadratic Lie superalgebras which have classified in [5]. Before giving the main result, we will list the elementary quadratic Lie superalgebras in [5]. There are exactly three superalgebras as follows.

(1) $\mathfrak{g}^s_{4,1} = (\mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}) \oplus (\mathbb{C}X_{\overline{1}} \oplus \mathbb{C}Y_{\overline{1}})$, where

$$\mathfrak{g}_{\overline{0}} = span\{X_{\overline{0}}, Y_{\overline{0}}\}, \mathfrak{g}_{\overline{1}} = span\{X_{\overline{1}}, Y_{\overline{1}}\}.$$

The bilinear form B is defined by

$$B\,(X_{\overline{0}}, Y_{\overline{0}}) = 1, B\,(X_{\overline{1}}, Y_{\overline{1}}) = 1,$$

the others are zero and the Lie super bracket is given by

$$[Y_{\overline{1}}, Y_{\overline{1}}] = -2X_{\overline{0}}, [Y_{\overline{0}}, Y_{\overline{1}}] = -2X_{\overline{1}}.$$

(2) $\mathfrak{g}^s_{4,2} = (\mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}) \oplus (\mathbb{C}X_{\overline{1}} \oplus \mathbb{C}Y_{\overline{1}})$, where

$$\mathfrak{g}_{\overline{0}} = span\{X_{\overline{0}}, Y_{\overline{0}}\}, \mathfrak{g}_{\overline{1}} = span\{X_{\overline{1}}, Y_{\overline{1}}\}.$$

The bilinear form B is defined by

$$B\,(X_{\overline{0}}, Y_{\overline{0}}) = 1, B\,(X_{\overline{1}}, Y_{\overline{1}}) = 1,$$

the others are zero and the Lie super bracket is given by

$$[X_{\overline{1}}, Y_{\overline{1}}] = X_{\overline{0}}, [Y_{\overline{0}}, X_{\overline{1}}] = X_{\overline{1}}, [Y_{\overline{0}}, Y_{\overline{1}}] = -Y_{\overline{1}}.$$

(3) $\mathfrak{g}^s_{6} = (\mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}) \oplus (\mathbb{C}X_{\overline{1}} \oplus \mathbb{C}Y_{\overline{1}} \oplus \mathbb{C}Z_{\overline{1}} \oplus \mathbb{C}T_{\overline{1}})$, where

$$\mathfrak{g}_{\overline{0}} = span\{X_{\overline{0}}, Y_{\overline{0}}\}, \mathfrak{g}_{\overline{1}} = span\{X_{\overline{1}}, Y_{\overline{1}}, Z_{\overline{1}}, T_{\overline{1}}\}.$$

The bilinear form B is defined by

$$B\,(X_{\overline{0}}, Y_{\overline{0}}) = 1, B\,(X_{\overline{1}}, Z_{\overline{1}}) = 1, B\,(Y_{\overline{1}}, T_{\overline{1}}) = 1,$$

the others are zero and the Lie super bracket is given by

$$[Z_{\overline{1}}, T_{\overline{1}}] = -X_{\overline{0}}, [Y_{\overline{0}}, Z_{\overline{1}}] = -Y_{\overline{1}}, [Y_{\overline{0}}, T_{\overline{1}}] = -X_{\overline{1}}.$$

2.3 The Main Result

Now we will introduce the main result of the paper. Namely, we will describe the second cohomology group of the elementary quadratic Lie superalgebras which have listed in Subsection 2.2.

Theorem 2.3.1. *With notations being as above in Subsection 2.2, the second cohomology group of the elementary quadratic Lie superalgebras are described as follows*

(i) $H^2(\mathfrak{g}^s_{4,1},\mathbb{C}) = span\left\{\left[Y^*_{\overline{0}}\otimes X^*_{\overline{1}}\right], \left[X^*_{\overline{1}}Y^*_{\overline{1}} - 2X^*_{\overline{0}}\wedge Y^*_{\overline{0}}\right]\right\}$
where $\{X^*_{\overline{0}}, Y^*_{\overline{0}}, X^*_{\overline{1}}, Y^*_{\overline{1}}\}$ *is the dual basis of* $\{X_{\overline{0}}, Y_{\overline{0}}, X_{\overline{1}}, Y_{\overline{1}}\}$.

(ii) $H^2(\mathfrak{g}^s_{4,2},\mathbb{C}) = \{0\}$.

(iii) $$H^2(\mathfrak{g}^s_{6},\mathbb{C}) = span\left\{\left[Y^*_{\overline{0}}\otimes X^*_{\overline{1}}\right], \left[Y^*_{\overline{0}}\otimes Y^*_{\overline{1}}\right], \left[\left(Z^*_{\overline{1}}\right)^2\right], \left[\left(T^*_{\overline{1}}\right)^2\right], \left[X^*_{\overline{1}}Z^*_{\overline{1}} - X^*_{\overline{0}}\wedge Y^*_{\overline{0}}\right], \left[Y^*_{\overline{1}}T^*_{\overline{1}} - X^*_{\overline{0}}\wedge Y^*_{\overline{0}}\right]\right\}$$

where
$\{X^*_{\overline{0}}, Y^*_{\overline{0}}, X^*_{\overline{1}}, Y^*_{\overline{1}}, Z^*_{\overline{1}}, T^*_{\overline{1}}\}$ *is the dual basis of* $\{X_{\overline{0}}, Y_{\overline{0}}, X_{\overline{1}}, Y_{\overline{1}}, Z_{\overline{1}}, T_{\overline{1}}\}$.

The Proof of Theorem 2.3.1

(i) Firstly, we consider $\mathfrak{g}^s_{4,1} = (\mathbb{C}X_{\overline{0}}\oplus\mathbb{C}Y_{\overline{0}})\oplus(\mathbb{C}X_{\overline{1}}\oplus\mathbb{C}Y_{\overline{1}})$, where $\mathfrak{g}_{\overline{0}} = span\{X_{\overline{0}}, Y_{\overline{0}}\}$, $\mathfrak{g}_{\overline{1}} = span\{X_{\overline{1}}, Y_{\overline{1}}\}$.
In view of [5], the associated 3-form of $\mathfrak{g}^s_{4,1}$ is given as follows

$$I = Y^*_{\overline{0}}\otimes\left(Y^*_{\overline{1}}\right)^2.$$

By a straightforward computation, we obtain

- $\left\{I, X^*_{\overline{0}}\right\} = \left(Y^*_{\overline{1}}\right)^2$, $\left\{I, Y^*_{\overline{0}}\right\} = 0$, $\left\{I, X^*_{\overline{1}}\right\} = 2Y^*_{\overline{0}}\otimes Y^*_{\overline{1}}$.
- $\left\{I, Y^*_{\overline{1}}\right\} = 0$, $\left\{I, X^*_{\overline{0}}\wedge Y^*_{\overline{0}}\right\} = Y^*_{\overline{0}}\otimes\left(Y^*_{\overline{1}}\right)^2$,
- $\left\{I, X^*_{\overline{0}}\otimes X^*_{\overline{1}}\right\} = X^*_{\overline{1}}\left(Y^*_{\overline{1}}\right)^2 + 2X^*_{\overline{0}}\wedge Y^*_{\overline{0}}\otimes Y^*_{\overline{1}}$.
- $\left\{I, X^*_{\overline{0}}\otimes Y^*_{\overline{1}}\right\} = -\left(Y^*_{\overline{1}}\right)^3$ $\left\{I, Y^*_{\overline{0}}\otimes X^*_{\overline{1}}\right\} = 0$.
- $\left\{I, Y^*_{\overline{0}}\otimes Y^*_{\overline{1}}\right\} = 0$, $\left\{I, \left(X^*_{\overline{1}}\right)^2\right\} = 4Y^*_{\overline{0}}\otimes X^*_{\overline{1}}Y^*_{\overline{1}}$.

- $\left\{I, \left(Y_{\overline{1}}^*\right)^2\right\} = 0$, $\left\{I, X_{\overline{1}}^* Y_{\overline{1}}^*\right\} = 2Y_{\overline{0}}^* \otimes \left(Y_{\overline{1}}^*\right)^2$.

Then we get

$$Im\delta_1 = span\left\{\left(Y_{\overline{1}}^*\right)^2, Y_{\overline{0}}^* \otimes Y_{\overline{1}}^*\right\}$$

and

$$Ker\delta_2 = span\left\{Y_{\overline{0}}^* \otimes X_{\overline{1}}^*, Y_{\overline{0}}^* \otimes Y_{\overline{1}}^*, \left(Y_{\overline{1}}^*\right)^2, X_{\overline{1}}^* Y_{\overline{1}}^* - 2X_{\overline{0}}^* \wedge Y_{\overline{0}}^*\right\}.$$

Therefore

$$H^2(\mathfrak{g}_{4,1}^s, \mathbb{C}) = Ker\delta_2 / Im\delta_1 = span\left\{\left[Y_{\overline{0}}^* \otimes X_{\overline{1}}^*\right], \left[X_{\overline{1}}^* Y_{\overline{1}}^* - 2X_{\overline{0}}^* \wedge Y_{\overline{0}}^*\right]\right\}.$$

(ii) Next, we consider $\mathfrak{g}_{4,2}^s = (\mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}) \oplus (\mathbb{C}X_{\overline{1}} \oplus \mathbb{C}Y_{\overline{1}})$, where

$$\mathfrak{g}_{\overline{0}} = span\{X_{\overline{0}}, Y_{\overline{0}}\},\ \mathfrak{g}_{\overline{1}} = span\{X_{\overline{1}}, Y_{\overline{1}}\}.$$

From [5], we obtain the associated 3-form $I = Y_{\overline{0}}^* \otimes X_{\overline{1}}^* Y_{\overline{1}}^*$. By a similar computation as above, we have

$$Ker\delta_2 = Im\delta_1 = span\left\{X_{\overline{1}}^* Y_{\overline{1}}^*, Y_{\overline{0}}^* \otimes X_{\overline{1}}^*, Y_{\overline{0}}^* \otimes Y_{\overline{1}}^*\right\}.$$

Therefore we get $H^2(\mathfrak{g}_{4,2}^s, \mathbb{C}) = \{0\}$.

(iii) Finally, we consider $\mathfrak{g}_6^s = (\mathbb{C}X_{\overline{0}} \oplus \mathbb{C}Y_{\overline{0}}) \oplus (\mathbb{C}X_{\overline{1}} \oplus \mathbb{C}Y_{\overline{1}} \oplus \mathbb{C}Z_{\overline{1}} \oplus \mathbb{C}T_{\overline{1}})$, where $\mathfrak{g}_{\overline{0}} = span\{X_{\overline{0}}, Y_{\overline{0}}\}$, $\mathfrak{g}_{\overline{1}} = span\{X_{\overline{1}}, Y_{\overline{1}}, Z_{\overline{1}}, T_{\overline{1}}\}$.

By a similar computation, we have

- $I = Y_{\overline{0}}^* \otimes Z_{\overline{1}}^* T_{\overline{1}}^*; Im\delta_1 = span\left\{Z_{\overline{1}}^* T_{\overline{1}}^*, Y_{\overline{0}}^* \otimes T_{\overline{1}}^*, Y_{\overline{0}}^* \otimes Z_{\overline{1}}^*\right\}$;
- $Ker\delta_2 =$

$$= span\left\{Y_{\overline{0}}^* \otimes X_{\overline{1}}^*, Y_{\overline{0}}^* \otimes Y_{\overline{1}}^*, Y_{\overline{0}}^* \otimes Z_{\overline{1}}^*, Y_{\overline{0}}^* \otimes T_{\overline{1}}^*, \left(Z_{\overline{1}}^*\right)^2, \left(T_{\overline{1}}^*\right)^2, Z_{\overline{1}}^* T_{\overline{1}}^*, X_{\overline{1}}^* Z_{\overline{1}}^* - X_{\overline{0}}^* \wedge Y_{\overline{0}}^*, Y_{\overline{1}}^* T_{\overline{1}}^* - X_{\overline{0}}^* \wedge Y_{\overline{0}}^*\right\}.$$

Therefore we get

$H^2(\mathfrak{g}_6^s, \mathbb{C}) = Ker\delta_2 / Im\delta_1 =$

$$= span\left\{\left[Y_{\overline{0}}^* \otimes X_{\overline{1}}^*\right], \left[Y_{\overline{0}}^* \otimes Y_{\overline{1}}^*\right], \left[\left(Z_{\overline{1}}^*\right)^2\right], \left[\left(T_{\overline{1}}^*\right)^2\right], \left[X_{\overline{1}}^* Z_{\overline{1}}^* - X_{\overline{0}}^* \wedge Y_{\overline{0}}^*\right], \left[Y_{\overline{1}}^* T_{\overline{1}}^* - X_{\overline{0}}^* \wedge Y_{\overline{0}}^*\right]\right\}.$$

The proof is complete. □

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