***East-West J. of Mathematics: Vol. 19, No 1*** (2017) pp. 23-31

# ON $*$-BASES FOR $QTAG$-MODULES

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**Abstract**

An $h$-reduced $QTAG$-module $M$ is called totally projective if it has a nice system. In this paper, we find a new characterization for totally projective $QTAG$-modules of cardinality not exceeding $\aleph_1$. This is in terms of the existence of a certain kind of basis which is called a $*$-basis. The question about the structure of larger modules having a $*$-basis is left open, but we establish some closure properties of such modules. We also study secure submodules with the help of these $*$-bases and prove that every secure submodule is nice.

## 1 Introduction and terminology

Let $R$ be any ring. A module $M_R$ is called a $TAG$-module if it satisfies the following two conditions:

$(I)$ Every finitely generated submodule of any homomorphic image of $M$ is a direct sum of uniserial modules.

$(II)$ Given any two uniserial submodules $U$ and $V$ of a homomorphic image of $M$, for any submodule $W$ of $U$, any non-zero homomorphism $f : W \to V$ can be extended to a homomorphism $g : U \to V$, provided the composition length $d(U/W) \leq d(V/f(W))$.

A module $M_R$ satisfying condition $(I)$ only is called a $QTAG$-module. The study of various structures for $QTAG$-module was started by Singh [12]. The structure theory of such modules has been developed on similar lines as that of torsion abelian groups. Several authors worked a lot on this module and

**Key words:** totally projective modules, $*$-bases, secure submodules, nice submodules.
2010 AMS Mathematics classification: 16K20

studied different notions and structures on $QTAG$-modules. Many interesting results have been surfaced, but there is a lot to explore.

Let all rings discussed here be associative with unity ($1 \neq 0$) and modules are unital $QTAG$-modules. A module in which the lattice of its submodule is totally ordered is called a serial module; in addition, if it has finite composition length, it is called a uniserial module. Let us recall some definitions from [10, 11]. An element $x \in M$ is uniform, if $xR$ is a non-zero uniform (hence uniserial) module, and for any $R$-module $M$ with a unique decomposition series, $d(M)$ denotes its decomposition length. For a uniform element $x \in M,\ e(x) = d(xR)$ and $H_M(x) = \sup\left\{ d\left(\frac{yR}{xR}\right) \mid y \in M,\ x \in yR \text{ and } y \text{ uniform}\right\}$ are the exponent and height of $x$ in $M$, respectively. $H_k(M)$ denotes the submodule of $M$ generated by the elements of height at least $k$ and $H^k(M)$ is the submodule of $M$ generated by the elements of exponents at most $k$ [3]. Let us denote by $M^1$, the submodule of $M$, containing elements of infinite height. As defined in [4], the module $M$ is $h$-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$. The module $M$ is $h$-reduced if it does not contain any $h$-divisible submodule. In other words, it is free from the elements of infinite height. $M$ is said to be bounded [11], if there exists an integer $n$ such that $H_M(x) \leq n$ for every uniform element $x \in M$.

A submodule $N$ of $M$ is $h$-pure in $M$ if $N \cap H_k(M) = H_k(N)$, for every integer $k \geq 0$. A submodule $N \subset M$ is nice [6] in $M$, if $H_\sigma(M/N) = (H_\sigma(M) + N)/N$ for all ordinals $\sigma$, i.e. every coset of $M$ modulo $N$ may be represented by an element of the same height.

A family $\mathcal{N}$ of nice submodules of $M$ is called a nice system [7] in $M$ if

(*i*) $0 \in \mathcal{N}$;

(*ii*) if $\{N_i\}_{i \in I}$ is any subset of $\mathcal{N}$, then $\sum_{i \in I} N_i \in \mathcal{N}$;

(*iii*) given any $N \in \mathcal{N}$ and any countable subset $X$ of $M$, there exists $K \in \mathcal{N}$ containing $N \cup X$, such that $K/N$ is countably generated.

An $h$-reduced $QTAG$-module $M$ is called totally projective if it has a nice system.

Imitating [8], the submodules $H_k(M), k \geq 0$ form a neighborhood system of zero, thus a topology known as $h$-topology arises. Closed modules are also closed with respect to this topology. Thus the closure of $N \subseteq M$ is defined as $\overline{N} = \bigcap_{k=0}^{\infty} (N + H_k(M))$. Therefore the submodule $N \subseteq M$ is closed with respect to $h$-topology if $\overline{N} = N$.

The sum of all simple submodules of $M$ is called the socle of $M$ and is denoted by $Soc(M)$. The cardinality of the minimal generating set of $M$ is denoted by $g(M)$. For all ordinals $\alpha$, $f_M(\alpha)$ is the $\alpha^{th}$-$Ulm$ invariant of $M$ (see [5]) and it is equal to $g\big(Soc(H_\alpha(M))/Soc(H_{\alpha+1}(M))\big)$.

Mehran et al. [9] proved that almost all of the results which hold for $TAG$-modules also hold good for $QTAG$-modules. Our notations and terminology are standard and follow essentially those from [1, 2].

## 2 $*$-basis

We begin with an explicit definition of our main term.

**Definition 2.1.** Let $M$ be a $QTAG$-module. For each ordinal $\sigma$, let $B_\sigma$ be a set of representatives of the nonzero cosets of $H_\sigma(M)$ mod $H_{\sigma+1}(M)$; in other words, $B_\sigma$ contains exactly one element from each of the nonzero cosets of $H_{\sigma+1}(M)$ in $H_\sigma(M)$. If each element $x$ in $M$ can be expressed as

$$x = b_1 + b_2 + \cdots + b_n \tag{1}$$

where $b_i \in B_{\sigma(i)}$ with $\sigma(1) < \sigma(2) < \cdots < \sigma(n)$, then $B = \bigcup B_\sigma$ is called a $*$-basis of $M$.

The expression (1) is called a representation of $x$ with respect to the $*$-basis $B$.

**Remark** In order to accommodate modules that are not $h$-reduced, we can adjoin $\infty$ to the class of ordinals with the understanding that $\sigma < \infty$ for every ordinal $\sigma$. It is convenient here, however, not to allow the usual $\infty < \infty$ in regard to the definition $B_\infty$. We, in fact, define $B_\infty$ to be the nonzero elements of $H_\infty(M) = \bigcap H_\alpha(M)$, where $\alpha$ ranges over all ordinals.

Now we prove the following lemma.

**Lemma 2.1.** *If $B = \bigcup B_\sigma$ is a $*$-basis of a $QTAG$-module $M$, then the representation* (1) *of each element $x$ in $M$ with respect to this basis is unique.*

*Proof.* Let $x = b_1 + b_2 + \cdots + b_n = b'_1 + b'_2 + \cdots + b'_m$ be two representations of $x$ with respect to the $*$-basis $B = \bigcup B_\sigma$. Hence, suppose that $b_i \in B_{\sigma(i)}$ and $b'_j \in B_{\mu(j)}$, where $\sigma(1) < \sigma(2) < \cdots < \sigma(n)$ and $\mu(1) < \mu(2) < \cdots < \mu(m)$. Since $\sigma(1) = H_M(x) = \mu(1)$, we quickly conclude that $\sigma(1) = \mu(1)$ and that $b_1 = b'_1$. If we replace $x$ by $x - b_1 = x - b'_1$, the proof then follows by induction on $n$. □

The following example demonstrates that not all $QTAG$-modules have a $*$-basis.

**Example 1:** Let $M$ be the closure of an unbounded, countably generated $QTAG$-module having no elements of infinite height. More specifically, we can take $M$ to be the torsion product of the uniserial modules. Let $U_n = \langle u_n \rangle$ be a uniserial module of exponent $n$, and let $M$ be the closed submodule of the product $\prod U_n$. As it is well known, in the specific case we are now considering or in the more general case which we began, $M$ has cardinality of the continuum and $H_\omega(M) = 0$. Moreover, for each nonnegative integer $n$, $H_n(M)/H_{n+1}(M)$ is countably generated. Now, suppose that $M$ has a $*$-basis $B = \bigcup B_n$. Since $H_n(M)/H_{n+1}(M)$ is countably generated, $B_n$ must be countably generated. Therefore, $B = \bigcup B_n$ is countably generated, and consequently $M$ must be countably generated. But this is not the case, and we conclude that $M$ cannot have a $*$-basis.

The next lemma shows that the class of $QTAG$-modules having a $*$-basis is closed with respect to direct sums.

**Lemma 2.2.** *If $M = \bigoplus_{i \in I} M_i$ is a direct sum of $QTAG$-modules $M_i$ that have $*$-bases, then $M$ itself has a $*$-basis.*

*Proof.* If $B_i = \bigcup B_{i,\sigma}$ be a $*$-basis for $M_i$. Then $B_{i,\sigma}$ is a set of representatives for the nonzero cosets of $H_{\sigma+1}(M_i)$ in $H_\sigma(M_i)$, and each element $x_i \in M_i$ has a unique representation

$$x_i = b_{i,1} + b_{i,2} + \cdots + b_{i,n}$$

where $b_{i,j} \in B_{i,\sigma(j)}$ with $\sigma(1) < \sigma(2) < \cdots < \sigma(n)$.

We define $B_\sigma = \bigoplus_{i \in I} B_{i,\sigma} = \sum_i b_{i,\sigma}$, where the sum is finite but not vacuous, and $b_{i,\sigma} \in B_{i,\sigma}$ and let $B = \bigcup B_\sigma$.

Our claim is that $B$ is a $*$-basis for $M$. To verify this, first observe that $B_\sigma$ is a set of representatives of the nonzero elements of $H_{\sigma+1}(M)$ in $H_\sigma(M)$. This is an immediate consequence of the fact that $H_\sigma(M) = \bigoplus_{i \in I} H_\sigma(M_i)$. It remains only to show that each element $x$ in $M$ can be written as

$$x = b_1 + b_2 + \cdots + b_n,$$

where $b_j \in B_{\sigma(j)} = \bigoplus_{i \in I} B_{i,\sigma(j)}$ with $\sigma(1) < \sigma(2) < \cdots < \sigma(n)$.

But we know that $x = \sum x_i$, where the sum is finite and $x_i \in M_i$. Moreover, since $B_i = \bigcup B_{i,\sigma}$ is a $*$-basis of $M_i$, we know that $x_i = \sum b_{i,j}$, where $b_{i,j} \in B_{i,\sigma(j)}$ with $\sigma_i(1) < \sigma_i(2) < \ldots$

Define

$$\Gamma = \{\sigma : \sigma = \sigma_i(j) \text{ for some } i \text{ and } j\},$$

and list this finite set in increasing order $\sigma(1) < \sigma(2) < \cdots < \sigma(n)$.

For each $\sigma(k) \in \Gamma$, we set $b_k = \sum b_{i,j}$, where $\sigma_i(j) = \sigma(k)$. Then $b = b_1 + b_2 + \cdots + b_n$, where $b_k \in B_{\sigma(k)} = \bigoplus_i B_{i,\sigma(k)}$, and $B$ is a $*$-basis of $M$. □

The next lemma can be interpreted to mean that any $QTAG$-module that has a basis also has a $*$-basis, which is different from the basis. The proof is trivial.

**Lemma 2.3.** *If $M$ is a QTAG-module such that $M = \bigoplus_{n<\omega} M_n$, where $M_n = \bigoplus \langle a_{n,i} \rangle$ is a direct sum of uniserial modules of exponent $n$. Define*

$$B_k = \{\sum_{n>k} t_{n,i} H(a'_{n,i}), \text{ where } d\left(\frac{a_{n,i}R}{a'_{n,i}R}\right) = k : t_{n,i} \geq 0\},$$

*where the sum is finite but not trivial. Then $B = \bigcup_{k<\omega} B_k$ is a $*$-basis of $M$. In particular, any bounded QTAG-module has a $*$-basis.*

Perhaps we should insert a word of caution here. The fact that $M$ has a $*$-basis does not imply that $B = \bigcup B_\sigma$ is a $*$-basis of $M$ for every choice $B_\sigma$ of representatives of $H_\sigma(M)$ mod $H_{\sigma+1}(M)$. The following simple example illustrates this fact.

**Example 2:** Let M=$\bigoplus_{n\geq 1} \langle c_n \rangle$ be a direct sum of uniserial modules where the exponent of $c_n$ is $n$ for each positive integer $n$. As in Lemma 2.3,

$$B_n = \{\sum_{i>n} t_i H(c'_i), \text{ where } d\left(\frac{c_i R}{c'_i R}\right) = n : t_i \geq 0\},$$

is a set of representatives of $H_n(M)$ mod $H_{n+1}(M)$ for all $n < \omega$. Then $B = \bigcup B_n$ is a $*$-basis of $M$. Now, let $N = \bigoplus \langle c_n - H(c'_{n+1}) \rangle$ where $d\left(\frac{c_{n+1}R}{c'_{n+1}R}\right) = 1$. Then $N$ is an $h$-pure submodule of $M$ with the property that $M/N$ is $h$-divisible.

Let $B' = \bigcup B'_n$ be a $*$-basis for $N$, where $B'_n$ is a set of representatives of $H_n(N)$ mod $H_{n+1}(N)$. Since $H_n(M) = H_n(N) + H_{n+1}(M)$, clearly $B'_n$ is also a set of representatives of $H_n(M)$ mod $H_{n+1}(M)$. But obviously $B'$ cannot be a $*$-basis of $M$ since $B' \subseteq N \neq M$.

The next lemma will prove to be a major component in the proof of our first major result, Theorem 2.1.

**Lemma 2.4.** *Let $M$ be a QTAG-module and $\beta$ an ordinal. If $H_\beta(M)$ and $M/H_\beta(M)$ both have a $*$-basis, then $M$ has a $*$-basis.*

*Proof.* For $\sigma < \beta$, let $\bar{A_\sigma} = A_\sigma/H_\beta(M)$ be a set of representatives of $H_\sigma(M/H_\beta(M)) = H_\sigma(M)/H_\beta(M) \mod H_{\sigma+1}(M/H_\beta(M)) = H_{\sigma+1}(M)/H_\beta(M)$, where $A_\sigma \subseteq H_\sigma(M)$. Certainly, $A_\sigma$ is a set representatives of $H_\sigma(M) \mod H_{\sigma+1}(M)$. Let $C_\sigma$ be a set representatives of $H_\sigma(H_\beta(M)) = H_{\beta+\sigma}(M) \mod H_{\sigma+1}(H_\beta(M)) = H_{\beta+\sigma+1}(M)$. Define $B_\sigma = A_\sigma$ if $\sigma < \beta$ and let $B_{\beta+\sigma} = C_\sigma$. In either case, $B_\mu$ is a set of representatives of $H_\mu(M) \mod H_{\mu+1}(M)$. Moreover, if we choose $A_\sigma$ and $C_\sigma$ such that $\bar{A} = \bigcup \bar{A_\sigma}$ and $C = \bigcup C_\sigma$ are $*$-bases of $M/H_\beta(M)$ and $H_\beta(M)$, respectively, then $B$ is a $*$-basis of $M$. □

And so, we prepare to state the following.

**Corollary 2.1.** *If $M$ is a QTAG-module and $H_1(M)$ has a $*$-basis, then so does $M$.*

Through the preceding series of lemmas, we have established the essentials for the proof of the following.

**Theorem 2.1.** *Let $M$ be a totally projective QTAG-module. Then $M$ has a $*$-basis.*

*Proof.* Since any $h$-divisible module $M$ has a $*$-basis $B = B_\infty = M \backslash 0$, we may assume, by virtue of Lemma 2.2, that $M$ is $h$-reduced. Let $M$ be $h$-reduced of length $\rho$, that is, let $\rho$ be the smallest ordinal for which $H_\rho(M) = 0$. The proof is by induction on $\rho$. If $\rho = 1$, in fact if $\rho = n < \omega$, then $M$ has a $*$-basis according to Lemma 2.3. If $\rho$ is infinite, there are two cases:

*Case I* : $\rho$ is a limit. As $M = \bigoplus_i M_i$, where $M_i$ is a totally projective module of smaller length than $\rho$. By the induction hypothesis $M_i$ has a $*$-basis. Hence, again by Lemma 2.2, $M$ has a $*$-basis.

*Case II* : $\rho$ is isolated. Then $H_{\rho-1}(M)$ being bounded has a $*$-basis according to Lemma 2.3. Moreover, $M/H_{\rho-1}(M)$ being a totally projective module of length $\rho - 1 < \rho$ has a $*$-basis according to the induction hypothesis. Finally, Lemma 2.4 implies that $M$ has a $*$-basis, which completes the proof of the theorem. □

# 3 Secure submodules

In this brief section, we start with the following useful concept.

**Definition 3.1.** A submodule $N$ of a $QTAG$-module $M$ with a $*$-basis $B$ is a called a secure submodule if for $0 \neq y \in N$, $y = b_1 + b_2 + \cdots + b_n$ is the unique representation of $y$ with respect to $B$, then $b_i \in N$ for each $i$.

The next lemma is crucial for our new characterization of totally projective $QTAG$-modules of cardinality not exceeding $\aleph_1$.

**Lemma 3.1.** *Let $N$ be a secure submodule of a QTAG-module $M$. Then $N$ is nice in $M$.*

*Proof.* Let $M$ be a $QTAG$-module with a $*$-basis $B = \bigcup B_\sigma$ and let $N$ be a secure submodule of $M$ with respect to $B$. Suppose that $x \in M \backslash N$. In order to show that $N$ is nice in $M$, it suffices to show that the coset $x + N$ has a proper element $x_0$, that is, to show that there is an element $x_0 \in x + N$ with the property that $H_M(x_0) \geq H_M(x_0 + y)$ for all $y \in N$. In order to do this, among all the elements in the coset $x + N$, choose $x_0$ to have the shortest possible representation

$$x_0 = b_1 + b_2 + \cdots + b_n.$$

Let $b_i \in B_{\sigma(i)}$, where $\sigma(1) < \sigma(2) \cdots < \sigma(n)$. Using the fact that $n$ is minimal, we will show that $x_0$ is proper. Suppose that $x_0$ is not proper, and let $H_M(x_0 - y) > H_M(x_0)$ where $y \in N$. Let

$$y = b'_1 + b'_2 + \cdots + b'_m$$

where $b'_i \in B_{\mu(i)}$, where $\mu(1) < \mu(2) \cdots < \mu(m)$. Since $H_M(x_0 - y) > H_M(x_0)$, it follows that $H_M(y) = H_M(x_0)$ and therefore $\mu(1) = H_M(y) = H_M(x_0) = \sigma(1)$. This implies that $b_1 = b'_1$. But $N$ is secure, so $b_1 = b'_1 \in N$. This, however, yields a contradiction since $x_0 - b_1 \in x + N$ has a shorter representation than $x_0$. Therefore, $N$ is nice in $M$, and the lemma is proved. □

We are now in a position to state and prove the second major result of this article.

**Theorem 3.1.** *Let $M$ be a QTAG-module of cardinality not exceeding $\aleph_1$. Then $M$ is totally projective if and only if $M$ has a $*$-basis.*

*Proof.* If $M$ is totally projective, it has a $*$-basis by Theorem 2.1.

Conversely, Suppose that $M$ has a $*$-basis $B = \bigcup B_\sigma$. It is well known that $M$ has a smooth chain of nice submodules

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_\alpha \subseteq \ldots$$

that leads up to $M = \bigcup N_\alpha$ with the property that $N_{\alpha+1}/N_\alpha$ is countably generated for each $\alpha$. Therefore, the theorem will be proved if we can show that there exists such a chain of nice submodules. If $M$ is countably generated, there is nothing more to prove. Hence, assume that $g(M) = \aleph_1$. In view of Lemma 3.1, we need only establish the desired chain

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_\alpha \subseteq \ldots$$

of secure submodules. The main advantage of considering secure submodules is that the property of being a secure submodule, unlike that of being a nice submodule, is inductive; all secure submodules here are understood to be with respect to the fixed $*$-basis $B$ of $M$. Since $g(M) = \aleph_1$, it now suffices to show that any countably generated submodule $K$ of $M$ is contained in a countably generated secure submodule $T$ of $M$.

Let $K$ be any countably generated submodule of $M$. Set $T_0 = K$. We define $T_n$ inductively as follows:

$$T_{n+1} = \langle b_i : x = b_1 + b_2 + \cdots + b_n, \text{ where } x \in T_n \rangle.$$

It should be understood in the preceding defining equation of $T_{n+1}$ that $b_1 + b_2 + \cdots + b_n$ is the representation of $x$ with respect to the $*$-basis $B$ of $M$. Finally, we set $T = \bigcup_{n<\omega} T_n$. Since $T$ is obviously secure and remains countably generated, the proof is finished. □

We close the study with

## 4 Concluding discussion

It remains to investigate the modules of cardinality larger than $\aleph_1$ that have $*$-bases. We have not been able to modify the proof of Theorem 3.1, so that it applies, but perhaps some other approach might work.

**Acknowledgements**

The author owes deep thanks to the referee for his/her careful reading of the paper, and to the Editor, for his/her valuable editorial work.

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