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Menger Algebras and Clones of Terms

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Abstract

Clones are sets of operations which are closed under superposition and contain all projections. The superposition operation maps to each $(n+1)-$tuple of n-ary operations a new n-ary operation and satisfies the so-called *superassociative law*. The corresponding algebraic structures are *Menger algebras of rank* n, *unitary Menger algebras of rank* n and *Menger algebras with infinitely many nullary operations.* Identities of clones of term operations of a given algebra correspond to hyperidentities of this algebra, i.e. to identities which are satisfied after any replacements of fundamental operations by derived operations ([10]). If any identity of an algebra is satisfied as a hyperidentity, the algebra is called solid ([3]). Solid algebras correspond to free clones. These connections will be extended to *strongly full clones*, to *generalized clones*, to *strong hyperidentities* and to *strongly solid* varieties. We prove that n-*hyperidentities*, SF-*hyperidentities* and strong hyperidentities correspond to identities in free unitary Menger algebras of finite rank, in Menger algebras of finite rank or to free unitary Menger algebras with infinitely many nullary operations, respectively.

1 Superposition of Total and Partial Operations

Let $f : A^n \to A$ be an n-ary operation defined on the set A, let $O^{(n)}(A)$ be the set of all n-ary operations on A and let $O(A) := \bigcup\limits_{n=1}^{\infty} O^{(n)}(A)$ be the set of all

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(finitary) operations on A. Then an $(n+1)$-ary operation $S^{n,A} : O^{(n)}(A)^{n+1} \to O^{(n)}(A)$ can be defined by setting

$$S^{n,A}(f, g_1, \ldots, g_n)(\underline{a}) := f(g_1(\underline{a}), \ldots, g_n(\underline{a})) \text{ for } \underline{a} := (a_1, \ldots, a_n).(*)$$

Together with the projections $e_i^{n,A}, 1 \leq i \leq n$, this gives an algebra $(O^{(n)}(A); S^{n,A}, (e_i^{n,A})_{1\leq i\leq n})$ of type $(n+1, 0, \ldots 0)$. For $n = 1$ the operation $S^{1,A}$ is the usual composition of unary operations and the algebra $(O^{(1)}(A); S^{1,A}, e_1^{1,A})$ is a monoid. An $(n+1)-$ary superposition operation $S^{n,A}$ can also be defined on the set $P^{(n)}(A)$ of all n-ary partial operations. In this case we request additionally that in (*) the left hand side is defined whenever the right hand side is defined and that both sides are equal. It is easy to see that the algebra $(O^{(n)}(A); S^{n,A}, (e_i^{n,A})_{1\leq i\leq n})$ satisfies the following identities:

(C1) $\tilde{S}^n(T, \tilde{S}^n(F_1, T_1, \ldots, T_n), \ldots, \tilde{S}^n(F_n, T_1, \ldots, T_n))$
$\approx \tilde{S}^n(\tilde{S}^n(T, F_1, \ldots, F_n), T_1, \ldots, T_n), n \in \mathbb{N}^+$.

(C2) $\tilde{S}^n(T, \lambda_1, \ldots, \lambda_n) = T, n \in \mathbb{N}^+$.

(C3) $\tilde{S}^n(\lambda_i, T_1, \ldots, T_n) = T_i$ for $1 \leq i \leq n, n \in \mathbb{N}^+$.

(Here $\tilde{S}^n$ is an operation symbol corresponding to the operations $S^n, \lambda_i, 1 \leq i \leq n$ are nullary operation symbols and T, T_j, F_i are variables.)
Indeed,

$$\begin{aligned}
&(S^{n,A}(S^{n,A}(f, g_1, \ldots, g_n), h_1, \ldots, h_n))(\underline{a}) \\
&\quad = f(g_1, \ldots, g_n)(h_1, \ldots, h_n)(\underline{a}) \\
&\quad = f(g_1(h_1, \ldots, h_n)(\underline{a}), \ldots, g_n(h_1, \ldots, h_n)(\underline{a})) \\
&\quad = S^{n,A}(f, S^{n,A}(g_1, h_1, \ldots, h_n), \ldots, S^{n,A}(g_n, h_1, \ldots, h_n))(\underline{a})
\end{aligned}$$

and this means that (C1) is satisfied. (C2) and (C3) can be easily checked.

The superposition operation $S^{n,A}$ can also be applied to partial operations defined on A. In this case one has to assume that the right hand side of (C1) ((C2), (C3)) is defined whenever the left hand side is defined and then both sides are equal.

2 Menger Algebras

Definition 2.1 An algebra $\mathcal{M} = (M; S^n)$ of type $(n+1)$, which satisfies (C1) is called a *Menger algebra of rank n*. An algebra $\mathcal{M} = (M; S^n, \lambda_1, \ldots, \lambda_n)$ of type $(n+1, 0, \ldots, 0)$ satisfying (C1), (C2), (C3) is called a *unitary Menger algebra* of rank n. (For more background on Menger algebras see e.g [9]).

If $(M; S^n)$ is an algebra of type $(n+1)$, then we introduce a binary operation $*$ on the cartesian power M^n by

$$(x_1, \ldots, x_n) * (y_1, \ldots, y_n) := (S^n(x_1, y_1 \ldots, y_n), \ldots, S^n(x_n, y_1, \ldots, y_n)).$$

Then it is easy to see that an algebra $(M; S^n)$ of type $n+1$ is a Menger algebra iff $(M^n; *)$ is a semigroup (see e.g. [Sch-T 65]).
We introduce the following generalization of the concept of a unitary Menger algebra of rank n:

Definition 2.2 An algebra $(M; S^m, (e_i)_{i\in\mathbb{N}^+})$ where S^m is $(m+1)$-ary and $e_i, i \in \mathbb{N}^+$ are nullary, is called a Menger algebra with infinitely many nullary operations if the following axioms (Cg1), (Cg2), (Cg3) and (Cg4) are satisfied:

(Cg1) $\tilde{S}^m(T, \tilde{S}^m(F_1, T_1, \ldots, T_m), \ldots, \tilde{S}^m(F_m, T_1, \ldots, T_m))$
$\approx \tilde{S}^m(\tilde{S}^m(T, F_1, \ldots, F_m), T_1, \ldots, T_m), m \in \mathbb{N}^+$.

(Cg2) $\tilde{S}^m(T, \lambda_1, \ldots, \lambda_m) = T, m \in \mathbb{N}^+$.

(Cg3) $\tilde{S}^m(\lambda_i, T_1, \ldots, T_m) = T_i$ for $1 \leq i \leq m, m \in \mathbb{N}^+$.

(Cg4) $\tilde{S}^m(\lambda_j, T_1, \ldots, T_m) = \lambda_j$ for $j > m, m \in \mathbb{N}^+$.

(Here $\tilde{S}^m$ is an $(m+1)$-ary operation symbol corresponding to S^m, $\lambda_i, i \in \mathbb{N}^+$ are nullary operation symbols and T, T_j, F_i are variables).

The connection between Menger algebras of rank n and semigroups defined on the cartesian power of its universe can also be generalized.
Menger algebras of rank n, unitary Menger algebras of rank n and unitary Menger algebras with infinitely many nullary operations form varieties which we want to denote by V_{M_n}, $V_{M_n^+}$ and V_M^∞, respectively. Our next aim is to determine the free objects with respect to these varieties.

3 Sets of Terms and Free Menger Algebras

We will call a type of algebras n-ary if all the operation symbols of the type are n-ary, for some fixed natural number n. Now we assume that τ_n is such a fixed n-ary type, with operation symbols $(f_i)_{i\in I}$ indexed by some set I.
We begin with some notation. We let $X = \{x_1, x_2, x_3, \ldots\}$ be a countably infinite set of individual variables, and for each $n \geq 1$ let $X_n = \{x_1, x_2, \ldots x_n\}$.
We denote by $W_{\tau_n}(X_n)$ the set of all n-ary terms of type τ_n built up from the operation symbols f_i of type τ_n and the alphabet X_n.
On the set $W_{\tau_n}(X_n)$ of all n-ary terms of type τ_n an algebra of the type $\tau = (n+1, 0, \ldots, 0)$ can be defined. Here the $(n+1)$-ary superposition operation S^n is defined inductively by
$S^n(x_j, t_1, \ldots, t_n) := t_j$, for $1 \leq j \leq n$; and
$S^n(f_i(s_1, \ldots, s_n), t_1, \ldots, t_n) := f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_n, t_1, \ldots, t_n))$.
Selecting the variable terms $x_1, \ldots, x_n$ for the nullary operations, we form the algebra

$$n - clone\tau_n \quad := \quad (W_{\tau_n}(X_n);\ S^n,\ x_1, \ldots, x_n).$$

Of particular interest are the terms of the form $f_i(x_1,\ldots,x_n)$, for each $i \in I$, which are usually called the fundamental terms of type τ_n. We shall denote by F_{τ_n} the set of these fundamental terms of type τ_n. We shall make frequent use of the fact that this set F_{τ_n} generates the algebra $n-clone\tau_n$.

Proposition 3.1 *The algebra $n-clone\tau_n$ is a unitary Menger algebra of rank n.*

Proof $(C3)$ is satisfied by the definition of S^n. We prove $(C1)$ by induction on the complexity of the term which is substituted for T. If we substitute for T a variable x_j, then by $(C3)$ we have $S^n(x_j, S^n(t_1,s_1,\ldots,s_n),\ldots,S^n(t_n,s_1,\ldots,s_n)) = S^n(t_j,s_1,\ldots,s_n) = S^n(S^n(x_j,t_1,\ldots,t_n),s_1,\ldots,s_n)$. If we substitute for T the term $t_0 = f_i(r_1,\ldots,r_n)$ and assume inductively that $(C1)$ is satisfied for $r_1,\ldots,r_n$, then

$$\begin{aligned}
&S^n(f_i(r_1,\ldots,r_n),S^n(t_1,s_1,\ldots,s_n),\ldots,S^n(t_n,s_1\ldots,s_n))\\
&= f_i(S^n(r_1,S^n(t_1,s_1,\ldots,s_n),\ldots,S^n(t_n,s_1\ldots,s_n)),\ldots,\\
&\qquad S^n(r_n,S^n(t_1,s_1,\ldots,s_n),\ldots,S^n(t_n,s_1\ldots,s_n)))\\
&= f_i(S^n(S^n(r_1,t_1,\ldots,t_n),s_1,\ldots,s_n),\ldots,S^n(S^n(r_n,t_1,\ldots,t_n),s_1,\ldots,s_n))\\
&= S^n(f_i(S^n(r_1,t_1,\ldots,t_n),\ldots,S^n(r_n,t_1,\ldots,t_n)),s_1,\ldots,s_n))\\
&= S^n(S^n(f_i(r_1,\ldots,r_n),t_1,\ldots,t_n),s_1,\ldots,s_n)\\
&= S^n(S^n(t_0,t_1,\ldots,t_n),s_1,\ldots,s_n).
\end{aligned}$$

$(C2)$ is a consequence of $(C3)$ and $(C1)$. □

We consider the free algebra $\mathcal{F}_{V_{M_n^+}}(\{Y_i \mid i \in I\})$ in the variety $V_{M_n^+}$, generated by a special alphabet $\{Y_i \mid i \in I\}$. The fact that this alphabet is in bijection with the set of fundamental operations $(f_i)_{i\in I}$ of type τ_n, and hence with the set F_{τ_n} of fundamental terms which generates $n-clone\tau_n$, will give us an isomorphism between this free algebra and $n-clone\tau_n$.

Theorem 3.2 *([4]) The algebra $n-clone\tau_n$ is isomorphic to $\mathcal{F}_{V_{M_n^+}}(\{Y_i \mid i \in I\})$, and therefore free with respect to the variety of unitary Menger algebras of rank n, and freely generated by the set F_{τ_n}.*

A Menger algebra of terms can be obtained in the following way:
We consider the concept of a term in a restricted setting. *Strongly full* terms of n-ary type τ_n are inductively defined by the following steps:

(i) $f_i(x_1,\ldots,x_n), i \in I$, is a strongly full term,

(ii) If $t_1,\ldots,t_n$ are strongly full terms, then $f_i(t_1,\ldots,t_n)$ is strongly full.

The superposition operation S^n is defined on $W^{SF}_{\tau_n}(X_n)$ as follows:

(i) $S^n(f_i(x_1,\ldots,x_n),t_1,\ldots,t_n) := f_i(t_1,\ldots,t_n)$,

(ii) $S^n(f_i(s_1,\ldots,s_n),t_1,\ldots,t_n) := f_i(S^n(s_1,t_1,\ldots,t_n),\ldots,S^n(s_n,t_1,\ldots,t_n))$.

The set $W^{SF}_{\tau_n}(X_n)$ is closed under the application of the $(n+1)$-ary superposition operation and $clone_{SF}\tau_n := (W^{SF}_{\tau_n}(X_n); S^n)$ is an algebra of type $\tau = (n+1)$ with F_n as a generating system. The algebra $clone_{SF}\tau_n$ is called *the clone of strongly full terms* of type τ_n. Clearly, $clone_{SF}\tau_n$ satisfies the axiom (C1) and therefore it is a Menger algebra of rank n. Let V_{M_n} be the variety of Menger algebras of rank n and let $\mathcal{F}_{V_{M_n}}(\{Y_i \mid i \in I\})$ be the free algebra with respect to V_{M_n}, freely generated by a special alphabet $\{Y_i \mid i \in I\}$. The fact that this alphabet is in bijection with the set of fundamental operations $(f_i)_{i\in I}$ of type τ_n, and hence with the set F_{τ_n} of fundamental terms which generates $clone_{SF}\tau_n$, will give us an isomorphism between this free algebra and $clone_{SF}\tau_n$.

Theorem 3.3 *The algebra $clone_{SF}\tau_n$ is isomorphic to $\mathcal{F}_{V_{M_n}}(\{Y_i \mid i \in I\})$, and therefore free with respect to the variety V_{M_n}, and freely generated by the set $\{f_i(x_1,\ldots,x_n) \mid i \in I\}$.*

Proof We define a mapping $\varphi : W^{SF}_{\tau_n}(X_n) \to F_{V_{M_n}}(\{Y_i \mid i \in I\})$ inductively as follows:

(i) $\varphi(f_i(x_1 \ldots x_n)) := Y_i$ for every $i \in I$.

(ii) $\varphi(f_i(t_1,\ldots,t_n)) := \widetilde{S}^n(Y_i, \varphi(t_1),\ldots,\varphi(t_n))$.

Since φ maps the generating system of $clone_{SF}\tau_n$ onto the generating system of $\mathcal{F}_{V_{M_n}}(\{Y_i \mid i \in I\})$ it is surjective. We prove the homomorphism property $\varphi(S^n(t_0,t_1,\ldots,t_n)) = \widetilde{S}^n(\varphi(t_0),\ldots,\varphi(t_n))$ by induction on the complexity of the term t_0. If $t_0 = f_i(x_1,\ldots,x_n)$, then $\varphi(S^n(f_i(x_1,\ldots,x_n),t_1,\ldots,t_n)) = \varphi(f_i(t_1,\ldots,t_n)) = \widetilde{S}^n(Y_i,\varphi(t_1),\ldots,\varphi(t_n)) = \widetilde{S}^n(\varphi(f_i(x_1,\ldots,x_n)),\varphi(t_1),\ldots,\varphi(t_n))$. Inductively, assume that $t_0 = f_i(s_1,\ldots,s_n)$ and that $\varphi(S^n(s_j,t_1,\ldots,t_n)) = \widetilde{S}^n(\varphi(s_j),\ldots,\varphi(t_n))$ for all $1 \le j \le n$. Then

$$\begin{aligned}
&\varphi(S^n(f_i(s_1,\ldots,s_n),t_1,\ldots,t_n))\\
=\ &\varphi(f_i(S^n(s_1,t_1,\ldots,t_n),S^n(s_2,t_1,\ldots,t_n),\ldots,S^n(s_n,t_1,\ldots,t_n))\\
=\ &\widetilde{S}^n(Y_i,\varphi(S^n(s_1,t_1,\ldots,t_n)),\varphi(S^n(s_2,t_1,\ldots,t_n)),\ldots,\varphi(S^n(s_n,t_1,\ldots,t_n)))\\
=\ &\widetilde{S}^n(Y_i,\widetilde{S}^n(\varphi(s_1),\varphi(t_1),\ldots,\varphi(t_n)),\ldots,\widetilde{S}^n(\varphi(s_n),\varphi(t_1),\ldots,\varphi(t_n)))\\
=\ &\widetilde{S}^n(\widetilde{S}^n(Y_i,\varphi(s_1),\ldots,\varphi(s_n)),\varphi(t_1),\ldots,\varphi(t_n)))\\
=\ &\widetilde{S}^n(\varphi(f_i(s_1,\ldots,s_n)),\varphi(t_1),\ldots,\varphi(t_n)))\\
=\ &\widetilde{S}^n(\varphi(t_0),\varphi(t_1),\ldots,\varphi(t_n)).
\end{aligned}$$

Thus φ is a homomorphism. The mapping φ is bijective since $\{Y_i \mid i \in I\}$ is a free independent set and therefore we have

$$Y_i = Y_j \quad\Rightarrow\quad i = j \quad\Rightarrow\quad f_i(x_1,\ldots,x_n) = f_j(x_1,\ldots,x_n).$$

Thus φ is a bijection between the generating sets of $clone_{SF}\tau_n$ and $\mathcal{F}_{V_{M_n}}(\{Y_i \mid i \in I\})$, and hence it is bijective on $W_{\tau_n}(X_n)$. Altogether, φ is an isomorphism. $\square$

Menger algebras with infinitely many nullary operations occur if we consider a generalized superposition operation of terms which is inductively defined as an $(m+1)$-ary operation $S^m, m \in \mathbb{N}^+ := \mathbb{N} \setminus \{0\}$ on $W_\tau(X)$ by the following steps:

Definition 3.4 (i) If $t = x_i, 1 \leq i \leq m,$ then $S^m(x_i, t_1, \ldots, t_n) := t_i$ for $t_1, \ldots, t_n \in W_\tau(X)$.

(ii) If $t = x_i,\ m < i$, then $S^m(x_i, t_1, \ldots, t_n) := x_i$.

(iii) If $t = f_i(s_1, \ldots, s_{n_i})$, then

$$S^m(t, t_1, \ldots, t_n) := f_i(S^m(s_1, t_1, \ldots, t_n), \ldots, S^m(s_{n_i}, t_1, \ldots, t_n)).$$

Then for the type τ_n we may consider the algebraic structure

$$clone_g\tau := (W_{\tau_n}(X); S^n, (x_i)_{i\in\mathbb{N}^+})$$

with the universe $W_{\tau_n}(X)$, with the $(n+1)$-ary operation S^n and with infinitely many nullary operations. Now we prove:

Theorem 3.5 *The algebra $clone_g\tau_n$ satisfies the following identities:*

(Cg1) $\tilde{S}^n(T, \tilde{S}^n(F_1, T_1, \ldots, t_n), \ldots, \tilde{S}^n(F_n, T_1, \ldots, t_n))$
$\approx \tilde{S}^n(\tilde{S}^n(T, F_1, \ldots, F_n), T_1, \ldots, t_n)$.

(Cg2) $\tilde{S}^n(T, \lambda_1, \ldots, \lambda_n) = T$.

(Cg3) $\tilde{S}^n(\lambda_i, T_1, \ldots, t_n) = T_i$ *for* $1 \leq i \leq n$.

(Cg4) $\tilde{S}^n(\lambda_j, T_1, \ldots, t_n) = \lambda_j$ *for* $j > n$.

(Here $\tilde{S}^n, \lambda_i$ are operation symbols corresponding to the operations S^n and $x_i, i \in \mathbb{N}^+$, respectively and T, T_j, F_i are new variables.)

Proof. (Cg1) We give a proof by induction on the complexity of the term t.

(i) If $t = x_j, 1 \leq j \leq n$, then
$$\begin{aligned} &S^n(x_j, S^n(s_1, t_1, \ldots, t_n), \ldots S^n(s_n, t_1, \ldots, t_n)) \\ &= \quad S^n(s_j, t_1, \ldots, t_n) \\ &= \quad S^n(S^n(x_j, s_1, \ldots, s_n), t_1, \ldots, t_n). \end{aligned}$$

(ii) If $t = x_j, j > n$, then
$$\begin{aligned} &S^n(x_j, S^n(s_1, t_1, \ldots, t_n), \ldots S^n(s_n, t_1, \ldots, t_n)) \\ &= \quad x_j \\ &= \quad S^n(x_j, t_1, \ldots, t_n) \\ &= \quad S^n(S^n(x_j, s_1, \ldots, s_n), t_1, \ldots, t_n). \end{aligned}$$

(iii) If $t = f_i(t'_1, \ldots, t'_n)$ and if we assume that our proposition is satisfied for $t'_1, \ldots, t'_n$, then

$$\begin{array}{rl} & S^n(f_i(t'_1,\ldots,t'_n), S^n(s_1,t_1,\ldots,t_n),\ldots,S^n(s_n,t_1,\ldots,t_n)) \\ = & f_i(S^n(t'_1, S^n(s_1,t_1\ldots,t_n),\ldots,S^n(s_n,t_1,\ldots,t_n)),\ldots, \\ & S^n(t'_n, S^n(s_1,t_1,\ldots,t_n),\ldots,S^n(s_n,t_1,\ldots,t_n))) \\ = & f_i(S^n(S^n(t'_1,s_1,\ldots,s_n),t_1,\ldots,t_n),\ldots,S^n(S^n(t'_n,s_1,\ldots,s_n),t_1,\ldots,t_n)) \\ = & S^n(S^n(f_i(t'_1,\ldots,t'_n),s_1,\ldots,s_n),t_1,\ldots,t_n) \\ = & S^n(S^n(t,s_1,\ldots,s_n),t_1,\ldots,t_n). \end{array}$$

(Cg2) If t contains variables from the set $\{x_1, \ldots, x_n\}$, then we substitute in t for these variables the same variables and obtain t. If t contains a variable which does not belong to the set $\{x_1, \ldots, x_n\}$, then this variable will not be replaced by another term. Therefore the result is t.
(Cg3) and (Cg4) correspond to (i) and (ii), respectively, from the definition of S^n. □

The class of all unitary Menger algebras with infinitely many nullary operations forms a variety V_M^∞ and the algebra $clone_g\tau$ belongs to this variety.
Let $\mathcal{F}_{V_M^\infty}(\{Y_i \mid i \in I\})$ be the free algebra with respect to the variety V_M, freely generated by $\{Y_i \mid i \in I\}$. Then we have:

Theorem 3.6 *([6]) The algebra $clone_g\tau$ is isomorphic to the free algebra $\mathcal{F}_{V_M^\infty}(\{Y_i \mid i \in I\})$ and therefore free with respect to the variety of unitary Menger algebras with infinitely many nullary operations.*

4 Hypersubstitutions, Clone Substitutions and Endomorphisms of free Menger Algebras

Hypersubstitutions are mappings which take n-ary operation symbols to n-ary terms.

Definition 4.1 Let τ_n be an n-ary type. A *hypersubstitution* of type τ_n is a mapping from the set $\{f_i \mid i \in I\}$ of operation symbols of type τ_n to the set $W_{\tau_n}(X_n)$ of all n-ary terms of type τ_n. Any hypersubstitution induces a mapping $\hat{\sigma}$ defined on $W_{\tau_n}(X_n)$ by the following steps:

(i) $\hat{\sigma}[x_j] := x_j, 1 \leq j \leq n,$

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^n(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]).$

Hypersubstitutions can be multiplied by $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ and together with the identity hypersubstitution σ_{id} which maps for every $i \in I$ the operation symbol f_i to the term $f_i(x_1, \ldots, x_{n_i})$ we have a monoid $(Hyp(\tau_n); \circ_h, \sigma_{id})$.

If we map the operation symbols of our n-ary type to strongly full n-ary terms, we get the subset of all *strongly full hypersubstitutions*.

Definition 4.2 A *strongly full hypersubstitution* of n-ary type τ_n is a mapping from the set $\{f_i \mid i \in I\}$ of n-ary operation symbols of the type τ_n to the set $W_{\tau_n}^{SF}(X_n)$ of all strongly full n-ary terms of type τ_n.

Any strongly full hypersubstitution σ induces a mapping $\hat{\sigma}$ defined on the set $W_{\tau_n}^{SF}(X_n)$ of all n-ary terms of the type τ_n, as follows:

Definition 4.3 Let σ be a strongly full hypersubstitution of type τ_n. Then σ induces a mapping $\hat{\sigma} : W_{\tau_n}^{SF}(X_n) \longrightarrow W_{\tau_n}^{SF}(X_n)$, by setting

(i) $\hat{\sigma}[f_i(x_1, \ldots, x_n)] := \sigma(f_i)$,

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_n)] := S_n^n(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n])$.

Let $Hyp^{SF}(\tau_n)$ be the set of all hypersubstitutions of type τ_n. Then the product $\sigma_1 \circ_h \sigma_2$ of two strongly full hypersubstitutions is again strongly full and σ_{id} is also strongly full. Therefore $(Hyp^{SF}(\tau_n); \circ_h, \sigma_{id})$ is a submonoid of the monoid of all hypersubstitutions of type τ_n. Let $\mathcal{M}$ be any submonoid of $Hyp(\tau_n)$. If $\mathcal{A}$ $= (A; (f_i^{\mathcal{A}})_{i \in I})$ is an n-ary algebra, then an identity $s \approx t$ in $\mathcal{A}$ is said to be an *M-hyperidentity* in $\mathcal{A}$ if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is an identity in $\mathcal{A}$ for every hypersubstitution $\sigma \in M$. In the special case that M is all of $Hyp^{SF}(\tau_n)$, an M-hyperidentity is usually called a *strongly full hyperidentity*. An identity is an *M-hyperidentity* of a variety V if it is an M-hyperidentity of every algebra in V. A variety in which every identity of the variety holds as an M-hyperidentity is called an *M-solid variety*, or a *SF-solid variety* in the special case $M = Hyp^{SF}(\tau_n)$. If the variety is of n-ary type and if the identity $s \approx t$ is built up of terms of arity n then $s \approx t$ is called an *n-hyperidentity* if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ is an identity for every hypersubstitution σ and V is called *n-solid* if every n-ary identity is an n-hyperidentity. For more detailed information on hyperidentities we refer the reader to [3].
Hypersubstitutions and strongly full hypersubstitutions preserve arities. A generalization was defined in [8].

Definition 4.4 A mapping $\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$ is called a *generalized hypersubstitution* of type τ. Generalized hypersubstitutions can be inductively extended to mappings $\hat{\sigma}$ defined on $W_\tau(X)$ by

(i) $\hat{\sigma}[x_i] := x_i \in X$.

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$

Here S^{n_i} means the generalized superposition introduced in section 3. Let $Hyp_G(\tau)$ be the set of all generalized hypersubstitutions of type τ.

In [6] was proved:

Proposition 4.5 *The extension of a generalized hypersubstitution is an endomorphism of the algebra $clone_g \tau_n$.*

The converse is also satisfied, i.e.

Proposition 4.6 *([6]) Every endomorphism of $clone_g\tau_n$ is the extension of a generalized hypersubstitution.*

It is easy to see that the set $Hyp_G(\tau_n)$ together with the binary operation $\circ_G$ defined by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ and σ_{id} forms also a monoid $(Hyp_G(\tau_n); \circ_G, \sigma_{id})$. Let $End(clone_g\tau_n)$ be the endomorphism monoid of $clone_g\tau_n$. Then we have:

Proposition 4.7 *([6]) The monoid $(Hyp_G(\tau_n); \circ_G, \sigma_{id})$ of all generalized hypersubstitutions is isomorphic with the endomorphism monoid $End(clone_g\tau_n)$.*

The algebra $clone_g\tau_n = (W_{\tau_n}(X); S^n, (x_i)_{i\in\mathbb{N}^+})$ is generated by the set $F_{\tau_n} = \{f_i(x_1, \ldots, x_{n_i}) \mid i \in I\}$. Any mapping from F_{τ_n} to $W_{\tau_n}(X)$ is called a *generalized clone substitution*. Since $clone_g\tau_n$ is free, every generalized clone substitution can be uniquely extended to an endomorphism of the algebra $clone_g\tau_n$. Let $Subst_G$ be the set of all generalized clone substitutions. We introduce a binary composition operation $\otimes$ on this set, by setting $\eta_1 \otimes \eta_2 := \overline{\eta_1} \circ \eta_2$ where $\circ$ denotes the usual composition of functions. Denoting by $id_{F_{\tau_n}}$ the identity mapping on F_{τ_n} we see that $(Subst_G; \otimes, id_{F_{\tau_n}})$ is a monoid. Further we have:

Proposition 4.8 *([6]) The monoids $(Subst_G; \otimes, id_{F_{\tau_n}})$ and $(Hyp_G(\tau_n); \circ_G, \sigma_{id})$ are isomorphic.* □

For the monoids $(Hyp(\tau_n); \circ_h, \sigma_{id})$ and $(Hyp^{SF}(\tau_n); \circ_h, \sigma_{id})$ we obtain similar results if we consider the monoids $(Subst; \otimes; id)$ of all substitutions $F_{\tau_n} \to W_{\tau_n}(X_n)$, $(Subst_{SF}; \otimes; id)$ of all substitutions $F_{\tau_n} \to W_{\tau_n}^{SF}(X_n)$ and the endomorphism monoids $End(n - clone\tau_n)$ and $End(clone_{SF}\tau_n)$, respectively.

Theorem 4.9 *Let τ_n be an n-ary type, then*

(i) $(Hyp(\tau_n); \circ_h, \sigma_{id}) \cong (Subst; \otimes; id) \cong End(n - clone\tau_n)$

(ii) $(Hyp^{SF}(\tau_n); \circ_h, \sigma_{id}) \cong (Subst_{SF}; \otimes; id) \cong End(clone_{SF}\tau_n)$.

Proof (i) The isomorphism $(Hyp(\tau_n); \circ_h, \sigma_{id}) \cong (Subst; \otimes; id)$ was proved in [4] and the isomorphism $(Subst; \otimes; id) \cong End(n - clone\tau_n)$ is clear since because of the freeness of $End(n - clone\tau_n)$ every mapping from the free generating system F_{τ_n} to $W_{\tau_n}(X_n)$ can be uniquely extended to an endomorphism of $n - clone\tau_n$. The mapping which maps each substitution to its extension is a bijection. The inverse mapping takes any endomorphism of $n - clone\tau_n$ to its restriction on F_{τ_n} which is a clone substitution. For two substitutions η_1, η_2 we have $\overline{\eta_1 \otimes \eta_2}(f_i(x_1, \ldots, x_{n_i})) = (\eta_1 \otimes \eta_2)(f_i(x_1, \ldots, x_{n_i})) = \overline{\eta_1}(\eta_2(f_i(x_1, \ldots, x_n))) = \overline{\eta_1}(\overline{\eta_2}(f_i(x_1, \ldots, x_n))) = (\overline{\eta_1} \circ \overline{\eta_2})(f_i(x_1, \ldots, x_n))$ and because of the freeness of $n - clone\tau_n$ we get$\overline{\eta_1 \otimes \eta_2} = \overline{\eta_1} \circ \overline{\eta_2}$. This shows that the mapping which takes any clone substitution to its extension is an isomorphism.

(ii) We define a mapping $\psi : Subst_{SF} \longrightarrow Hyp^{SF}(\tau_n)$ by $\psi(\eta) := \eta \circ \sigma_{id}$. This gives a well-defined mapping between $Subst_{SF}$ and $Hyp^{SF}(\tau_n)$. The mapping is surjective, since any strongly full hypersubstitution σ can be obtained as $\psi(\eta)$ for $\eta = \sigma \circ \sigma_{id}^{-1}$. The mapping ψ is also injective, since

$$\psi(\eta_1) = \psi(\eta_2) \quad \Rightarrow \quad \eta_1 \circ \sigma_{id} = \eta_2 \circ \sigma_{id} \quad \Rightarrow \quad \eta_1 = \eta_2,$$

since σ_{id} is a bijection. To show that ψ is a homomorphism, we first verify the following additional property:

$$(\eta_1 \circ \sigma_{id})\hat{}\,[t] \quad = \quad \overline{\eta}_1[t], \qquad\qquad (*)$$

where $\overline{\eta}$ is the unique extension of η. For the fundamental terms $t = f_i(x_1, \ldots, x_n)$, we have
$(\eta_1 \circ \sigma_{id})\hat{}\,[f_i(x_1, \ldots, x_n)] = (\eta_1 \circ \sigma_{id})(f_i) = \eta_1(f_i(x_1, \ldots, x_n)) = \overline{\eta}_1(f_i(x_1, \ldots, x_n))$,
by (C3) and the definition of the extension of a hypersubstitution. The claimed property then follows by induction. Now for the homomorphism property for ψ we have

$$\begin{aligned}
\psi(\eta_1) \circ_h \psi(\eta_2) &= (\eta_1 \circ \sigma_{id}) \circ_h (\eta_2 \circ \sigma_{id}) \\
&= (\eta_1 \circ \sigma_{id})\hat{} \circ (\eta_2 \circ \sigma_{id}) \\
&= \overline{\eta}_1 \circ (\eta_2 \circ \sigma_{id}), && \text{by property (*) above,} \\
&= (\overline{\eta_1} \circ \eta_2) \circ \sigma_{id}, && \text{by associativity} \\
&= (\eta_1 \otimes \eta_2) \circ \sigma_{id}, && \text{by definition of } \otimes, \\
&= \psi(\eta_1 \otimes \eta_2).
\end{aligned}$$

The isomorphism $(Subst_{SF}; \otimes; id) \cong End(clone_{SF}\tau_n)$ can be proved in a similar way as we proved the corresponding isomorphism $n-clone\tau_n \cong (Subst; \otimes, id)$ in (i). $\square$

5 Hyperidentities and Clone Identities

Generalized hypersubstitutions can be used to define strong hyperidentities in algebras or in varieties ([8]).

Definition 5.1 Let V be a variety of type τ and let IdV be the set of all identities satisfied in V. An identity $s \approx t \in IdV$ is called a *strong hyperidentity* in V (Lee-D;00]) if $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$ for all generalized hypersubstitutions $\sigma \in Hyp_G(\tau)$. If every identity of a variety V is satisfied as a strong hyperidentity, the variety is called *strongly solid.*

An example of a strongly solid variety of semigroups is the variety

$$Rec := Mod\{x_1(x_2x_3) \approx (x_1x_2)x_3 \approx x_1x_3\}([8]).$$

The set IdV of all identities satisfied in a variety V forms a fully invariant congruence relation of the absolutely free algebra of the given type. But we have also:

Theorem 5.2 ([6]) *Let V be a variety of type τ_n and let IdV be the set of all identities satisfied in V. Then IdV is a congruence relation on $clone_g\tau_n$.*

Further we have:

Theorem 5.3 ([6]) Let V be a variety of type τ_n. Then V is strongly solid if and only if IdV is a fully invariant congruence relation on $clone_g\tau_n$.

Since IdV is a congruence relation on $clone_g\tau_n$, we may form the quotient algebra $clone_gV := clone_g\tau_n/IdV$. The operations $\tilde{S}^m$ of this algebra are defined as usual by

$$\tilde{S}^m([t]_{IdV}, [t_1]_{IdV}, \ldots, [t_n]_{IdV}) := [S^m(t, t_1, \ldots, t_n)]_{IdV}.$$

The nullary operations are $[x_i]_{IdV}, i \in \mathbb{N}^+$. Since for a strongly solid variety V the relation IdV is fully invariant on $clone_g\tau_n$, it corresponds to a fully invariant congruence on the absolutely free algebra of the type of unitary Menger algebras with infinitely many nullary operations (see [1]). Fully invariant congruences on absolutely free algebras of a given type correspond to equational theories, i.e. to sets of identities of certain varieties. Therefore we have:

Theorem 5.4 ([6]) Let V be a variety of type τ_n and let $s \approx t \in IdV$. Then $s \approx t$ is a strong hyperidentity in V iff $s \approx t$ is an identity in $clone_gV$.

Finally from this result one obtains a nice characterization of strongly solid varieties.

Corollary 5.5 *Let V be a variety of type τ_n. Then V is strongly solid iff $clone_gV$ is free with respect to itself, freely generated by the set $\{[f_i(x_1, \ldots, x_{n_i})]_{IdV} \mid i \in I\}$, meaning that every mapping from $\{[f_i(x_1, \ldots, x_{n_i})]_{IdV} \mid i \in I\}$ to the universe of $clone_gV$ can be extended to an endomorphism of $clone_gV$.*

Let Id_nV be the set of all n-ary identities of a variety V of n-ary type τ_n, i.e. the set $IdV \cap W_{\tau_n}(X_n)^2$. The variety V is called *n-solid* if every identity from Id_nV is satisfied as an n-hyperidentity. Then we have ([5]):

Theorem 5.6 *Let V be a variety of type τ_n. Then*

(i) *The set Id_nV of all n-ary identities satisfied in V is a congruence relation on $n - clone\tau_n$.*

(ii) *V is n-solid if and only if Id_nV is a fully invariant congruence relation on $n - clone\tau_n$.*

(iii) *An equation $s \approx t$ is an n-hyperidentity in V iff $s \approx t$ is an identity in $n - cloneV$ where $n - cloneV$ is the quotient algebra $n - clone\tau_n/Id_nV$.*

(iv) *The variety V is n-solid iff $n - cloneV$ is free with respect to itself, freely generated by the set $\{[f_i(x_1, \ldots, x_{n_i})]_{IdV} \mid i \in I\}$, meaning that every mapping from $\{[f_i(x_1, \ldots, x_{n_i})]_{IdV} \mid i \in I\}$ to the universe of $n - cloneV$ can be extended to an endomorphism of $n - cloneV$.*

(For a generalization of this theorem see [2]). For a variety V of n-ary type we form the set $SF_n^E(V)$ of all n-ary identities consisting of strongly full terms. An equation $s \approx t \in SF_n^E(V)$ is said to be a SF_n-hyperidentity if $\hat{\sigma}[s] \approx \hat{\sigma}[t]$ for every $\sigma \in Hyp^{SF}(\tau_n)$ and V is called SF_n*-solid* if every identity from $SF_n^E(V)$ is a SF_n-hyperdentity. SF_n-hyperidentities and SF_n-solid varieties can be characterized as follows:

Theorem 5.7 *Let V be a variety of type τ_n. Then*

(i) *The set $SF_n^E(V)$ is a congruence relation on $clone_{SF}\tau_n$.*

(ii) *V is SF_n-solid if and only if $SF_n^E(V)$ is a fully invariant congruence relation on $clone_{SF}\tau_n$.*

(iii) *An equation $s \approx t$ is a SF_n-hyperidentity in V iff $s \approx t$ is an identity in $clone_{SF}V$ where $SF - cloneV$ is the quotient algebra $clone_{SF}\tau_n/SF_n(V)$.*

(iv) *The variety V is SF_n-solid iff $clone_{SF}V$ is free with respect to itself, freely generated by the set $\{[f_i(x_1, \ldots, x_{n_i})]_{SF_n(V)} \mid i \in I\}$, meaning that every mapping from $\{[f_i(x_1, \ldots, x_{n_i})]_{SF_n(V)} \mid i \in I\}$ to the universe of $clone_{SF}V$ can be extended to an endomorphism of $clone_{SF}V$.*

Proof (i) Let $t_1 \approx s_1, \ldots, t_n \approx s_n \in IdSF_n^E(V)$. Then we show by induction on the complexity of the strongly full n-ary term t that $S^n(t, t_1, \ldots, t_n) \approx S^n(t, s_1, \ldots, s_n) \in SF_n^E(V)$. Assume that $t = f_i(x_1, \ldots, x_n)$. Then

$$\begin{aligned} & S^n(f_i(x_1, \ldots, x_n), t_1, \ldots, t_n) \\ &\quad = f_i(t_1, \ldots, t_n) \\ &\quad \approx f_i(s_1, \ldots, s_n) \\ &\quad = S^n(f_i(x_1, \ldots, x_n), s_1, \ldots, s_n) \in SF_N^E(V) \end{aligned}$$

since $f_i(t_1, \ldots, t_n) \approx f_i(s_1, \ldots, s_n) \in IdV$ and since $f_i(t_1, \ldots, t_n), f_i(s_1, \ldots, s_n)$ are strongly full n-ary terms of type τ_n. Assume now that $t = f_i(l_1, \ldots, l_n)$ and that for $l_j,\ 1 \leq j \leq n$, we have already $S^n(l_j, t_1, \ldots, t_n) \approx S^n(l_j, s_1, \ldots, s_n) \in SF_n^E(V),\ 1 \leq j \leq n$. Then

$$\begin{aligned} & S^n(t, t_1, \ldots, t_n) \\ &\quad = f_i(S^n(l_1, t_1, \ldots, t_n), \ldots, S^n(l_{n_i}, t_1, \ldots, t_n)) \\ &\quad \approx f_i(S^n(l_1, s_1, \ldots, s_n), \ldots, S^n(l_n, s_1, \ldots, s_n)) \in SF_n^E(V), \end{aligned}$$

since IdV is compatible with the operations corresponding to $f_i, i \in I$, in the absolutely free algebra of type τ_n and since $S^n(l_j, t_1, \ldots, t_n), S^n(l_j, s_1, \ldots, s_n)$ are n-ary SF-terms.

The next step consists in showing

$$t \approx s \Rightarrow S^n(t, s_1, \ldots, s_n) \approx S^n(s, s_1, \ldots, s_n) \in SF_n^E(V).$$

Since IdV is a fully invariant congruence on $\mathcal{F}_\tau(X)$ from $t \approx s \in IdV$ we obtain $S^n(t, s_1, \ldots, s_n) \approx S^n(s, s_1, \ldots, s_n) \in IdV$ by substitution. The terms $S^n(t, s_1, \ldots, s_n), S^n(s, s_1, \ldots, s_n)$ are strongly full and n-ary and thus $S^n(t, s_1, \ldots, s_n) \approx S^n(s, s_1, \ldots, s_n) \in SF_n^E(V)$.
Assume now that $t \approx s, t_1 \approx s_1, \ldots, t_n \approx s_n \in IdV$. Then

$$S^n(t, t_1, \ldots, t_n) \approx S^n(t, s_1, \ldots, s_n) \approx S^n(s, s_1, \ldots, s_n) \approx S^n(s, t_1, \ldots, t_n) \in SF_n^E(V).$$

(ii) Let V be SF_n-solid, let $s \approx t \in SF_n^E(V)$ and let $\varphi : clone_{SF}\tau_n \to clone_{SF}\tau_n$ be an endomorphism of $clone_{SF}\tau_n$ ($\varphi \in End(clone_{SF}\tau_n)$). Then we have

$$\varphi(s) = (\varphi/F_{\tau_n} \circ \sigma_{id})\,\hat{}\,[s] \approx (\varphi/F_{\tau_n} \circ \sigma_{id})\,\hat{}\,[t] = \varphi(t) \in SF_n^E(V)$$

since $\varphi/F_{\tau_n} \circ \sigma_{id}$ is a SF- hypersubstitution with $\varphi = (\varphi/F_{\tau_n} \circ \sigma_{id})\,\hat{}$. Therefore IdV is fully invariant.
If conversely $SF_n^E(V)$ is fully invariant, $s \approx t \in SF_n^E(V)$ and let $\sigma \in Hyp^{SF}(\tau_n)$, then $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in SF_n^E(V)$ since the extension of a SF-hypersubstitution is a clone endomorpism. This shows that every identity $s \approx t \in SF_n^E(V)$ is satisfied as a $SF-$hyperidentity and then V is SF_n-solid.

(iii) We first assume that $s \approx t$ is a SF_n-hyperidentity of V. This means that for every $\sigma \in Hyp^{SF}(\tau_n)$ we have $nat(SF_n^E(V))(\hat{\sigma}[s]) = nat(SF_n^E(V))(\hat{\sigma}[t])$, where $nat(SF_n^E(V)) : W_{\tau_n}^{SF} \to W_{\tau_n}^{SF}/SF_n^E(V)$ is the natural mapping. To show that $s \approx t$ holds in $clone_{SF}\tau_n$, we will show that $\overline{v}(s) = \overline{v}(t)$ for every valuation $F_{\tau_n} \to clone_{SF}V$. Since $nat(SF_n^E(V))$ is surjective, there exists a clone substitution η_v such that $v = nat(SF_n^E(V)) \circ \eta_v$. Then $\eta_v \circ \sigma_{id}$ is a hypersubstitution, which we shall denote by σ_v and we have:

$$\overline{v}(s) = (nat(SF_n^E(V)) \circ \overline{\eta}_v)(s) = (nat(SF_n^E(V)) \circ (\eta_v \circ \sigma_{id})\,\hat{}\,)(s) = nat(SF_n^E(V))(\hat{\sigma}_v[s]).$$

Similarly, we have $\overline{v}(t) = nat(SF_n^E(V))(\hat{\sigma}_v[t])$. Since by our assumption we have $nat(SF_n^E(V))(\hat{\sigma}_v[s]) = nat(SF_n^E(V))(\hat{\sigma}_v[t])$, we get $\overline{v}(s) = \overline{v}(t)$, as required.
Conversely, let $s \approx t \in Id(clone_{SF}\tau_n)$, so that $s, t \in W_{\tau_n}^{SF}(X_n)$ and for every valuation mapping v we have $\overline{v}(s) = \overline{v}(t)$. Let σ be any hypersubstitution. Then there is a clone substitution η_σ such that $\eta_\sigma \circ \sigma_{id} = \sigma$. We take v to be the valuation $nat(SF_n^E(V)) \circ \overline{\eta_\sigma}$. Then $[\hat{\sigma}[s]]_{SF_n^E(V)} = nat(SF_n^E(V))(\hat{\sigma}[s]) = (nat(SF_n^E(V)) \circ (\eta_\sigma \circ \sigma_{id})\,\hat{}\,)(s)$
$= (nat(SF_n^E(V)) \circ \overline{\eta}_\sigma)(s) = \overline{v}[s]$. Similarly, we have $[\hat{\sigma}[t]]_{SF_n^E(V)} = \overline{v}[t]$, and our assumption that $\overline{v}(s) = \overline{v}(t)$ gives the desired equality.
(iv) Using the equivalence from (iii), we will show that $clone_{SF}V$ is free iff

every identity $s \approx t \in SF_n^E(V)$ is also an identity in $clone_{SF}V$. Suppose first that $clone_{SF}V$ is free with respect to itself, freely generated by the set $\{[f_i(x_1,\ldots,x_n)]_{SF_n^E(V)} \mid i \in I\}$. Let $s \approx t$ be any identity in $SF_n^E(V)$. To show that $s \approx t$ is an identity in $clone_{SF}V$, we will show that $\overline{v}(s) = \overline{v}(t)$ for any valuation mapping $v : F_{\tau_n} \longrightarrow clone_{SF}V$. Given v, we define a mapping $\alpha_v : \{[f_i(x_1,\ldots,x_n)]_{SF_n^E(V)} \mid i \in I\} \longrightarrow W_{\tau_n}(X_n)/SF_n^E(V)$ by $\alpha_v([f_i(x_1,\ldots,x_n)]_{SF_n^E(V)}) = v(f_i(x_1,\ldots,x_n))$. Since
$[f_i(x_1,\ldots,x_n)]_{SF_n^E(V)} \;=\; [f_j(x_1,\ldots,x_n)]_{SF_n^E(V)} \Longrightarrow i = j$
$\Longrightarrow \quad f_i(x_1,\ldots,x_n) = f_j(x_1,\ldots,x_n)$
$\Longrightarrow \quad v(f_i(x_1,\ldots,x_n)) \;=\; v(f_j(x_1,\ldots,x_n))$
$\Longrightarrow \quad \alpha_v([f_i(x_1,\ldots,x_n)]_{SF_n^E}) \;=\; \alpha_v([f_j(x_1,\ldots,x_n)]_{SF_n^E(V)})$,
the mapping α_v is well-defined. Since the set $\{[f_i(x_1,\ldots,x_n)]_{SF_n^E(V)} \mid i \in I\}$ generates the free algebra $clone_{SF}V$, the mapping v can be uniquely extended to $\overline{v}$ on the set $W_{\tau_n}^{SF}(X_n)/SF_n^E(V)$. Then we have

$$[s]_{SF_n^E(V)} = [t]_{SF_n^E(V)} \Longrightarrow \overline{\alpha}_v([s]_{SF_n^E(V)}) = \overline{\alpha}_v([t]_{SF_n^E(V)}) \Longrightarrow \overline{v}(s) = \overline{v}(t),$$

showing that $s \approx t \in SF_n^E(V))$.
For the converse direction, we show that when V is SF_n-solid, any mapping $\alpha : \{[f_i(x_1,\ldots,x_n)]_{SF_n^E(V)} \mid i \in I\} \longrightarrow W_{\tau_n}^{SF}(X_n)/SF_n^E(V)$ can be extended to an endomorphism of $clone_{SF}V$. We consider the mapping $\overline{\alpha} = \alpha \circ nat(SF_n^E(V)) : F_{\tau_n} \longrightarrow W_{\tau_n}^{SF}(X_n)/SF_n^E(V)$, which is a valuation of terms. Then for any terms $s,t \in W_{\tau_n}^{SF}(X_n)$, it follows from $[s]_{SF_n^E(V)} = [t]_{SF_n^E(V)}$, i.e. $nat(SF_n^E(V))(s) = nat(SF_n^E(V))(t)$ that $\overline{\alpha}(nat(SF_n^E(V))(s)) = \overline{\alpha}(nat(SF_n^E(V))(t))$ and $(\overline{\alpha} \circ (nat(SF_n^E(V)))(s) = (\overline{\alpha} \circ (nat(SF_n^E(V))(t))$ since $\overline{\alpha} \circ \overline{n}at(SF_n^E(V)$ is a valuation and every SF_n-identity is a $clone_{SF}V$-identity. This shows that $\overline{\alpha}$ is well-defined. It is also an endomorphism since
$\overline{\alpha}(S^{n,clone_{SF}V}([s]_{SF_n^E(V)}, [t_1]_{SF_n^E(V)}, \ldots, [t_n]_{SF_n^E(V)}))$
$= (\overline{\alpha} \circ nat(SF_n^E(V)))(S^n(s,t_1,\ldots,t_n))$
$= S_n^{n,clone_{SF}V}((\overline{\alpha} \circ nat(SF_n^E(V)))(s), (\overline{\alpha} \circ nat(SF_n^E(V)))(t_1), \ldots,$
$(\overline{\alpha} \circ nat(SF_n^E(V)))(t_n))$
$= S_n^{n,clone_{SF}V}(\overline{\alpha}([s]_{SF_n^E(V)}), \overline{\alpha}([t_1]_{SF_n^E(V)}), \ldots, \overline{\alpha}([t_n]]_{SF_n^E(V)}))$,
using the fact that $\overline{\alpha} \circ nat(SF_n^E(V))$ is the homomorphism extending the valuation $\alpha \circ nat(SF_n^E(V))$ defined on the generating set of the free algebra $clone_{SF}V$. Finally, $\overline{\alpha}$ extends α since $\overline{\alpha}([f_i(x_1,\ldots,x_n)]_{SF_n^E(V)}) = (\alpha \circ nat(SF_n^E(V)))(f_i(x_1,\ldots,x_n)) = \alpha([f_i(x_1,\ldots,x_n)]_{SF_n^E(V)})$ for each $i \in I$. □

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