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A NOTE ON LEFT SYMMETRIC ALGEBRAS

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Abstract

In this work we study left-symmetric algebra over a field K with characteristic $\neq 2$, which are power-associative algebras.

1. PRELIMINARIES

Let A be a nonassociative algebra over a field K. We call A left-symmetric algebra if it satisfies the identity:

$$(x, y, z) = (y, x, z) \tag{1}$$

where $(x, y, z) = (xy)z - x(yz)$. Right-symmetric algebras are defined by the identity $(x, y, z) = (x, z, y)$. Right-symmetric algebras are sometimes called Vinberg-algebras (see, [8]).

If A is a left-symmetric algebra, then A is a left Novikov algebra if the identity $(xy)z = (xz)y$ is valid in A. We call A right Novikov algebra it the identities $(x, y, z) = (x, z, y)$ and $x(yz) = y(xz)$ are valid in A. Right Novikov algebras were introduced by Balinskii and Novikov in [1] , and have also been studied

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by Zelmanov [9] and Fillipov [3]. Left Novikov algebras were investigated by Cherkashin [2] and Osborn [4], [5], [6].

We define the right principal powers of $x \in A$ by $x^1 = x$ and $x^{n+1} = x^n x$ for all $n \geq 1$. An element $x \in A$ is called right nilpotent if there exists $n \geq 1$ such that $x^n = 0$, and $x \in A$ is called right nilpotent with right nilindex $n \geq 2$ if $x^n = 0$ and $x^{n-1} \neq 0$. If any element in A is right nilpotent, then A is called a right nilalgebra. A is called a right nilalgebra with right nilindex $n \geq 2$ if $x^n = 0$ for all $x \in A$ and there exists $y \in A$ such that $y^{n-1} \neq 0$. It is known that A is a power-associative algebra if for all $x \in A$ we have $x^i x^j = x^{i+j}$ for all $i,\ j \geq 1$. A is a flexible algebra, if $(x, y, x) = 0$ for all $x,\ y \in A$. A is a right alternative algebra, if $(y, x, x) = 0$ for all $x,\ y \in A$. Similarly, we define left alternative algebra.

If B, D are subspaces of A then BD is the subspace of A spanned by all products bd with b in B, d in D. We define the right principal powers of B by $B^1 = B$ and $B^{n+1} = B^n B$ for all $n \geq 1$. If there exists an element $k \geq 1$ such that $B^k = 0$ then B is called right nilpotent.

A is called nilpotent if for some integer positive n the product of any n elements from the algebra A, with any arrangement of parentheses, equals zero.

An element e of A is called an idempotent in case $e^2 = e \neq 0$. An idempotent $e \in A$ is called principal in case there is no idempotent $u \in A$ such that $eu = ue = 0$.

2. POWER-ASSOCIATIVE LEFT-SYMMETRIC ALGEBRAS

In this section, A is a left-symmetric algebra over a field K with characteristic $\neq 2$. It is known that when A is a commutative algebra, then A is associative. Also it is known that, left-symmetric algebras are Lie-admissibbe, i.e., under the commutator $[a, b] = ab - ba$, we obtain a Lie algebra.

Proposition 2.1 *The following conditions are equivalent:*

- (a) $xx^2 = x^3$ *for all* $x \in A$, *where* $x^3 = x^2 x$.
- (b) A *is a power-associative algebra.*

Proof Suppose that (a) is valid. That is, $xx^2 = x^2 x$ for all $x \in A$. The linearized form of the identity $xx^2 = x^2 x$ is $(x, z, y) + (z, x, y) + (x, y, z) + (z, y, x) + (y, x, z) + (y, z, x) = 0$. Using this last relation and since A is a left-symmetric algebra, we obtain that the following identities are valid in A :

$$(x, z, y) + (x, y, z) + (y, z, x) = 0 \qquad (2)$$

$$(x, y, z) + (z, x, y) + (z, y, x) = 0 \qquad (3)$$

For $x \in A$, we will prove first that $xx^n = x^{n+1}$ for all $n \geq 1$, where $x^{n+1} = x^n x$. We consider $n \geq 2$ and suppose that $xx^k = x^{k+1}$ for all k with $1 \leq k \leq n$.

Replacing z by x, y by x^{n-1} in (2) and using the inductive hypothesis we obtain that $x^{n-1}x^2 = x^2x^{n-1}$. Now $(x, x^2, x^{n-1}) = (x^2, x, x^{n-1})$ implies $x(x^2x^{n-1}) = x^2x^n$,

$(x^{n-1}, x, x) = (x, x^{n-1}, x)$ implies $x^{n-1}x^2 = xx^n = x^{n+1}$, and $(x, x^n, x) = (x^n, x, x)$ implies $xx^{n+1} = x^nx^2$. Thus we get $xx^{n+1} = x(x^{n-1}x^2) = x(x^2x^{n-1}) = x^2x^n$, which implies that $xx^{n+1} = x^nx^2 = x^2x^n$. Replacing z by x and y by x^n in (2), we obtain that $xx^{n+1} = x^{n+1}x$. So we prove that $xx^n = x^{n+1}$ for all $n \geq 1$.

Finally we will prove that $x^ix^j = x^{i+j}$ for all $i, j \geq 1$. If $j = 1$, then we know that $x^ix = x^{i+1}$. If we suppose that $x^ix^j = x^{i+j}$, then $(x^i, x, x^j) = (x, x^i, x^j)$ implies $x^ix^{j+1} = xx^{i+j} = x^{i+j+1}$. It is clear that (b) implies (a).

Proposition 2.2 *The following conditions are equivalent:*

- (a) *A is a right alternative algebra.*
- (b) *A is a flexible algebra.*
- (c) *A is a left alternative algebra.*
- (d) *A is a associative algebra.*

Proof Since $(x, y, x) = (y, x, x)$ for all $x, y \in A$, then (a) and (b) are equivalent. We observe that if (a), (b) or (c) are valid, then by proposition 2.1, A is a power-associative algebra.

Replacing z by y and y by x in (3), we obtain that $2(y, x, x) = -(x, x, y)$. Thus clearly (a) and (c) are equivalent. Suppose that A is flexible. The linearized form of the flexible law is $(x, y, z) + (z, y, x) = 0$ for all x, y, z in A. Using the identity (3) we obtain that $(z, x, y) = 0$ for all x, y, z in A, and therefore A is a associative algebra. Finally we conclude that (a), (b), (c) and (d) are equivalent.

Proposition 2.3 *If A is a power-associative algebra, which contains an idempotent $e \neq 0$, then A is the vector space direct sum $A = A_{11} \oplus A_{10} \oplus A_{01} \oplus A_{00}$, where $A_{11} = \{\ x \in A\ /\ ex = xe = x\ \}$, $A_{10} = \{\ x \in A\ /\ ex = x,\ xe = 0\ \}$, $A_{01} = \{\ x \in A\ /\ ex = 0,\ xe = x\ \}$ and $A_{00} = \{\ x \in A\ /\ ex = xe = 0\ \}$.*

Proof Replacing x by e and y by e in (3), we get $\frac{1}{2}(L_e^2 - L_e) = R_e^2 - R_e$, and therefore $L_e(R_e^2 - R_e) = (R_e^2 - R_e)L_e$. Now $(z, e, e) = (e, z, e)$ implies $L_eR_e = R_eL_e + R_e - R_e^2$. We have $L_e(R_e^2 - R_e) = (L_eR_e)R_e - L_eR_e = (R_eL_e + R_e - R_e^2)R_e - L_eR_e = R_eL_eR_e + R_e^2 - R_e^3 - L_eR_e = R_e(R_eL_e + R_e - R_e^2) + R_e^2 - R_e^3 - L_eR_e = R_e^2L_e + 2R_e^2 - 2R_e^3 - (R_eL_e + R_e - R_e^2) = R_e^2L_e - R_eL_e - 2R_e^3 + 3R_e^2 - R_e = (R_e^2 - R_e)L_e$, which implies that $2R_e^3 - 3R_e^2 + R_e = 0$. That is, $f(R_e) = 0$ where $f(\lambda) = (\lambda - 1)(2\lambda - 1)\lambda$. Hence A is the vector space direct sum $A = U_1 \oplus U_{\frac{1}{2}} \oplus U_0$, where $U_1 = \{ x \in A \ / \ xe = x \}$, $U_{\frac{1}{2}} = \{ x \in A \ / \ xe = \frac{1}{2}x \}$ and $U_0 = \{ x \in A \ / \ xe = 0 \}$. We will prove that $U_{\frac{1}{2}} = 0$. We consider $y \in U_{\frac{1}{2}}$ and $ey = y_0$. Now $L_e^2 - L_e = 2(R_e^2 - R_e)$ implies $ey_0 = y_0 - \frac{1}{2}y$ and $(e, y, e) = (y, e, e)$ implies $y_0 e = \frac{1}{2}y_0 - \frac{1}{4}y$. Using the above results we have that $(e, y_0, e) = (y_0, e, e)$ implies $y = 2y_0$. Therefore $y_0 e = 0$ and $y = 2ye = 4y_0 e = 0$. Hence we prove that $U_{\frac{1}{2}} = 0$, and thus $A = U_1 \oplus U_0$. We obtain now that $R_e^2 = R_e$, $L_e^2 = L_e$ and $L_e R_e = R_e L_e$ (i.e., L_e and R_e are commuting projections). It follows that A is the vector space direct sum $A = A_{11} \oplus A_{10} \oplus A_{01} \oplus A_{00}$, where $A_{ij} = \{ x_{ij} \ / \ ex_{ij} = ix_{ij},\ x_{ij}e = jx_{ij}\}$, i, $j \in \{0, 1\}$.

Proposition 2.4 *If A is a power-associative algebra, $e \in A$ an idempotent and $A = A_{11} \oplus A_{10} \oplus A_{01} \oplus A_{00}$, then:* $A_{11}^2 \subset A_{11}$, $A_{11}A_{10} \subset A_{10}$, $A_{10}A_{11} = 0$, $A_{11}A_{01} \subset A_{00}$, $A_{01}A_{11} \subset A_{01} + A_{11}A_{01} \subset A_{01} + A_{00}$, $A_{11}A_{00} = A_{00}A_{11} = 0$, $A_{10}^2 = 0$, $A_{10}A_{01} \subset A_{11}$, $A_{01}A_{10} \subset A_{00}$, $A_{00}A_{10} \subset A_{11}$, $A_{10}A_{00} \subset A_{00}A_{10} + A_{10} \subset A_{11} + A_{10}$, $A_{01}^2 = 0$, $A_{01}A_{00} = 0$, $A_{00}A_{01} \subset A_{01}$ *and* $A_{00}^2 \subset A_{00}$.

Proof For to prove that $A_{11}^2 \subset A_{11}$, we consider $x, y \in A_{11}$. Thus $ex = xe = x$ and $ey = ye = y$. Replacing z by e in (2), we obtain $(xy)e = xy$, and $(x, e, y) = (e, x, y)$ implies $e(xy) = xy$. Hence $A_{11}^2 \subset A_{11}$. To prove that $A_{11}A_{10} \subset A_{10}$ and $A_{10}A_{11} = 0$, we consider $x \in A_{11}$ and $y \in A_{10}$. Thus $ex = xe = x$, $ey = y$ and $ye = 0$. Since $(y, e, x) = (e, y, x)$, then $e(yx) = 2yx$. But we know that the characteristic roots of L_e are 1 and 0, and so $e(yx) = 2yx$ implies that $yx = 0$. Therefore $A_{10}A_{11} = 0$. Now $(x, e, y) = (e, x, y)$ implies $e(xy) = xy$. Replacing z by e in (3) and since $yx = 0$, $e(xy) = xy$, we get $(xy)e = 0$. Therefore we conclude that $A_{11}A_{10} \subset A_{10}$.

To prove that $A_{11}A_{01} \subset A_{00}$ and $A_{01}A_{11} \subset A_{01} + A_{11}A_{01} \subset A_{01} + A_{00}$, we consider $x \in A_{11}$ and $y \in A_{01}$. Thus $ex = xe = x$, $ey = 0$ and $ye = y$. Now $(e, x, y) = (x, e, y)$ implies $e(xy) = 0$, and $(e, y, x) = (y, e, x)$ implies $e(yx) = 0$. Replacing z by e in (3) we get $(xy)e = 0$, and replacing z by e, x by y, y by x in (3), we obtain $yx = (yx)e + xy$. We note that $0 = e(yx) = e((yx)e + xy) = e((yx)e)$ and $(yx)e = ((yx)e + xy)e = ((yx)e)e$, which implies that $(yx)e \in A_{01}$. With the above results we get that $A_{11}A_{01} \subset A_{00}$ and $A_{01}A_{11} \subset A_{01} + A_{11}A_{01} \subset A_{01} + A_{00}$. In a similar form, it is possible to prove the relations of the remaining cases.

Lemma 2.5 *Let A be a finite-dimensional power-associative algebra, $e \in A$ an idempotent and $A = A_{11} \oplus A_{10} \oplus A_{01} \oplus A_{00}$. Then e is principal idempotent of A if and only if the subalgebra A_{00} is a nilalgebra.*

Proof Suppose that $e \in A$ is a principal idempotent. If A_{00} is not a nilalgebra, then there exists an idempotent $u \in A_{00}$. Since $e \in A_{11}$ and $A_{11}A_{00} = 0$, we obtain that $eu = ue = 0$, which is a contradiction. Conversely, suppose that

A_{00} is a nilalgebra. If $e \in A$ is not a principal idempotent, then there exists an idempotent $u \in A$ such that $eu = ue = 0$. We consider $u = u_{11} + u_{10} + u_{01} + u_{00}$ where $u_{ij} \in A_{ij}$ with $i, j \in \{0, 1\}$. Now $0 = eu = u_{11} + u_{10}$ and $0 = ue = u_{11} + u_{01}$ imply $u_{11} = u_{10} = u_{01} = 0$, and so $u = u_{00} \in A_{00}$, a contradiction.

Proposition 2.6 *If A is a power-associative algebra, $e \in A$ an idempotent and $A = A_{11} \oplus A_{10} \oplus A_{01} \oplus A_{00}$, then the subspace $B = (A_{10}A_{01} + A_{00}A_{10}) + A_{10} + A_{01} + (A_{01}A_{10} + A_{11}A_{01})$ is an ideal of A.*

Proof We consider u_{ij} in A_{ij} with $i, j \in \{0, 1\}$. We will prove that for all $i, j \in \{0, 1\}$, $A_{ij}(A_{10}A_{01})$ and $(A_{10}A_{01})A_{ij}$are subsets of B. Using the relations of proposition 2.4, we obtain that: $(u_{11}, u_{01}, u_{10}) = (u_{11}u_{01})u_{10} - u_{11}(u_{01}u_{10}) \in A_{00}A_{10} + A_{11}A_{00} = A_{00}A_{10}$ and $(u_{01}, u_{10}, u_{11}) = (u_{01}u_{10})u_{11} - u_{01}(u_{10}u_{11}) = 0$. Now using (3), we get $(u_{11}, u_{10}, u_{01}) = -(u_{11}, u_{01}, u_{10}) - (u_{01}, u_{10}, u_{11}) \in A_{00}A_{10} \subset B$, which implies that $u_{11}(u_{10}u_{01}) - (u_{11}u_{10})u_{01} \in B$. Hence $u_{11}(u_{10}u_{01}) \in B$, and so $A_{11}(A_{10}A_{01}) \subset B$. Since $(u_{10}, u_{01}, u_{11}) = (u_{01}, u_{10}, u_{11}) = 0$, then $(u_{10}u_{01})u_{11} = u_{10}(u_{01}u_{11}) \in A_{10}A_{01} + A_{10}A_{00} \subset A_{10}A_{01} + A_{00}A_{10} + A_{10} \subset B$, and thus $(A_{10}A_{01})A_{11} \subset B$.

Now $A_{10}(A_{10}A_{01}) \subset A_{10}A_{11} = 0$, $(A_{10}A_{01})A_{10} \subset A_{11}A_{10} \subset A_{10} \subset B$, $A_{01}(A_{10}A_{01}) \subset A_{01}A_{11} \subset A_{01} + A_{11}A_{01} \subset B$, $(A_{10}A_{01})A_{01} \subset A_{11}A_{01} \subset B$ and $A_{00}(A_{10}A_{01}) = (A_{10}A_{01})A_{00} = 0$. Similarly, it is possible to prove that the subspaces $A_{ij}(A_{00}A_{10})$, $(A_{00}A_{10})A_{ij}$, $A_{ij}A_{10}$, $A_{10}A_{ij}$, $A_{ij}A_{01}$, $A_{01}A_{ij}$, $A_{ij}(A_{01}A_{10})$, $(A_{01}A_{10})A_{ij}$, $A_{ij}(A_{11}A_{01})$ and $(A_{11}A_{01})A_{ij}$ are subsets of B. Therefore we conclude that B is an ideal of A.

Corolario 2.7 *If A is of finite-dimensional simple power-associative algebra with idempotent $e \neq 1$ and $A = A_{11} \oplus A_{10} \oplus A_{01} \oplus A_{00}$ is the Peirce decomposition of A relative to e, then $A_{11} = A_{10}A_{01} + A_{00}A_{10}$ and $A_{00} = A_{01}A_{10} + A_{11}A_{01}$.*

Proof By proposition 2.6, we know that $B = (A_{10}A_{01} + A_{00}A_{10}) + A_{10} + A_{01} + (A_{01}A_{10} + A_{11}A_{01})$ is an ideal of A. Since A is a simple algebra, then we must to have that either $B = 0$ or $B = A$. If $B = 0$ then $A = A_{11} \oplus A_{00}$, and $A^2 = A$ implies $A_{11}^2 = A_{11}$ and $A_{00}^2 = A_{00}$. Since by hypothesis $e \neq 1$, then $A_{00} \neq 0$. Moreover in this case A_{00} is an ideal of A, and so $A_{00} = A$, a contradiction. Therefore $B = A$ which implies that $A_{11} = A_{10}A_{01} + A_{00}A_{10}$ and $A_{00} = A_{01}A_{10} + A_{11}A_{01}$.

Proposition 2.8 *If A is a power-associative algebra and I is and ideal of A, then I^2 is an ideal of A.*

Proof We consider x, y in I and $z \in A$. Now $(z, x, y) = (x, z, y)$ implies $z(xy) \in I^2$. Since (x, z, y), (y, z, x) are elements in I^2, then using (2) we get $(x, y, z) \in I^2$, which implies that $(xy)z \in I^2$.

Proposition 2.9 *Let A be a finite-dimensional power-associative algebra over K of characteristic 0. If x is nilpotent, then R_x is nilpotent.*

Proof Since the identity $(y,z,x)=(z,y,x)$ is valid in A, then:

$$R_xL_y - L_yR_x = R_xR_y - R_{yx} \tag{4}$$

for all $x, y \in A$. Now as $trace(R_xL_y)=trace(L_yR_x)$, we obtain that $trace(R_xR_y) = trace(R_{yx})$ for all x, $y \in A$. We will prove that $trace(R_x^nR_y)=trace(R_{R_x^n(y)})$ for all $n \geq 1$. Suppose that $trace(R_x^nR_y)=trace(R_{R_x^n(y)})$ for all x, $y \in A$. We observe that $trace(R_x^nR_xL_y) = trace(R_x(R_x^nL_y)) = trace(R_x^nL_yR_x)$. Therefore using (4) and the inductive hypothesis we get that $trace(R_x^{n+1}R_y) = trace(R_x^nR_{yx}) = trace(R_{R_x^n(yx)})$, as desired. Now it is clear that $trace(R_x^m) = trace(R_{x^m})$ for all $m \geq 1$. Since there exists $n \geq 1$ such that $x^n=0$, then for all $i \geq 1$ we have $trace((R_x^n)^i)=0$, which implies that R_x^n is nilpotent. Clearly we get that R_x is nilpotent.

We consider the algebra A^+, with multiplication defined by $x \cdot y = \frac{1}{2}(xy+yx)$ for x, y in A. It is known that when A is power-associative, then A^+ is a commutative power-associative algebra.

Proposition 2.10 *If A is a power-associative algebra, then the following conditions are equivalent:*

(*a*) *A^+ is a Jordan algebra.*

(*b*) *$R_xR_{x^2}=R_{x^2}R_x$ for all $x \in A$.*

Proof We note first that $(x,x^2,y)=(x^2,x,y)$ implies $x(x^2y)=x^2(xy)$, that is $L_xL_{x^2}=L_{x^2}L_x$. If (*a*) is valid, then $(x\cdot x)\cdot(y \cdot x) = ((x\cdot x)\cdot y)\cdot x$ for all x, $y \in A$, which implies that $x^2(yx)+x^2(xy)+(yx)x^2+(xy)x^2 = x(x^2y)+x(yx^2)+(x^2y)x+(yx^2)x$. Hence $L_{x^2}R_x+L_{x^2}L_x+R_{x^2}R_x+R_{x^2}L_x = L_xL_{x^2}+L_xR_{x^2}+R_xL_{x^2}+R_xR_{x^2}$. Since $L_xL_{x^2}=L_{x^2}L_x$ and replacing $L_{x^2}R_x = R_xL_{x^2}-R_xR_{x^2}+R_{x^3}$ and $L_xR_{x^2}=R_{x^2}L_x-R_{x^2}R_x+R_{x^3}$ in this last relation, we obtain (*b*). It is easy to prove that (*b*) implies (*a*).

Proposition 2.11 Let A be a finite-dimensional power-associative algebra over K of characteristic 0, $e \in A$ an principal idempotent and we consider $\omega: A \to K$ defined by $\omega(x)=trace(R_x)$, which clearly is a linear map. If $Ker(\omega)$ is a subalgebra of A, then A is a baric algebra.

Proof We note that $\omega(e)=trace(R_e)=\dim_K(A_{11})+\dim_K(A_{10}) \neq 0$, and so $A = Ke \oplus Ker(\omega)$. To prove that $Ker(\omega)$ is an ideal of A, we consider $x \in Ker(\omega)$. Thus $trace(R_x)=0$. Let $x = x_{11}+x_{10}+x_{01}+x_{00} \in A_{11} \oplus A_{10} \oplus A_{01} \oplus A_{00}$. Since x_{10}, x_{01} and x_{00} are nilpotent (By Lemma 2.5, A_{00} is a nilalgebra), then proposition 2.9 implies that $trace(R_x)=trace(R_{x_{11}})=0$. Using (4) we get that $trace(R_eR_x)=trace(R_{ex})=trace(R_{x_{11}})+trace(R_{x_{10}}) = trace(R_{x_{11}})=0$. We conclude that $\omega(ex)=0$, and thus $ex \in Ker(\omega)$. Similarly, it is possible to prove that $xe \in Ker(\omega)$, and therefore $Ker(\omega)$ is an ideal of

A. Finally, since $e = e^2 \in A^2$ and $e \notin Ker(\omega)$, we conclude that A is a baric algebra.

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