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STRUCTURE OF CERTAIN PERIODIC RINGS AND NEAR-RINGS

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Abstract

Using commutativity of rings satisfying $(xy)^{n(x,y)} = xy$ proved by Searcoid and MacHale [16], Ligh and Luh [13] have given a direct sum decomposition for rings with the mentioned condition. Further Bell and Ligh [9] sharpened the result and obtained a decomposition theorem for rings with the property $xy = (xy)^2 f(x,y)$ where $f(X,Y) \in Z < X,Y >$, the ring of polynomials in two noncommuting indeterminates. In the present paper we continue the study and investigate structure of certain rings and near rings satisfying the following condition which is more general than the mentioned conditions : $xy = p(x,y)$, where $p(x,y)$ is an admissible polynomial in $Z < X,Y >$. Moreover we deduce the commutativity of such rings.

1 Introduction

Throughout the paper R will denote an associative ring (may be without unity 1) and N the set of all nilpotent elements of R. A ring R is called zero-commutative if for all $x, y \in R$, $xy = 0$ implies that $yx = 0$. An element $x \in R$ with the property $x = x^{n(x)}$ for some integer $n(x) > 1$ will be called potent. The set of all potent elements will be denoted by P. If $P = R$, we shall call R a J-ring. By the well known $x = x^{n(x)}$ theorem of Jacobson [12], J-rings are necessarily commutative.

Let $Z < X,Y >$ denote the ring of polynomials with integer coefficients in two noncommuting indeterminates X and Y. The symbol $w(X,Y)$ will

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denote a word in X and Y-ie. an element of $Z < X, Y >$ of the form $Y^{j_1}X^{k_1}Y^{j_2}X^{k_2}\cdots Y^{j_s}X^{k_s}$ where the j_i and k_i are nonnegative integers such that $\sum_{i=1}^{s} j_i + k_i > 0$; and the symbols $|w|_x$ and $|w|_y$ will denote $\sum_{i=1}^{s} k_i$ and $\sum_{i=1}^{8} j_i$ respectively. We shall call $P(X, Y) \in Z < X, Y >$ an admissible polynomial if $P(X, Y) = \sum_{i=1}^{k} n_i w_i(X, Y)$, where each n_i is an integer and each $w_i(X, Y)$ is a word with $|w_i|_x \geq 2$ and $|w_i|_y \geq 2$.

A sufficient condition for R to be periodic is Chacron's criterion: For each $x \in R$ there exists an integer $m = m(x) \geq 1$ and a polynomial $f(X) \in Z[X]$ such that $x^m = x^{m+1}f(x)$ ([10]). It is shown in [5] that if R is a periodic ring, then every element $x \in R$ can be written in the form $x = a + u$, where $a \in P$ and $u \in N$. In a very surprising structural result (signified as theorem B in sequel) Bell [7] remarked that if in a periodic ring R every element has a unique representation as above, then P and N both are ideals and $R = P \bigoplus N$.

2. A decomposition theorem for rings

Theorem 2.1 *Let R be a ring such that for each $x, y \in R$ there exists an admissible $p(X, Y) \in Z < X, Y >$ for which*

$$xy = p(x, y). \tag{P}$$

Then R is a direct sum of a J-ring and a zero ring.

Proof By taking $y = x$ in conditon (P), we see that R satisfies Chacron's condition for periodicity and $u^2 = 0$ for all $u \in N$. Now let $u \in N$ and $x \in R$. Then $(ux)u = P(ux, u)$, where $P(X, Y) = \sum_{i=1}^{t} n_i w_i(X, Y)$ as mentioned in section 1. If Y precedes an X in $w_i(X, Y)$, then clearly $w_i(ux, u) = 0$; otherwise $w_i(X, Y) = X^j Y^k$ with $j, k \geq 2$, and again $w_i(ux, u) = 0$. Thus $(ux)u = 0$, and it follows easily that $xu = ux = 0$. Thus

$$RN = NR = \{0\}. \tag{2.1}$$

Since R is periodic, every element x in R can be written in the form $x = a + u$, where $a \in P$ and $u \in N$; and by [7, Theorem B] it is sufficient to show that the above representation is unique. Let $a + u = b + v$ where $a, b \in P$ and $u, v \in N$. Then

$$a - b = v - u. \tag{2.2}$$

Since $a, b \in P$, there exist odd integers $p = p(a) > 1$ and $q = q(b) > 1$ such that $a^p = a$ and $b^q = b$; and letting $k = (p-1)q - (p-2) = (q-1)p - (q-2)$, we see that $a^k = a$ and $b^k = b$. Note that $e = a^{k-1}$ and $f = b^{k-1}$ are idempotents such that $ea = a$ and $fb = b$. Multiplying (2.2) by a and b from both sides gives $a^2 = ab = ba$ and $b^2 = ab = ba$, which yields that $a^2 = b^2$; and since $k-1$ is even, we have $e = f$. Now multiplying (2.2) by e shows that $a = b$. Hence the proof is complete. □

In view of (2.1), we conclude that the nilpotent elements of R annihilate R on both sides and hence are central. Since J-rings are commutative, a consequence of the above theorem is the following corollary.

Corollary 2.1 *Let R be a ring such that for each $x, y \in R$ there exists an admissible $p(X,Y) \in Z < X, Y >$ for which condition (P) is satisfied. Then R is commutative.*

3. A decomposition theorem for near-rings

For the purpose of this section R will denote a left-near ring with multiplicative center Z. An element $x \in R$ is said to be distributive if $(y+z)x = yx + zx$ for all $y, z \in R$. If every element of R is distributive, then R is said to be a distributive near ring. A near ring R is said to be distributively generated $(d-g)$ if it contains a multiplicative subsemigroup of distributive elements which generates the additive group $(R, +)$.

An ideal of a near ring R is a normal subgroup I of $(R,+)$ such that (i) $RI \subseteq I$ and (ii) $(x+s)y - xy \in I$ for $x, y \in R$ and $s \in I$.

It is natural to question whether the analogous hypotheses yield the direct sum decomposition in the case of near rings. An example due to Clay (cf. [11, Example H-29, page 342]) shows that it is not possible to obtain such a decomposition.

Consider the non-abelian additive group $(R,+)$, isomorphic to the symmetric group S_3, and define addition and multiplication in R as follows :

+	0	a	b	c	u	v
0	0	a	b	c	u	v
a	a	0	v	u	c	b
b	b	u	0	v	a	c
c	c	v	u	0	b	a
u	u	b	c	a	v	0
v	v	c	a	b	0	u

.	0	a	b	c	u	v
0	0	0	0	0	0	0
a	0	a	a	a	0	0
b	0	a	a	a	0	0
c	0	a	a	a	0	0
u	0	0	0	0	0	0
v	0	0	0	0	0	0

It is easy to see that $(R, +, \cdot)$ is a commutative (distributive) near ring satisfying $xy = xy^2x$, for all $x, y \in R$. However $P = \{0, a\}$ is not an ideal of R.

Hence, following [9], we define a weaker notion of orthogonal sum. A near ring R is an orthogonal sum of subnear rings A and B denoted by $R = A \uplus B$ if $AB = BA = \{0\}$ and each element of R has a unique representation in the form $a+b$ with $a \in A$ and $b \in B$.

In this section we shall obtain an orthogonal sum decomposition for certain near rings R satisfying condition (P).

Theorem 3.1 *Let R be a d-g near ring such that for each $x, y \in R$ there exists an admissible $p(X,Y) \in Z < X, Y >$ for which condition (P) is satisfied. Then R is periodic and commutative. Moreover, $R = P \uplus N$, where P is a subring and N is a subnear ring with trivial multiplication.*

Before starting the proof of the above theorem we state the following known results which are essentially found in [2],[4],[6] and [9] respectively.

Lemma 3.1 *Let R be a zero symmetric near ring satisfying properties :*

(i) For each x in R, there exists an integer $n(x) > 1$ such that $x^{n(x)} = x$.

(ii) Every non-trivial homomorphic image of R contains a non zero central idempotent.

Then $(R,+)$ is abelian.

Lemma 3.2 *If R is a d-g near ring such that $N \subseteq Z$, then N is an ideal.*

Lemma 3.3 *Let R be a d-g near ring such that for each $x \in R$, there exist a positive integer $n = n(x)$ and an element s in the subnear ring generated by x for which $x^n = x^n s$. If $N \subseteq Z$, then R is periodic and commutative.*

Lemma 3.4 *Let R be a near ring in which idempotents are multiplicatively central. If e and f are any idempotents, there exists an idempotent g such that $ge = e$ and $gf = f$.*

Lemma 3.5 *If R is a zero commutative periodic near ring, then $R = P + N$.*

Proof of theorem 3.1 Let R satisfy condition (P). Using the fact that $d-g$ near rings are zero symmetric and arguing in the similar manner as we have done in the proof of theorem 2.1, we obtain

$$NR = RN = \{0\}. \tag{3.1}$$

Thus using (3.1), we have $N \subseteq Z$ and $N^2 = \{0\}$. Replacing y by x in (P) we get an element R in the subnear ring generated by x such that $x^2 = x^2 r$. Hence by Lemma 3.3 R is periodic and commutative. Every element $x \in R$ can be expressed in the form $x = a + u$, where $a \in P$ and $u \in N$.

Next we show that P is a subring. Let $a, b \in P$ and choose integers $p = p(a) > 1$ and $q = q(b) > 1$ such that $a^p = a$ and $b^q = b$. Let $r = (p-1)q - (p-2) = (q-1)p - (q-2)$. Then it is clear that $a^r = a$ and $b^r = b$. Note that $e = a^{r-1}$ and $f = b^{r-1}$ are idempotents with $ea = a$ and $fb = b$. Obviously $(ab)^r = a^r b^r = ab$, hence $ab \in P$. Moreover, by Lemma

3.2 N is an ideal. Since R/N has the $x^t = x$ property we have an integer $j > 1$ such that

$$(a-b)^j = a - b + u \ ; \ u \in N \tag{3.2}$$

Using Lemma 3.4 we can choose an idempotent g for which $ge = e$ and $gf = f$. Therefore, $ga = a$ and $gb = b$. Now multiplying (3.2) by g we have $(a-b)^j = a - b$ i.e. $a - b \in P$. Also by Lemma 3.1 $(P, +)$ is abelian. Hence P is a subring.

Trivially $P \cap N = \{0\}$. Let $a + u = b + v$, where $a, b \in P$ and $u, v \in N$. Then $a - b = v - u \in P \cap N = \{0\}$, which yields $a = b$ and $v = u$. Hence $R = P \uplus N$.

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