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CHOQUET THEOREM FOR THE SPACE OF CONTINUOUS REAL-VALUED FUNCTIONS

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Abstract

We prove that Choquet Theorem does not hold for $C[0,1]$, the space of all continuous real-valued functions on the unit interval $[0,1]$.

1 Introduction

Choquet Theorem for locally compact separable Hausdorff spaces is foundation in several areas of applied mathematics, including probability. However, for non-locally compact space, Choquet Theorem does not hold. In fact, a negative version of Choquet Theorem for the Hilbert space

$$l^2 = \{x = (x_n) : ||x|| = (\sum_{n=1}^{\infty} x_n^2)^{\frac{1}{2}} < \infty\}$$

was given in [4].

In this note we consider Choquet Theorem for $C[0,1]$, the space of all continuous real-valued functions on the unit interval $[0,1]$. As one of the most important domains in probability, see Billingsley [1], it is of interest to know whether Choquet Theorem holds for $C[0,1]$. Our aim is to seek a version of

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Choquet Theorem for the space $C[0,1]$. As in [4] our result again shows that Choquet Theorem fails for this space.

To state our result for the space $C[0,1]$, we first describe the Choquet Theorem in the locally compact setting. Let E be a locally compact separable Hausdorff space. Let $\mathcal{K}, \mathcal{F}, \mathcal{G}$ and $\mathcal{B}$ denote the classes of all compact subsets, closed subsets, open subsets and Borel subsets of E, respectively. For $A \subset E$, we denote

$$\mathcal{F}_{\mathcal{A}} = \{F \in \mathcal{F} : F \cap A \neq \emptyset\} \text{ and } \mathcal{F}^A = \{F \in \mathcal{F} : F \cap A = \emptyset\}.$$

To topologize the space $\mathcal{F}$ we define a base of open neighborhoods for $F \in \mathcal{F}$ of the form

$$\{\mathcal{F}^K_{G_1 \dots G_n} : K \in \mathcal{K}; G_1, ..., G_n \in \mathcal{G}; \text{ and } n \in \mathbb{N}\},$$

where

$$\mathcal{F}^K_{G_1 \dots G_n} = \mathcal{F}^K \cap \mathcal{F}_{G_1} \cap \dots \cap \mathcal{F}_{G_n}.$$

Then $\mathcal{U}(F) = \mathcal{F}^K_{G_1 \dots G_n}$ is a neighborhood of $F \in \mathcal{F}$ if and only if $F \cap K = \emptyset$ and $F \cap G_i \neq \emptyset$ for $i = 1, \dots, n$. We will refer to this as the *miss-and-hit topology.*

It was shown in [3] that for a locally compact separable Hausdorff space E, the space $\mathcal{F}$ with the miss-and-hit topology is compact, Hausdorff and separable. Let $\sigma(\mathcal{F})$ denote the family of all Borel sets of $\mathcal{F}$ in the miss-and-hit topology. A *random closed set* on E is a probability measure P on the measurable space $(\mathcal{F}, \sigma(\mathcal{F}))$.

By a *capacity functional* $T : \mathcal{K} \to [0,1]$ we mean a monotone set function defined on $\mathcal{K}$ with values in $[0,1]$ satisfying the following conditions:

(i) $T(\emptyset) = 0$;

(ii) T is alternating of infinite order: For any compact sets $K_i,\ i = 1, 2, \dots, n$; $n \geq 2$, we have

$$T(\bigcap_{i=1}^{n} K_i) \leqslant \sum_{I \in \mathcal{I}(n)} (-1)^{\#I+1} T(\bigcup_{i \in I} K_i),$$

where $\mathcal{I}(n) = \{I \subset \{1, \dots, n\},\ I \neq \emptyset\}$ and $\#I$ denotes the cardinality of I;

(iii) T is upper continuous in the miss-and-hit topology on $\mathcal{K}$, i.e., $K_{n+1} \subset K_n$ for every $n \in \mathbb{N}$ and $K = \bigcap_{n=1}^{\infty} K_n$, then $T(K_n) \searrow T(K)$.

The following result, known as Choquet Theorem, provides ways to specify probability measures on the measurable space $(\mathcal{F}, \sigma(\mathcal{F}))$ as follows.

1.1. Choquet Theorem ([2],[3]). *Let E be a locally compact separable Hausdorff space. Then there exists a bijection between probability measures P on $\sigma(\mathcal{F})$ and capacity functional $T : \mathcal{K} \to [0, \infty)$ satisfying*

$$P(\mathcal{F}_K) = T(K) \textit{ for every } K \in \mathcal{K}.$$

Choquet Theorem is equivalent to the theorem below, called *an open version of Choquet Theorem.*

1.2. An open version of Choquet Theorem ([2],[3]). *Let E be a locally compact separable Hausdorff space and let $T : \mathcal{G} \to [0, \infty)$ be a set function. Then there exists unique a probability measure P on $\sigma(\mathcal{F})$ satisfying*

$$P(\mathcal{F}_G) = T(G) \text{ for every } G \in \mathcal{G}$$

if and only if T is a capacity functional on $\mathcal{G}$, i.e.,

(i) $T(\emptyset) = 0, 0 \leqslant T \leqslant 1, T$ is monotone;

(ii) T is alternating of infinite order on $\mathcal{G}$, and

(iii) If $G_n \nearrow G$ on $\mathcal{G}$ (i.e.,$G_n \subset G_{n+1}$ for every $n \in \mathbb{N}$ and $G = \bigcup_{n=1}^{\infty} G_n$), then $T(G_n) \nearrow T(G)$.

1.3. Remark By a *Polish space* we mean a complete, separable metric space. It was shown in [5] that if E is a non-locally compact Polish space, then the miss-and-hit topology on $\mathcal{F}$ is no longer Hausdorff. However, the space of all bounded closed subsets of a metric space equipped with the Hausdorff metric is Hausdorff. In the next section, we will use this topology to obtain a negative version of Choquet Theorem for the space $C[0, 1]$.

2 A negative version of Choquet Theorem

In what follows E will denote the closed unit ball of $C[0, 1]$, i.e.,

$$E = \{x \in C[0, 1] : ||x|| = \sup_{0 \leqslant t \leqslant 1} |x(t)| \leqslant 1\}.$$

Note that E is a bounded Polish space which is not locally compact. Let $\mathcal{F}, \mathcal{G}$ and $\mathcal{B}$ denote the families of all closed subsets, open subsets and Borel subsets of E, respectively. The space $\mathcal{F}$ will be equipped with the Hausdorff metric

$$d(S, T) = \begin{cases} \max\{\sup_{x \in S} ||x - T||, \sup_{x \in T} ||x - S||\} & \text{if } S \neq \emptyset, T \neq \emptyset \\ 0 & \text{if } S = T = \emptyset \\ 2 & \text{otherwise,} \end{cases} \tag{2.1}$$

where $||x - S|| = \inf\{||x - y|| : y \in S\}$. Let $\sigma(\mathcal{F})$ denote the family of all Borel sets (with respect to the Hausdorff metric) of $\mathcal{F}$.

2.1. Proposition. *The topology induced by the Hausdorff metric is stronger than the miss-and-hit topology.*

Proof. We equip $\mathcal{F}$ with the topology induced by the Hausdorff metric. It suffices to prove that $\mathcal{F}_G$ and $\mathcal{F}^K$ are open in $\mathcal{F}$ for $G \in \mathcal{G}$ and $K \in \mathcal{K}$.

Let $S \in \mathcal{F}_G$. Then $S \cap G \neq \emptyset$. Since G is open in E, there exist $r > 0$ and $x \in S$ such that the ball $B(x, r) \subset G$. Then it is easy to show that $B(S, r) \subset \mathcal{F}_G$, where $B(S, r)$ denotes the ball in $\mathcal{F}$ (in the Hausdorff metric) with center S and radius r. That means $\mathcal{F}_G$ is open in $\mathcal{F}$.

For $F \in \mathcal{F}^K$ we have $F \cap K = \emptyset$. By the compactness of K we get

$$\epsilon = \inf\{||x - F|| : x \in K\} > 0.$$

We claim that the ball $B(F, \epsilon) \subset \mathcal{F}^K$. Indeed, let $T \in B(F, \epsilon)$. If $T \cap K \neq \emptyset$, then

$$\begin{aligned} \sup_{y \in T} ||y - F|| &\geq \sup_{y \in T \cap K} ||y - F|| \\ &\geq \inf_{y \in T \cap K} ||y - F|| \\ &\geq \inf_{y \in K} ||y - F|| = \epsilon, \end{aligned}$$

which contradicts $T \in B(F, \epsilon)$. Therefore, $T \cap K = \emptyset$, i.e., $T \in \mathcal{F}^K$. Consequently, $\mathcal{F}^K$ is open in $\mathcal{F}$. The proposition is proved. □

Note that the space $\mathcal{F}$ equipped with the miss-and-hit topology is not Hausdorff. Therefore, this topology is strictly weaker than the topology induced by the Hausdorff metric.

Our main result in this paper is the following theorem.

2.2. Theorem (Main Theorem). *There exists a monotone set function $T : \mathcal{G} \to [0, 1]$ with the following properties:*

(i) $T(\emptyset) = 0, T(E) = 1$;

(ii) T is lower continuous in the Hausdorff metric, i.e., if $G_n \nearrow G$ (in the Hausdorff metric), then $T(G_n) \nearrow T(G)$;

(iii) T is alternating of infinite order on $\mathcal{G}$;

(iv) No probability measure P on $\sigma(\mathcal{F})$ satisfies the conditions

$$P(\mathcal{F}_G) = T(G) \text{ for every } G \in \mathcal{G}.$$

2.3. Remark. Obviously Theorem 2.2 is not a counter-example of Choquet Theorem for the space $C[0, 1]$, since we use the topology induced by Hausdorff metric (2.1) instead of the miss-and-hit topology. However, in some sense, Theorem 2.2 can be viewed as a "negative version" of Choquet Theorem for the space $C[0, 1]$.

To prove Theorem 2.2, we are going to define a set function $T : \mathcal{G} \to [0, 1]$ satisfying the conditions *(i)-(iv)* of the theorem. The proof of this theorem is based on the proof of the Theorem 1 in [4].

3 Some auxiliary lemmas

Denote $B(x, r) = \{y \in E : ||x - y|| < r\}$. For every $G \in \mathcal{G}$, let

$$T_n(G) = \inf \{r > 0 : G \subset B(x_1, r) \cup \ldots \cup B(x_n, r)\},$$

for $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$.

It is easy to see that $\{T_n(G)\}$ is a descreasing sequence. Define

$$T(G) = \lim_{n\to\infty} T_n(G) \text{ for every } G \in \mathcal{G}. \tag{3.1}$$

Clearly $T(G) = 0$ if and only if $\overline{G}$ is compact. Accordingly $T(G)$ is known as *Kuratowski measure* of non-compactness of $G \in \mathcal{G}$. It is easy to see that T is monotone and $0 \leqslant T \leqslant 1$.

First we are going to construct a sequence $\{e_n\} \subset E$ of continuous functions on the unit interval $[0, 1]$. Let $\Delta = [a, a + \delta], a \in \mathbb{R}, \delta > 0$. We first define functions $\alpha_\Delta, \beta_\Delta : \mathbb{R} \to [-1, 1]$ by

$$\alpha_\Delta(t) = \begin{cases} -1 & \text{if } a \leqslant t \leqslant a + \delta/3 \\ 6(t-a)/\delta - 3 & \text{if } a + \delta/3 < t < a + 2\delta/3 \\ 1 & \text{if } a + 2\delta/3 \leqslant t \leqslant a + \delta \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\beta_\Delta(t) = \begin{cases} -6(t-a)/\delta + 1 & \text{if } a < t < a + \delta/3 \\ -1 & \text{if } a + \delta/3 \leqslant t \leqslant a + 2\delta/3 \\ 6(t-a)/\delta - 5 & \text{if } a + 2\delta/3 < t < a + \delta \\ 0 & \text{otherwise.} \end{cases}$$

Clearly,

$$\alpha_\Delta(t) = \begin{cases} |3t - 3a - \delta|/\delta - |3t - 3a - 2\delta|/\delta & \text{if } a \leqslant t \leqslant a + \delta \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

and

$$\beta_\Delta(t) = \begin{cases} |3t - 3a - \delta|/\delta + |3t - 3a - 2\delta|/\delta - 2 & \text{if } a < t < a + \delta \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

For each $n \in \mathbb{N}, (n \geq 1)$, we divide the unit interval $[0, 1]$ into 3^{n-1} equal subintervals of length 3^{-n+1}, and denote

$$\Delta_{n,i} = [\ \frac{i-1}{3^{n-1}}, \frac{i}{3^{n-1}}\ ] \text{ for } i = 1, \dots, 3^{n-1},$$

$$I_n(k) = \{i \in \{1, \dots, 3^{n-1}\} :\ i = 4l + k, l \in \mathbb{N}\} \text{ for } k = 1, 2, 3, 4. \qquad (3.4).$$

For each $n \in \mathbb{N}, n \geq 1$, let

$$e_n = \sum_{i\in I_n(1)} \alpha_{\Delta_{n,i}} + \sum_{i\in I_n(2)} \beta_{\Delta_{n,i}} - \sum_{i\in I_n(3)} \alpha_{\Delta_{n,i}} - \sum_{i\in I_n(4)} \beta_{\Delta_{n,i}},$$

where α_Δ and β_Δ are defined by (3.2) and (3.3), respectively. From the definition of α_Δ and β_Δ it follows that e_n are continuous functions on $[0,1]$ and $||e_n|| = 1$ for every n. Moreover, observe that

$$e_n(t) = \begin{cases} -1 & \text{for } t \in \bigcup_{i\in I_{n+1}(1)} \Delta_{n+1,i} \\ 1 & \text{for } t \in \bigcup_{i\in I_{n+1}(3)} \Delta_{n+1,i}. \end{cases} \tag{3.5}$$

We prove the following lemmas.

3.1. Lemma *a) Let $m,n \in \mathbb{N}$ and $1 \leqslant m < n$. Then for any $i \in I_{m+1}(1)$ (or $i \in I_{m+1}(3)$) there exist $j \in I_{n+1}(1)$ and $k \in I_{n+1}(3)$ such that*

$$\Delta_{n+1,j} \subset \Delta_{m+1,i} \text{ and } \Delta_{n+1,k} \subset \Delta_{m+1,i}. \tag{3.6}$$

b) For any $r \in (0,1)$ and for any $x \in E$, the ball $B(x,r)$ contains at most one e_n.

Proof of a). Clearly that it is sufficient to prove in the case $n = m+1$. If $i \in I_{m+1}(1)$, then $i = 4l+1, l \in \mathbb{N}$ (see (3.4)). The interval $\Delta_{m+1,i}$ is divided into three equal subintervals denoted by $\Delta_{m+2,j}, \Delta_{m+2,j+1}$ and $\Delta_{m+2,j+2}$. We have

$$j = 3(i-1)+1 = 3(4l+1-1)+1 = 4(3l)+1 \in I_{m+2}(1) = I_{n+1}(1).$$

Hence $j \in I_{n+1}(1)$ and $k = j+2 \in I_{n+1}(3)$ satisfy inclusions (3.6). If $i \in I_{m+1}(3)$, then $i = 4l+3, l \in \mathbb{N}$ (see (3.4)). In the same way, the interval $\Delta_{m+1,i}$ is divided into three equal subintervals $\Delta_{m+2,j}, \Delta_{m+2,j+1}$ and $\Delta_{m+2,j+2}$. Observe that

$$j = 3(i-1)+1 = 3(4l+3-1)+1 = 4(3l+1)+3 \in I_{m+2}(3) = I_{n+1}(3).$$

Hence $j+2 \in I_{n+1}(1)$ and $k = j \in I_{n+1}(3)$ satisfy inclusions (3.6).
Proof of b). If b) does not hold, there are $e_m, e_n \in B(x,r)$ with $n \neq m$. Assume that $n \geq m+1$. Let $i \in I_{m+1}(3)$. By a) there exists $j \in I_{n+1}(1)$ such that $\Delta_{n+1,j} \subset \Delta_{m+1,i}$. For $t \in \Delta_{n+1,j}$, from (3.5) we have

$$\begin{aligned} 2 = |e_m(t) - e_n(t)| &\leqslant ||e_n - e_m|| \\ &\leqslant ||e_n - x|| + ||e_m - x|| < 2r < 2, \end{aligned}$$

a contradiction. The lemma is proved. □

3.2. Lemma. $T(G) = 1$ *for any open subset G of E containing $\{e_n : n \in \mathbb{N}\}$.*
Proof. Asumme on the contrary that $T(G) < 1$. Take $r > 0$ with $T(G) < r < 1$. By the definition of T there exists $N \in \mathbb{N}$ such that

$$G \subset \bigcup_{i=1}^{N} B(x_i, r).$$

Hence, there exists $i_0 \in \{1, \dots, N\}$ such that $B(x_{i_0}, r)$ contains infinitely many e_n. By Lemma 3.1(b) we get a contradiction. Consequently $T(G) = 1$, and the proof of the lemma is completed. $\square$

Let $\mathcal{E}$ be a class of subsets of a set E which is closed under finite unions and intersections. We say that a set function T is *maxitive* on $\mathcal{E}$ if

$$T(A \cup B) = \max\{T(A), T(B)\} \text{ for } A, B \in \mathcal{E}.$$

The following lemma was shown in [4].

3.3. Lemma. *Any maxitive set function is alternating of infinite order.*

4. Proof of the main theorem

Our aim is to show that the set function T defined by (3.1) satisfies the conditions of Theorem 2.2.

Proof of (i). Obviously $T(\emptyset) = 0$, and the fact that $T(E) = 1$ is a special case of Lemma 3.2.

Proof of (ii). Asumme that

$$G_n \nearrow G \subset E \text{ and } \sup_{x \in G} ||x - G_n|| \to 0 \text{ as } n \to \infty. \tag{4.1}$$

Since T is monotone, $\lim_{n\to\infty} T(G_n) \leqslant T(G)$. We claim that $lim_{n\to\infty} T(G_n) = T(G)$. Suppose it is not the case, say $\lim_{n\to\infty} T(G_n) < \alpha < T(G)$, then we take $\delta > 0$ such that $\alpha + \delta < T(G)$. By (4.1) there exists $N \in \mathbb{N}$ such that

$$G \subset (G_N)_\delta, \tag{4.2}$$

where $(G_N)_\delta = \{x \in E : ||x - G_N|| < \delta\}$.

Since $\{G_n\}$ is increasing, $T(G_N) \leqslant \lim_{n\to\infty} T(G_n) < \alpha$. By definition

$$G_N \subset B(x_1, \alpha) \cup \ldots \cup B(x_k, \alpha) \text{ for sufficiently large } k \in \mathbb{N},$$

which implies, see (4.2)

$$G \subset B(x_1, \alpha + \delta) \cup \ldots \cup B(x_k, \alpha + \delta).$$

This follows that

$$T(G) \leqslant \alpha + \delta < T(G),$$

a contradiction. Consequently,

$$T(G_n) \nearrow T(G),$$

and (ii) is established.

Proof of (iii). In order to prove that T is alternating of infinite order, it suffices to show by Lemma 3.3 that T is maxitive, i.e.,

$$T(U \cup V) = \max\{T(U), T(V)\} \text{ for every } U, V \in \mathcal{G}.$$

In fact, we have

$$T_{2n}(U \cup V) \leqslant \max\{T_n(U), T_n(V)\} \text{ for every } U, V \in \mathcal{G}.$$

Hence

$$T(U \cup V) \leqslant \max\{T(U), T(V)\} \text{ for every } U, V \in \mathcal{G},$$

which follows that

$$T(U \cup V) = \max\{T(U), T(V)\} \text{ for every } U, V \in \mathcal{G}.$$

Consequently (iii) holds.

Proof of (iv). From Proposition 2.1 we have $\mathcal{F}_G \in \sigma(\mathcal{F})$ for any open set G of E. For every $n \in \mathbb{N}$, we put

$$B_n = \text{conv}\{e_1, \ldots, e_n\} \text{ and } B = \bigcup_{n=1}^{\infty} B_n. \tag{4.3}$$

Observe that B_n is a compact convex set of E for every $n \in \mathbb{N}$. We claim that.

Claim. For each $n \in \mathbb{N}$, there exists an open set $G_n \supset B_n$ such that

$$T(G_n) < 2^{-n-1}.$$

Indeed, let $\epsilon = 2^{-n-1}$. Since B_n is compact, there are $\{B(x_i, \epsilon), i = 1, \ldots, k\}$ such that $B_n \subset \bigcup_{i=1}^{k} B(x_i, \epsilon)$. Denote $G_n = \bigcup_{i=1}^{k} B(x_i, \epsilon)$. Then G_n is an open set containing B_n and

$$T_k(G_n) \leqslant \epsilon,$$

which implies

$$T(G_n) \leqslant \epsilon = 2^{-n-1}.$$

The claim is proved.

We now prove (iv). Assume on the contrary that there is a probability measure P on $\sigma(\mathcal{F})$ such that

$$P(\mathcal{F}_G) = T(G) \text{ for every } G \in \mathcal{G}.$$

Then for each $n \in \mathbb{N}$, by the above claim there is an open set $G_n \supset B_n$ such that $T(G_n) < 2^{-n-1}$. Denote $G = \bigcup_{n=1}^{\infty} G_n$. Then G is an open set containing

B. Hence, from Lemma 3.2 and the above claim we get

$$\begin{aligned}
P(\mathcal{F}_G) &= P(\mathcal{F}_{\bigcup_{n=1}^{\infty} G_n}) = P(\bigcup_{n=1}^{\infty} \mathcal{F}_{G_n}) \\
&\leqslant \sum_{n=1}^{\infty} P(\mathcal{F}_{G_n}) = \sum_{n=1}^{\infty} T(G_n) \\
&< \sum_{n=1}^{\infty} 2^{-n-1} = \frac{1}{2} < 1 = T(G),
\end{aligned}$$

which contradicts Lemma 3.2. Therefore, (iv) holds. Consequently, the proof of our main theorem is finished. $\square$

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