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ON A BOUNDARY VALUE PROBLEM FOR A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract

New sufficient conditions of the existence and uniqueness of the solution of a boundary problem for a system of ordinary differential equations with certain functional boundary conditions are constructed by the method of a priori estimates.

Introduction

In this paper we give new sufficient conditions for the existence and the uniqueness of the solution of the problem

$$x_i'(t) = f_i(t, x_1, \dots, x_n) \quad (i = 1, \dots, n) \tag{1}$$

$$\Phi_{0i}(x_i) = \varphi_i(x_1, \dots, x_n) \quad (i = 1, \dots, n) \tag{2}$$

where for each $i \in \{1, \dots, n\}$ $\quad f_i : \langle a, b\rangle \times R^n \to R$ satisfies the Carathéodory conditions, Φ_{0i} - the linear nondecreasing continuous functional on $C(\langle a, b\rangle)$ is concentrated on $\langle a_i, b_i\rangle \subseteq \langle a, b\rangle$ (i.e. the value of Φ_{0i} depends only on functions

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restricted to $\langle a_i, b_i\rangle$ and the segment can be degenerated to a point) and φ_i is a continuous functional on $C_n(\langle a, b\rangle)$. In general $\Phi_{0i}(1) = C_i \quad (i = 1, \dots, n)$. Without loss of generality we can suppose $\Phi_{0i} = 1 \quad (i = 1, \dots, n)$, which simplifies the notation.

Special cases of the conditions (2) are presented by the series of formerly investigated problems e.g.

$$x_i(t_i) = \varphi_i(x_1, \dots, x_n) \quad (i = 1, \dots, n) \tag{3}$$

$$\Phi_{0i}(x_i) = c_i \quad (c_i \in R) \quad (i = 1, \dots, n) \tag{4}$$

and more specialised - Cauchy-Nicoletti problem

$$x_i(t_i) = c_i \quad (c_i \in R) \quad (i = 1, \dots, n) \tag{5}$$

or periodical problem

$$x_i(a) = x_i(b) \quad (i = 1, \dots, n) \tag{6}$$

Problems (1), (5) and (1), (6) were studied in the papers [4], [5]. Problem (1), (3) was studied in [5], [6], [8] and [9], problem (1), (4) in [2], [3], similar results are also published in [1].

Main result

We adopt the following notation:
$\langle a, b\rangle$ - a segment, $-\infty < a \le a_i \le b_i \le b < +\infty \quad (i = 1, \dots, n)$, $\mathbb{R}^n$, the n-dimensional real space with points $x = (x_i)_{i=1}^n$ normed by $\|x\| = \sum\limits_{i=1}^n |x_i|$,

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i \ge 0,\ i = 1, \dots, n\},$$

$C_n(\langle a, b\rangle)$ and $AC_n(\langle a, b\rangle)$ are, respectively, the spaces of continuous and absolutely continuous n-dimensional vector-valued functions on $\langle a, b\rangle$ with the norm

$$\|x\|_{C_n(\langle a,b\rangle)} = \max\Big\{\sum_{i=1}^n |x_i(t)| : a \le t \le b\Big\},$$

$$C^+(\langle a, b\rangle) = \big\{x \in C(\langle a, b\rangle) : x(t) \ge 0, a \le t \le b\big\}$$

$L^p(\langle a, b\rangle)$ is space of functions integrable on $\langle a, b\rangle$ in p-th power with the norm

$$\|u\|_{L^p} = \begin{cases} \big[\int\limits_a^b |u(t)|^p dt\big]^{1/p} & \text{for } 1 \le p < \infty \\ \text{vrai } \max\{|u(t)| : a \le t \le b\} & \text{for } p = +\infty \end{cases}$$

If $x = (x_i(t))_{i=1}^n \in C_n(\langle a, b\rangle)$ and $y = (y_i(t))_{i=1}^n \in C_n(\langle a, b\rangle)$, then $x \le y$ if and only if $x_i(t) \le y_i(t)$ for all $t \in \langle a, b\rangle$ and $i = 1, \dots, n$. $K(\langle a, b\rangle)$ is the set of functions $g : \langle a, b\rangle \times R^n \to R$ satisfying local Carathédory conditions, i.e. if $g \in K(\langle a, b\rangle)$, $g(., x)$ is measurable on $\langle a, b\rangle$ for each $x \in \mathbb{R}^n$, $g(t, .)$ is continuous in $\mathbb{R}^n$ for almost all $t \in \langle a, b\rangle$, and

$$\sup\{|g(., x)| : \|x\| \le \varrho\} \in L(\langle a, b\rangle) \text{ for } \varrho \in (0, +\infty)$$

Let us consider the problem (1), (2). Under the solution we understand absolutely continuous n-dimensional vector-valued function on $\langle a, b\rangle$, which satisfies the equation (1) for almost all $t \in \langle a, b\rangle$ and fulfils the boundary conditions (2).

Definition Let $G = (g_i)_{i=1}^n : C\langle a, b\rangle \to \mathbb{R}^n$, $H = (h_{ij})_{i,j=1}^n : \langle a, b\rangle \to \mathbb{R}_+^{n\times n}$ and $\Psi = (\psi_i)_{i=1}^n : C_n(\langle a, b\rangle) \to \mathbb{R}_+^n$ is a positively homogeneous nondecreasing operator. We say that

$$(G, H, \Psi) \in Nic_0(\langle a, b\rangle; a_1, \dots, a_n, b_1, \dots, b_n) \tag{7}$$

if the system of differential inequalities

$$|x_i'(t) - g_i(t)x_i(t)| \le \sum_{j=1}^n h_{ij}(t)|x_j(t)| \text{ for } a \le t \le b \quad (i = 1, \dots, n) \tag{8}$$

with boundary conditions

$$\min\{|x_i(t)| : a_i \le t \le b_i\} \le \psi_i(|x_1(t)|, \dots, |x_n(t)|) \quad (i = 1, \dots, n) \tag{9}$$

has only trivial solution

Theorem 1. *Let the inequalities*

$$[f_i(t, x_1, \dots, x_n) - g_i(t)x_i] \text{ sign } x_i \le \sum_{j=1}^n h_{ij}(t)|x_j| + \omega_i\big(t, \sum_{j=1}^n |x_j|\big) \quad \text{if } t \in \langle a_i, b\rangle, x \in \mathbb{R}^n \quad (i = 1, \dots, n) \tag{10_1}$$

$$[f_i(t, x_1, \dots, x_n) - g_i(t)x_i] \text{ sign } x_i \ge -\sum_{j=1}^n h_{ij}(t)|x_j| - \omega_i\big(t, \sum_{j=1}^n |x_j|\big) \quad \text{if } t \in \langle a, b_i\rangle, x \in \mathbb{R}^n \quad (i = 1, \dots, n) \tag{10_2}$$

$$|\varphi_i(x_1, \dots, x_n)| \le \psi_i(|x_1|, \dots, |x_n|) + r_i\Big(\sum_{j=1}^n |x_j|\Big) \quad \text{for all } x = (x_i)_{i=1}^n \in C_n(\langle a, b\rangle) \quad (i = 1, \dots, n). \tag{11}$$

hold, where $G = (g_i)_{i=1}^n$, $H = (h_{ij})_{i,j=1}^n$ *and* $\Psi = (\psi_i)_{i=1}^n$ *satisfy the condition (7), the functions* $\omega_i : \langle a, b\rangle \times \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 1, 2, \dots, n)$ *are measurable with regard to the first and nondecreasing to the second argument,* $r_i : \mathbb{R}_+ \to \mathbb{R}_+$ *are nondecreasing and*

$$\lim_{\varrho\to+\infty} \frac{1}{\varrho}\int_a^b \omega_i(t,\varrho)dt = 0 = \lim_{\varrho\to+\infty} \frac{1}{\varrho} r_i(\varrho) \quad (i = 1, \dots, n) \tag{12}$$

then the problem (1), (2) has at least one solution.

For the proof of the Theorem 1 we need two following assertions and the first is similar to lemma 4.1 from [4] about differential inequality with boundary conditions of Cauchy type.

Lemma 1. *Let* $g_i^*(t, y_1, \dots, y_n) \in K(\langle a, b\rangle)$, $g_i^*(t, y_1, \dots, y_n) sign(t - t_i)$ *be nondecreasing to arguments* $y_1, y_2, \dots, y_{i-1}, y_{i+1}, \dots, y_n$ *and each solution of the problem*

$$y_i' = g_i^*(t, y_1, \dots, y_n) \qquad (i = 1, \dots, n) \tag{13}$$

$$y_i(t_i) = c_i \qquad (i = 1, \dots, n) \tag{14}$$

where $t_i \in \langle a, b\rangle$, $c_i \in R \quad (i = 1, \dots, n)$ *can be extended in the whole segment* $\langle a, b\rangle$. *Then for each solution* $(x_i(t))_{i=1}^n \in AC_n(\langle a, b\rangle)$ *of the problem (15), (16)*

$$\begin{aligned} x_i'(t) sign(t - t_i) &\le g_i^*(t, x_1, \dots, x_n) \times \\ &(\ge) \\ &\times sign(t - t_i) \text{ for } a \le t \le b \quad (i = 1, \dots, n) \end{aligned} \tag{15}$$

$$\begin{aligned} x_i(t_i) &\le c_i \quad (i = 1, \dots, n) \\ &(\ge) \end{aligned} \tag{16}$$

there exists a solution $(y_i)_{i=1}^n$ *defined in the segment* $\langle a, b\rangle$ *of the problem (13), (14) such that*

$$\begin{aligned} x_i(t) &\le y_i(t) \text{ for } a \le t \le b \quad (i = 1, \dots, n) \\ &(\ge) \end{aligned} \tag{17}$$

Lemma 2. *Let the condition (7) be satisfied. Then there exists a constant* $\varrho > 0$ *such that the estimate*

$$\|x\|_{C_n(\langle a,b\rangle)} \le \varrho\Big[r_0 + \int_a^b \omega_0(t)dt\Big] \tag{18}$$

holds for each constant $r_0 \geq 0$, $\omega_0 \in L(\langle a,b\rangle, \mathbb{R}_+)$ *and for each solution* $x \in AC_n(\langle a,b\rangle)$ *of the differential inequalities*

$$[x_i'(t) - g_i(t)x_i(t)] \text{ sign } x_i(t) \leq \leq \sum_{j=1}^{n} h_{ij}(t)|x_j(t)| + \omega_0(t) \text{ if } a_i \leq t \leq b \quad (i=1,\dots,n) \tag{19_1}$$

$$[x_i'(t) - g_i(t)x_i(t)] \text{ sign } x_i(t) \geq \geq -\sum_{j=1}^{n} h_{ij}(t)|x_j(t)| - \omega_0(t) \text{ if } a \leq t \leq b_i \quad (i=1,\dots,n) \tag{19_2}$$

with boundary conditions

$$\min\{|x_i(t)| : a_i \leq t \leq b_i\} \leq \leq \psi_i(|x_1(t)|,\dots,|x_n(t)|) + r_0 \quad (i=1,\dots,n) \tag{20}$$

Proof. By contradiction let $r_k \in \mathbb{R}_+$, $\omega_k \in L(\langle a,b\rangle, \mathbb{R}_+)$ and $x_k = (x_{ik})_{i=1}^n \in AC_n(\langle a,b\rangle)$ exist for any natural k, such that

$$\|x_k\|_{C_n(\langle a,b\rangle)} \geq k\Big[r_k + \int_a^b \omega_k(t)dt\Big]$$

$$[x_{ik}'(t) - g_i(t)x_{ik}(t)] \text{ sign } x_{ik}(t) \leq \sum_{j=1}^{n} h_{ij}(t)|x_{jk}(t)| + \omega_k(t) \quad \text{if } a_i \leq t \leq b \quad (i=1,\dots,n) \tag{21_1}$$

$$[x_{ik}'(t) - g_i(t)x_{ik}(t)] \text{ sign } x_{ik}(t) \geq -\sum_{j=1}^{n} h_{ij}(t)|x_{jk}(t)| - \omega_k(t) \quad \text{if } a \leq t \leq b_i \quad (i=1,\dots,n) \tag{21_2}$$

and

$$\min\{|x_{ik}(t)| : a_i \leq t \leq b_i\} \leq \psi_i(|x_{1k}|,\dots,|x_{nk}|) + r_k \quad (i=1,\dots,n) \tag{22}$$

We denote

$$\tilde{x}_{ik}(t) = \frac{x_{ik}(t)}{\|x_k\|_{C_n(\langle a,b\rangle)}} \quad (i=1,\dots,n), \qquad \tilde{\omega}_k(t) = \frac{\omega_k(t)}{k[r_k + \int_a^b \omega_k(t)dt]}, \tag{1}$$

we get

$$\|\tilde{x}_k\|_{C_n(\langle a,b\rangle)} = 1 \text{ and } \|\tilde{w}_k\|_{L(\langle a,b\rangle)} \le \frac{1}{k} \tag{23}$$

On the other hand according to (21), (22)

$$\begin{gathered}[\tilde{x}'_{ik}(t) - g_i(t)\tilde{x}_{ik}(t)] \text{ sign } \tilde{x}_{ik}(t) \le \\ \le \sum_{j=1}^{n} h_{ij}(t)|\tilde{x}_{jk}(t)| + \tilde{\omega}_k(t) \text{ if } a_i \le t \le b \quad (i = 1, \dots n)\end{gathered} \tag{24_1}$$

$$\begin{gathered}[\tilde{x}'_{ik}(t) - g_i(t)\tilde{x}_{ik}(t)] \text{ sign } \tilde{x}_{ik}(t) \ge \\ \ge -\sum_{j=1}^{n} h_{ij}(t)|\tilde{x}_{jk}(t)| - \tilde{\omega}_k(t) \text{ if } a \le t \le b_i \quad (i = 1, \dots, n)\end{gathered} \tag{24_2}$$

and

$$\begin{gathered}\min\{|\tilde{x}_{ik}(t)| : a_i \le t \le b_i\} \le \\ \le \psi_i(|\tilde{x}_{1k}|, \dots, |\tilde{x}_{nk}|) + \frac{1}{k} \quad (i = 1, \dots, n)\end{gathered} \tag{25}$$

Now for any $i \in \{1, \dots, n\}$ and a natural k we choose a point $t_{ik} \in \langle a_i, b_i\rangle$ such that

$$|\tilde{x}_{ik}(t_{ik})| = \min\{|\tilde{x}_{ik}(t)| : a_i \le t \le b_i\} \tag{26}$$

then from (24), (25) and (26), we have

$$\begin{gathered}[\tilde{x}'_{ik}(t) - g_i(t)\tilde{x}_{ik}(t)] \text{ sign } [(t - t_{ik})\tilde{x}_{ik}(t)] \le \\ \le \sum_{j=1}^{n} h_{ij}(t)|\tilde{x}_{jk}(t)| + \tilde{\omega}_k(t) \\ \text{for } a \le t \le b \quad (i = 1, \dots, n)\end{gathered} \tag{27}$$

and

$$|\tilde{x}_{ik}(t_{ik})| \le \psi_i(|\tilde{x}_{1k}|, \dots, |\tilde{x}_{nk}|) + \frac{1}{k} \quad (i = 1, \dots, n) \tag{28}$$

Let $(y_{ik})_{i=1}^{n}$ be the solution of the Cauchy-Nicoletti problem

$$\begin{gathered}y'_{ik}(t) = g_i(t)y_{ik}(t) + \Big[\sum_{j=1}^{n} h_{ij}(t)|\tilde{x}_{ik}(t)| + \\ +\tilde{\omega}_k(t)\Big] \text{ sign } (t - t_{ik}) \quad (i = 1, \dots, n)\end{gathered} \tag{29}$$

$$y_{ik}(t_{ik}) = |\tilde{x}_{ik}(t_{ik})| \quad (i = 1, \dots, n) \tag{30}$$

then according to Lemma 1 and to the condition (27)

$$|\tilde{x}_{ik}(t)| \le y_{ik}(t) \text{ for } a \le t \le b \quad (i = 1, \dots, n) \tag{31}$$

Formulae (29), (30) and (31) yield

$$\begin{gathered} y_{ik}(t) \le \exp\Big(\int\limits_{t_{ik}}^{t} g_i(s)\, ds\Big)|\tilde{x}_{ik}(t_{ik})| + \\ +\Big|\int\limits_{t_{ik}}^{t} e^{\int_\tau^t g_i(s) ds}\Big[\sum_{j=1}^{n} h_{ij}(\tau)|\tilde{x}_{ik}(\tau)| + \tilde{\omega}_k(\tau)\Big]d\tau\Big| \end{gathered} \tag{32}$$

and

$$\begin{gathered} y_{ik}(t) \le \exp\Big(\int\limits_{t_{ik}}^{t} g_i(s) ds\Big) y_{ik}(t_{ik}) + \\ +\Big|\int\limits_{t_{ik}}^{t} e^{\int\limits_\tau^t g_i(s)\, ds}\Big[\sum_{j=1}^{n} h_{ij}(\tau) y_{ik}(\tau) + \tilde{\omega}_k(\tau)\Big]d\tau\Big| \end{gathered} \tag{33}$$

According to (23), (29) and (32), we obtain

$$|y_{ik}(t)| \le r \text{ for } a \le t \le b, \quad (i = 1, \dots, n) \quad (k = 1, 2, \dots) \tag{34}$$

and

$$|y'_{ik}(t)| \le \tilde{g}(t) + \tilde{\omega}_k(t) \text{ for } a \le t \le b \quad (i = 1, \dots, n), \quad (k = 1, \dots, n) \tag{35}$$

where

$$r = \Big(2 + \sum_{i,j=1}^{n} \int\limits_a^b h_{ij}(t) dt\Big) \max\Big\{ \exp \int\limits_a^b |g_i(t)| dt : i = 1, \dots, n\Big\}$$

and

$$\tilde{g}(t) = \sum_{i,j=1}^{n} h_{ij}(t) + r \max\big\{|g_i(t)| : i = 1, \dots, n\big\}$$

Formulae (23), (28), (30) and (31) imply, that

$$\|(y_{ik})_{i=1}^n\|_{C_n(\langle a,b\rangle)} \ge 1, \quad (k = 1, 2, \dots) \tag{36}$$

$$|y_{ik}(t_{ik})| \le \psi_i(y_{ik}, \dots, y_{nk}) + \frac{1}{k} \quad (i = 1, \dots, n), \quad (k = 1, 2, \dots) \tag{37}$$

From (34) and (35), it follows that the sequences $\{y_{ik}\}_{k=1}^{\infty}$ $(i = 1, \dots, n)$ are uniformly bounded and uniformly continuous. According to the Lemma of Arzela-Ascoli, we can suppose without the loss of generality that these sequences uniformly converge. The sequences of points $\{t_{ik}\}_{k=1}^{\infty}$ $(i = 1, \dots, n)$ can be taken convergent as well. Denoting

$$\lim_{k \to +\infty} t_{ik} = t_{i0} \quad (i = 1, \dots, n) \tag{38}$$

and

$$\lim_{k \to +\infty} y_{ik}(t) = y_{i0}(t) \quad (i = 1, \dots, n)$$

Clearly

$$t_{i0} \in \langle a_i, b_i \rangle \quad (i = 1, \dots, n) \tag{39}$$

Passing to the limit in the inequalities (33) and (37), using (23) we obtain

$$\begin{aligned} y_{i0}(t) \le \exp\Big(\int_{t_{i0}}^{t} g_i(s)ds\Big) y_{i0}(t_{i0}) + \\ +\Big|\int_{t_{i0}}^{t} e^{\int_{\tau}^{t} g_i(s)ds} \Big[\sum_{j=1}^{n} h_{ij}(\tau) y_{i0}(\tau)\Big] d\tau\Big| \quad (i = 1, \dots, n) \end{aligned} \tag{40}$$

$$y_{i0}(t_{i0}) \le \psi_i(y_{10}, \dots, y_{n0}) \quad (i = 1, \dots, n) \tag{41}$$

Let us introduce the functions

$$\begin{aligned} y_i(t) = \exp\Big(\int_{t_{i0}}^{t} g_i(s)ds\Big) y_{i0}(t_{i0}) + \\ +\Big|\int_{t_{i0}}^{t} e^{\int_{\tau}^{t} g_i(s)ds} \Big[\sum_{j=1}^{n} h_{ij}(\tau) y_{i0}(\tau)\Big] d\tau\Big| \quad (i = 1, \dots, n), \end{aligned}$$

then

$$y_i(t_{i0}) = y_{i0}(t_{i0}) \quad (i = 1, \dots, n) \tag{42}$$

and

$$\begin{aligned} y_i'(t) = g_i(t) y_i(t) + \Big[\sum_{j=1}^{n} h_{ij}(t) y_{i0}(t)\Big] \times \\ \times \text{ sign } (t - t_{i0}) \quad (i = 1, \dots, n) \end{aligned} \tag{43}$$

From (39) - (43) it follows that $(y_i(t))_{i=1}^n$ is a solution of the problem (8), (9). Therefore according to the condition (7)

$$y_i(t) \equiv 0 \qquad (i = 1, \dots, n)$$

On the other hand, (36) and (40) imply

$$\|(y_i(t)_{i=1}^n\|_{C_n(\langle a,b\rangle)} \geq 1,$$

which is a contradiction and the lemma is proved. □

Proof of Theorem 1. Let ϱ be a constant from Lemma 1. Firstly, we want to show that there exists a constant $\varrho_0 > 0$ such that

$$\varrho\Big[r(2\varrho_0) + \int_a^b \omega(t, 2\varrho_0)dt\Big] \leq \varrho_0 \tag{44}$$

where for any $\eta \in (0, +\infty)$

$$r(\eta) = \max\{r_i(\eta) : i = 1, \dots, n\}$$

$$\omega(t, \eta) = \max\{\omega_i(t, \eta) : i = 1, \dots, n\}$$

Suppose (44) is not valid, then for any $\eta \in (0, +\infty)$

$$\eta \leq \varrho\Big[r(2\eta) + \int_a^b \omega(t, 2\eta)dt\Big]$$

On the other hand, (12) implies that for any $k \geq \varrho$ there exists a constant $\eta_0 > k$ such that

$$\varrho\Big[r(2\eta) + \int_a^b \omega(t, 2\eta)dt\Big] < \frac{\varrho}{k}.\eta < \eta$$

for all $\eta \geq \eta_0$, which is a contradiction and (44) is valid.
Now we put

$$\chi(s) = \begin{cases} 1 & \text{if} \quad |s| \leq \varrho_0 \\ 2 - \dfrac{|s|}{\varrho_0} & \text{if} \quad \varrho_0 < |s| < 2\varrho_0 \\ 0 & \text{if} \quad |s| \geq 2\varrho_0 \end{cases}$$

$$\begin{gathered} \tilde{f}_i(t, x_1, \dots, x_n) = \chi(\|x\|)[f(t, x_1, \dots, x_n) - \\ -g_i(t)x_i] \text{ for } x = (x_i)_{i=1}^n \in \mathbb{R}^n, \quad (i = 1, \dots, n) \end{gathered} \tag{45}$$

$$\begin{aligned}\tilde{\varphi}_i(x_1,\dots,x_n) = \chi(\|x\|_{C_n(\langle a,b\rangle)})\varphi_i(x_1,\dots,x_n)\\ \text{for } x=(x_i)_{i=1}^n \in C_n(\langle a,b\rangle), \quad (i=1,\dots,n)\end{aligned} \tag{46}$$

We consider the problem

$$y_i' = g_i(t)y_i + \tilde{f}_i(t,y_1,\dots,y_n), \quad (i=1,\dots,n) \tag{47}$$

$$\Phi_{0i}(y_i) = \tilde{\varphi}_i(y_1,\dots,y_n) \quad (i=1,\dots,n) \tag{48}$$

From (45) and (46), it follows immediately that $\tilde{f}_i : \langle a,b\rangle \times \mathbb{R}^n \to R \quad (i = 1,\dots,n)$ satisfy the local Carathéodory conditions, $\tilde{\varphi}_i : C_n(\langle a,b\rangle) \to R \quad (i = 1,\dots,n)$ are continuous functionals,

$$\begin{aligned}f_{0i}(t) = \mathrm{suf}\{|\tilde{f}_i(t,x_1,\dots,x_n)| : (x_i)_{i=1}^n \in \mathbb{R}^n\} \in \\ \in L(\langle a,b\rangle) \quad (i=1,\dots,n)\end{aligned} \tag{49}$$

and

$$r_i = \sup\{|\tilde{\varphi}_i(x_1,\dots,x_n)| : x \in C_n(\langle a,b\rangle)\} < +\infty \quad (i=1,\dots,n) \tag{50}$$

We want to show that the homogeneous problem

$$y_i' = g_i(t)y_i \qquad (i=1,\dots,n) \tag{47_0}$$

$$\Phi_{0i}(y_i) = 0 \qquad (i=1,\dots,n) \tag{48_0}$$

has only trivial solution. Let $\tilde{y} = (\tilde{y}_i)_{i=1}^n$ be an arbitrary solution of this problem. Then $\tilde{y}_i(t) = c_i \exp\Big(\int_a^t g_i(\tau)d\tau\Big)$, where $c_i = const \quad (i=1,\dots,n)$. According to (48_0)

$$c_i.\Phi_{0i}\Big(\exp\int_a^t g_i(\tau)d\tau\Big) = 0 \quad (i=1,\dots,n)$$

However, if $\Phi_{0i} \quad (i=1,\dots,n)$ are nondecreasing functionals and $\Phi_{0i}(1) = 1 \quad (i=1,\dots,n)$, we have

$$\Phi_{0i}\Big(\exp\int_a^t g_i(\tau)d\tau\Big) \geq \exp\Big(-\int_a^b |g_i(t)|dt\Big)\Phi_{0i}(1) > 0 \quad (i=1,\dots,n)$$

Consequently

$$\tilde{y}_i(t) \equiv 0 \quad (i=1,\dots,n)$$

Using Lemma 2.1 from [3], we obtain that the conditions (49) and (50) and the unicity of trivial solution of the problem (47_0), (48_0) guarantee the existence of solutions of the problem (47), (48). Let $(y_i(t))_{i=1}^n$ be the solution of the problem (47), (48), then

$$[y_i'(t) - g_i(t)y_i(t)] \text{ sign } y_i(t) = \tilde{f}_i(t, y_1(t), \dots, y_n(t)) \text{ sign } y_i(t)$$
$$= \chi\Big(\sum_{i=1}^n |y_i(t)|\Big)[f_i(t, y_1(t), \dots, y_n(t)) - g_i(t)y_i(t)] \text{ sign } y_i(t)$$
$$\text{for } a \le t \le b, \quad (i = 1, \dots, n)$$

and

$$\min\{|y_i(t)| : a_i \le t \le b_i\} \le |\Phi_{0i}(y_i(t))| = |\tilde{\varphi}_i(y_1, \dots, y_n)|$$
$$= \chi(\|y\|_{C_n(\langle a,b\rangle)})|\varphi_i(y_1, \dots, y_n)| \qquad (i = 1, \dots, n)$$

From here taking into consideration ($10_{1,2}$) and (11), we obtain inequalities ($19_{1,2}$) and (20), where

$$\omega_0(t) = \chi\Big(\sum_{i=1}^n |y_i(t)|\Big)\omega\Big(t, \sum_{i=1}^n |y_i(t)|\Big) \le \omega(t, 2\varrho_0)$$

and

$$r_0 = \chi(\|y\|_{C_n(\langle a,b\rangle)})r\Big(\sum_{i=1}^n |y_i(t)|\Big) \le r(2\varrho_0)$$

Therefore by Lemma 2 and the inequality (44) we get

$$\|y\|_{C_n(\langle a,b\rangle)} \le \varrho[r(2\varrho_0) + \int_a^b \omega(t, 2\varrho_0)dt] \le \varrho_0$$

Consequently $\chi(\sum_{i=1}^n |y_i(t)|) = 1$ when $a \le t \le b$ and

$$\chi(\|y\|_{C_n(\langle a,b\rangle)}) = 1$$

Putting these equalities into (45) - (48), we obtain that $(y_i)_{i=1}^n$ is a solution of the problem (1), (2). The Theorem 1 is proved. □

Theorem 2. *Let the inequalities*

$$\{[f_i(t, x_{11}, \dots, x_{1n}) - f_i(t, x_{21}, \dots, x_{2n})] -$$
$$-g_i(t)[x_{1i} - x_{2i}]\} \textit{ sign } [x_{1i} - x_{2i}] \le \sum_{j=1}^n h_{ij}(t)|x_{1j} - x_{2j}| \qquad (51_1)$$

if $a_i \le t \le b$, $x_1 = (x_{1i})_{i=1}^n$, $x_2 = (x_{2i})_{i=1}^n \in \mathbb{R}^n \quad (i = 1, \dots, n)$,

$$\{[f_i(t, x_{11}, \dots, x_{1n}) - f_i(t, x_{21}, \dots, x_{2n})] - \\ -g_i(t)[x_{1i} - x_{2i}]\} \text{ sign } [x_{1i} - x_{2i}] \ge -\sum_{j=1}^{n} h_{ij}(t)|x_{1j} - x_{2j}| \tag{51_2}$$

if $a \le t \le b_i$, $x_1 = (x_{1i})_{i=1}^n$, $x_2 = (x_{2i})_{i=1}^n \in \mathbb{R}^n \quad (i = 1, \dots, n)$

$$|\varphi_i(x_{11}, \dots, x_{1n}) - \varphi_i(x_{21}, \dots, x_{2n})| \le \psi_i(|x_{11} - x_{12}|, \dots, |x_{1n} - x_{2n}|) \\ \text{for all } x_1 = (x_{1i})_{i=1}^n, x_2 = (x_{2i})_{i=1}^n \in C_n(\langle a, b\rangle) \quad (i = 1, \dots, n) \tag{52}$$

hold, where $G = (g_i)_{i=1}^n$, $H = (h_{ij})_{i,j=1}^n$ *and* $\Psi = (\psi_i)_{i=1}^n$ *satisfy the condition (7). Then the problem (1), (2) has unique solution.*

Proof. From ($51_{1,2}$) and (52) the conditions ($10_{1,2}$) and (11) follow, where $\omega_i(t, \varrho) = |f_i(t, 0, \dots, 0)|$ and $r_i(\varrho) = |\varphi_i(0, \dots, 0)| \quad (i = 1, \dots, n)$. Therefore, by Theorem 1 the problem (1), (2) has a solution. We shall prove its uniqueness.

Let $(x_{1i})_{i=1}^n$ and $(x_{2i})_{i=1}^n$ be arbitrary solutions of the problem (1), (2). Let us put

$$y_i(t) = x_{1i}(t) - x_{2i}(t) \quad (i = 1, \dots, n)$$

The assumptions ($51_{1,2}$) guarantee that vector function $(y_i)_{i=1}^n$ is a solution of the system of the differential inequalities (8) satisfying the conditions

$$|\Phi_{0i}(y_i)| \le \psi_i(|y_1|, \dots, |y_n|) \quad (i = 1, \dots, n)$$

However

$$|\Phi_{0i}(y_i)| \ge \Phi_{0i}(1) \min\{|y_i(t)| : a_i \le t \le b_i\} = \\ = \min\{|y_i(t)| : a_i \le t \le b_i\} \quad (i = 1, \dots, n)$$

Consequently the inequalities (9) are satisfied and according to the condition (7) the equalities

$$y_i(t) \equiv 0 \quad (i = 1, \dots, n)$$

hold, i.e $(x_{1i})_{i=1}^n = (x_{2i})_{i=1}^n$. Theorem is proved. □

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