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A GENERALIZATION OF PROJECTIVITY IN CATEGORIES

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Abstract

In this paper, the concept of projectivity is generalized by the notion of projective structures. The bases and subbases of a projective structure with their existences are discussed. The regular and finest form of projective structures and their characteristic properties are also studied.

1. INTRODUCTION

In this note, we introduce the projective structure in categories. The idea of a projective structure is actually a generelization of idea of projectivity and enough projectives in categories. For developing the structure, the first two sections deal with classes of M-projectives and P-morphisms and their properties. In Section 4, we provide the concept of projective structure and observe that a projective structure in a category exists if and only if it exists in its representative subcategory. In Section 5, the concept of bases and subbases and the existence of bases and subbases has been attempted. Finally, we define and characterize a regular projective structure and show how we can obtain such a structure in the finest form. For basic knowledge of category and functor theory, we can refer to [1], [3] and [4].

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2. CLASS OF M-PROJECTIVES

Definition 2.1 An object P in the category $\mathcal{C}$ will be called *projective with respect to the morphism* $\alpha : A \longrightarrow B$ if the morphism

$$Hom(P, \alpha) : Hom(P, A) \to Hom(P, B)$$

is an epimorphism in the category Ens of all sets. Therefore, for any morphism $\beta : P \to B$, there exists a morphism $\eta : P \to A$ such that

$$P \xrightarrow{\eta} A \xrightarrow{\alpha} B = P \xrightarrow{\beta} B.$$

Definition 2.2 Let M be a class of morphisms of the category $\mathcal{C}$. The class of all objects in $\mathcal{C}$ which are projective with respect to all morphisms in M, is called the *class of M-projectives in $\mathcal{C}$* and is denoted by $\pi(M)$.

Proposition 2.3 *Every retract of an M-projective object is also M-projective.*

Proof Suppose $\alpha : A \to B$ is a retraction in $\mathcal{C}$ and $A \in \pi(M)$. By the definition of retractions, there exists a morphism $\alpha' : B \to A$ such that $\alpha\alpha' = I_B$. Let $f : X \to Y$ be any morphism in M. Since $A \in \pi(M)$, we have $Hom(A, f) : Hom(A, X) \to Hom(A, Y)$ is an epimorphism in Ens. We now consider the morphism $Hom(B, f) : Hom(B, X) \to Hom(B, Y)$. For any morphism $\beta \in Hom(B, Y)$, we have $\beta\alpha \in Hom(A, Y)$. By the projectivity of A, there exists a morphism $\gamma \in Hom(A, X)$ such that $Hom(A, f)(\gamma) = \beta\alpha$, i.e., $f\gamma = \beta\alpha$. If we consider the morphism $\eta = \gamma\alpha' \in Hom(B, X)$, then $Hom(B, f)(\eta) = f\eta = f(\gamma\alpha') = (f\gamma)\alpha' = (\beta\alpha)\alpha' = \beta(\alpha\alpha') = \beta I_B = \beta$. This shows that $Hom(B, f)$ is an epimorphism. Therefore, B is projective with respect to an arbitrary $f \in M$, proving that B is M-projective.

Proposition 2.4 *Let $\mathcal{C}$ be a category with zero object. An object $A = \sum A_i$ is M-projective if and only if each A_i is M-projective.*

Proof Suppose that A is M-projective. Let $f : X \to Y$ be in M and $\alpha \in Hom(A_i, Y)$ for some $i \in I$, be arbitrary morphisms. Since the category $\mathcal{C}$ has a zero object, every injection $u_i : A_i \to A$ is a coretraction. Thus, there exists a morphism $\eta_i : A \to A_i$ such that $\eta_i u_i = I_{A_i}$. By the M-projectivity of A, the morphism $Hom(A, f) : Hom(A, X) \to Hom(A, Y)$ is an epimorphism in Ens. Therefore, for $\alpha\eta_i \in Hom(A, Y)$, there exists a $\gamma \in Hom(A, X)$ such that $Hom(A, f)(\gamma) = \alpha\eta_i$, i.e, $f\gamma = \alpha\eta_i$. If we take $\gamma_i = \gamma u_i \in Hom(A_i, X)$, then

$$Hom(A_i, f)(\gamma) = f\gamma_i = f(\gamma u_i) = (f\gamma)u_i = (\alpha\eta_i)u_i = \alpha(\eta_i u_i) = \alpha I_{Ai} = \alpha.$$

Since $\alpha \in Hom(A_i, Y)$ is arbitrary, we can see that $Hom(A_i, f)$ is an epimorphism. It follows that A_i is M-projective for all $i \in I$.

Conversely, suppose that each A_i is M-projective. Let $f : X \to Y$ in M and $\alpha \in Hom(A, Y)$ be an arbitrary morphism. By the M-projectivity of A_i, there exists a morphism $\eta_i : A_i \to X$ for all $i \in I$ such that

$$Hom(A_i, f)(\eta_i) = f\eta_i = \alpha u_i$$

Furthermore, by the definition of co-products, there exists a morphism $\eta : A \to X$ such that $\eta u_i = \eta_i$ for all $i \in I$. Therefore $(f\eta)u_i = \alpha u_i$ for all $i \in I$. Again, by the definition of coproducts, we can get $f\eta = \alpha$. Therefore $Hom(A, f)$ is an epimorphism and hence, A is M-projective, proving our proposition. □

Proposition 2.5 *Let $T : \mathcal{C} \to \mathcal{D}$ be a covariant functor which admits a left adjoint $S : \mathcal{D} \to \mathcal{C}$. If P is M-projective in $\mathcal{D}$, then $S(P)$ is $T^{-1}(M)$-projective in $\mathcal{C}$.*

Proof Suppose that $P \in \mathcal{D}$ is M-projective. Let $f : X \to Y$ in $T^{-1}(M)$ and $\alpha \in Hom(S(P), Y)$ be arbitrary morphisms. Since S and T are adjoint functors, there exists a natural adjunction $\theta : S \to T$. This induces an isomorphism $\theta : Hom(S(P), Y) \to Hom(P, T(Y))$ for each P in $\mathcal{D}$ and Y in $\mathcal{C}$. Thus we get $T(f) : T(X) \to T(Y)$ in M and $\beta = \theta(\alpha) \in Hom(P, T(Y))$. By the M-projectivity of P, the morphism $Hom(P, T(f)) : Hom(P, T(X)) \to Hom(P, T(Y)$ is an epimorphism in Ens. Therefore, there exists γ in $Hom(P, T(X))$ such that $Hom(P, T(f))(\gamma) = \beta$, i.e., $T(f)\gamma = \beta$. By the definition of natural adjunctions, there exists a morphism δ morphisms $Hom(S(P), X)$ such that $\theta(\delta) = \gamma$ and the following composite morphisms

$$Hom(S(P), X) \overset{Hom(S(P),f)}{\longrightarrow} Hom(S(P), Y) \overset{\theta}{\longrightarrow} Hom(P, T(Y)$$

and

$$Hom(S(P), X) \overset{\theta}{\longrightarrow} Hom(P, T(X)) \overset{Hom(P,T(F))}{\longrightarrow} Hom(P, T(Y))$$

are equal. For any δ in $Hom(S(P), X)$, we get $\theta(f\delta) = \theta(\alpha)$. It follows that $f\delta = \alpha$, i.e, $Hom(S(P), f)(\delta) = \alpha$. But $f \in T^{-1}$ (M) and $\alpha \in Hom(S(P), Y)$ are arbitrary, we can conclude that $S(P)$ is T^{-1} (M)-projective. □

We now can prove the following results easily.

Proposition 2.6 *If M_1 and M_2 are two classes of morphisms in the category $\mathcal{C}$ such that $M_1 \subseteq M_2$, then $\pi(M_1) \supseteq \pi(M_2)$.*

Proposition 2.7 *If $\{M_i\}_{i\in I}$ is a family of classes of morphisms in the category $\mathcal{C}$, then the following statements hold:*

(*i*) $\pi(\cup_{i\in I} M_i) = \cap_{i\in I} \pi(M_i)$

(*ii*) $\pi(\cap_{i\in I} M_i) \supseteq \cup_{i\in I} \pi(M_i)$.

3. Class of P-Morphisms

Definition 3.1 Let P be a class of objects of the category $\mathcal{C}$. The class of all morphisms of $\mathcal{C}$ with respect to which, every object in class P is projective, is called the *class of P-morphisms* and it is denoted by $\mu(P)$.

The following propositions are straightforwards.

Proposition 3.2 *Every retraction in $\mathcal{C}$, lies in $\mu(P)$ for any class of objects P.*

Proposition 3.3 *Any morphism in $\mu(P)$ with range in P is a retraction.*

Proposition 3.4 *The class of P-morphisms, $\mu(P)$ is closed with respect to composition.*

A partial converse of the proposition 3.4 in given as follows:

Proposition 3.5 *If $\beta\alpha \in \mu(P)$, then $\beta \in \mu(P)$. Further, if β is a monomorphism, then also $\alpha \in \mu(P)$.*

Proof Suppose that $\alpha : A \rightarrow B, \beta : B \rightarrow C$ are morphisms in $\mathcal{C}$ such that $\beta\alpha \in \mu(P)$. Let $X \in$ P and $\gamma \in Hom(X, C)$ be arbitrary. By the projectivity of X, we get $Hom(X, \beta\alpha) : Hom(X, A) \rightarrow Hom(X, C)$ is an epimorphism in the category Ens. This implies that there exists an $\eta \in Hom(X, A)$ such that $Hom(X, \beta\alpha)(\eta) = \gamma$, i.e., $(\beta\alpha)\eta = \gamma$. If we take $\xi = \alpha\eta \in Hom(X, B)$, then $\beta\xi = \beta(\alpha\eta) = (\beta\alpha)\eta = \gamma$, i.e., $Hom(X, \beta)(\xi) = \gamma$. Thus, X is projective with respect to β. Since $X \in P$ is arbitrary, we have $\beta \in \mu(P)$. We now assume that β is a monomorphism. Let X in P and δ in $Hom(X, B)$ be arbitrary. Since X is projective with respect to $\beta\alpha$, we get that

$$Hom(X, \beta\alpha) : Hom(X, A) \rightarrow Hom(X, C)$$

is an epimorphism in Ens. Hence, there exists a θ in $Hom(X, A)$ such that $Hom(X, \beta\alpha)(\theta) = \beta\delta$, i.e., $(\beta\alpha)\theta = \beta(\alpha\theta) = \beta\delta$. Since β is a monomorphism, it implies that $\alpha\theta = \delta$. This shows that X is projective with respect to α. Furthermore, since X in P is arbitrary, we can conclude that α in $\mu(P)$. □

We now can easily check the following propositions.

Proposition 3.6 *If P_1 and P_2 are two classes of objects in $\mathcal{C}$ such that $P_1 \supseteq P_2$, then $\mu(P_1) \subseteq \mu(P_2$.*

Proposition 3.7 *If $\{P_i\}_{i \in I}$ be a family of classes of objects in $\mathcal{C}$, then the following statements hold.*

(*i*) $\mu(\cup_{i \in I} P_i) = \cap_{i \in I} \mu(P_i)$;

(*ii*) $\mu(\cap_{i \in I} P_i) \supseteq U_{i \in I} \mu(P_i)$.

Remark 3.1 For any class of objects P in $\mathcal{C}$, there exists the largest class of objects in $\mathcal{C}$, $P' = \pi(\mu(P))$ such that $\mu(P) = \mu(P')$.

Remark 3.2 For any class of morphisms M in $\mathcal{C}$, there exists the largest class of morphisms $M' = \eta(\pi(M))$ in $\mathcal{C}$ such that $\pi(M) = \pi(M')$.

4. Projective Structure

Definition 4.1 A pair (M, P), where M is a class of morphisms and P is a class of objects in $\mathcal{C}$, is called a *projective structure* of $\mathcal{C}$ if the following conditions hold:

$PS_1 : \pi(M) = P$;

$PS_2 : \mu(P) = M$;

PS_3 : For any object $A \in \mathcal{C}$, there exists a morphism; $f : X \to A$ in M, where $X \in P$.

We denote a projective object by $\mathcal{P}(M, P)$ or simply by $(M, \mathcal{P})$. The classes M and P will be called *the class of proper morphisms* and *the class of projectives,* respectively.

Examples

(4.1) For any category $\mathcal{C}$, if we take M to be the class of all retractions in $\mathcal{C}$ and P, the class of all objects of $\mathcal{C}$, then (M, P) is a projective structure of $\mathcal{C}$.

(4.2) For any category $\mathcal{C}$ with zero object, if we take M to be the class of all morphisms in $\mathcal{C}$ and P consists only the zero object of $\mathcal{C}$, then (M, P) is a projective structure of $\mathcal{C}$.

(4.3) Consider the category Mod-R of right R-modules. If we take M to be the class of all epimorphisms in Mod-R and P, the class of all projective R-modules, then (M, P) is a projective structure of Mod-R.

(4.4) Consider the category Ab of abelian groups. If we take M to be the class of all epimorphisms in Ab and P, the class of all free abelian groups, then (M, P) is a projective structure of Ab.

Remark 4.1 The projective structure given in Example 3 and 4 are called *exact projective structures.*

Theorem 4.2 *If (M, P) is a projective structure of $\mathcal{C}$, then an object B is in P if and only if every morphism in M with range B is a retraction.*

The proof of this Theorem is followed from Proposition 2.3, PS_3 and Proposition 3.3. □

Theorem 4.3 *Let $\mathcal{C}'$ be a full representative subcategory of $\mathcal{C}$. If $\mathcal{P}'(M', P')$ is a projective structure of $\mathcal{C}'$, then there exists a projective structure of $\mathcal{C}$ in which every projective is isomorphic to some projective in (M', P').*

Proof Let $\pi(M') = P$ be the class of all objects in $\mathcal{C}$ which are M'-projectives and $\mu(P) = M$, the class of all morphisms in $\mathcal{C}$ such that every object in P is M-projective. We now show that (M, P) is the required projective structure of $\mathcal{C}$. Since $M' \subseteq M$, by Proposition 2.6 we get $P = \pi(M') \supseteq \pi(M)$.

Let $X \in P$ be arbitrary. Since $\mu(P) = M$, it implies that X is M-projective, i.e., $X \in \pi(M)$. This gives $P \subseteq \pi(M)$. Hence $P = \pi(M)$, this leads the Axiom PS_1. The Axiom PS_2, is clear from the above construction.

For the Axiom PS_3, we take any object $A \in \mathcal{C}$. Since $\mathcal{C}'$ is a representative subcategory of $\mathcal{C}$, there exists an object $A' \in \mathcal{C}$ and an isomorphism $\theta : A' \cong A$. Again, since $\mathcal{P}'(M', P')$ is a projective structure of $\mathcal{C}'$, there exists an object $X' \in P' \subseteq P$ and a morphism $f' : X' \longrightarrow A'$ which lies in $M' \subseteq M$. Note that if an object is projective with respect to f', then it is also projective with respect to $f = \theta f'$. Thus, we get an object $X' \in P$ and morphism $f : X' \longrightarrow A$ in M. This proves the axiom PS_3. Hence (M, P) is a projective structure of $\mathcal{C}$.

Furthermore, for any projective $X \in P$, there exists an object $X' \in \mathcal{C}'$ such that $X \cong X'$. By our construction, it is clear that $X' \in P'$. Hence (M, P) is the required projective structure of $\mathcal{C}$.

Conversely, we can prove the following theorem.

Theorem 4.4 *Let $\mathcal{C}'$ be a full representative subcategory of the category $\mathcal{C}$. If (M, P) is a projective structure of $\mathcal{C}$, then $(M, \cap\mathcal{C}', P \cap \mathcal{C}')$ is a projective structure of $\mathcal{C}'$.*

Remark 4.5. A category has a projective structure if and only if its full representative subcategory has a projective structure.

The proofs of the following theorems are straightforward, and therefore will be omitted.

Theorem 4.6 *Let $\mathcal{C}_1, \mathcal{C}_2$ be two categories and (M_1, P_1), (M_2, P_2) be their projective structures, respectively. Then $(M_1 \times M_2, P_1 \times P_2)$ is a projective structure of $\mathcal{C}_1 \times \mathcal{C}_2$. Conversely, the canonical projection of any projective structure of $\mathcal{C}_1 \times \mathcal{C}_2$ is also a projective structure.*

Theorem 4.7 *Let $F : \mathcal{C} \to \mathcal{D}$ be a full embedding functor such that every object of the category $\mathcal{D}$ is the image of some object of the category $\mathcal{C}$. If (M, P) is a projective structure of $\mathcal{C}$, then $(F(M), F(P))$ is a projective structure of $\mathcal{D}$.*

5. Bases and Subbases

Definition 5.1 Let (M, P) be a projective structure of the category $\mathcal{C}$. A subclass $P' \subseteq P$ will be called a *base of projectives* if every object in P is a retract of some object in P'.

Definition 5.2 Let (M, P) be a projective structure of the category $\mathcal{C}$. A subclass $P'' \subseteq P$ is called a *sub-base* of projectives if the class of all products of objects in P'' forms a base of projectives of (M, P).

Remark 5.1 Every base of projectives is also a sub-base.

Proposition 5.3 *Let P'', P' and P be classes of objects of the category $\mathcal{C}$. If every object in P is a coproduct of objects in P'' and every object in P is a retract of an object in P', then $\mu(P'') \subseteq \mu(P') \subseteq \mu(P)$.*

Proof Since $P'' \subseteq \pi(\mu(P''))$. By Proposition 2.4, the coproduct of projectives is also projective, we get $P' \subseteq \pi(\mu(P''))$. Moreover, by Proposition 3.6 and Remark 3.2, we get

$$\mu(P') \supseteq \mu(\pi(\mu(P''))) = \mu(P'').$$

Again, since $P' \subseteq \pi(\mu(P'))$, by Proposition 2.3, every retract of projective is also projective, we get $P \subseteq \pi(\mu(P'))$. By Proposition 2.4 and Remark 3.2, we get

$$\mu(P) \supseteq \mu(\pi(\mu(P'))) = \mu(P').$$

Hence, $\mu(P'') \subseteq \mu(P') \subseteq \mu(P)$, and the proof is now completed. □

Using Proposition 3.6 and Proposition 5.3 we get

Corollary 5.1 *Every sub-base of projectives is a generating class of projectives.*

Theorem 5.4 *A class P' of objects of the category $\mathcal{C}$ is a base of projectives of a projective structure (M, P) of $\mathcal{C}$ if and only if for any object $A \in \mathcal{C}$, there exists a morphism $\alpha' : X \to A$ in $\mu(P')$, where $X \in P'$.*

Proof Suppose that P' is a base of projectives of a projective structure (M, P) of $\mathcal{C}$. By Axiom PS_3, for any object $A \in \mathcal{C}$, there exists an object $Y \in P$ and a morphism $\alpha : Y \to A$ in $M = \mu(P) \subseteq \mu(P')$. Further, by the definition of bases, there is an object $X \in P'$ such that Y is the retract of X, i.e., there exists a retraction $\gamma : X \to Y$ which lies in $\mu(P')$, by Proposition 6. Thus $\alpha, \gamma \in \mu(P')$ and by Proposition 7, $\alpha' = \alpha\gamma : X \to A \in \mu(P')$ where $X \in P'$.

Conversely, let P be the class of all retracts of the objects in P' and $M = \mu(P')$. It is obvious that $P' \subseteq P$. By Proposition 3.6, we have $\mu(P') \supseteq \mu(P)$. Further, by Proposition 5.3, we have $\mu(P') \subseteq \mu(P)$. It implies that $M = \mu(P') = \mu(P)$. In order to prove the theorem, we have to show that $\pi(M) = P$. Evidently,

$$P \subseteq \pi(\mu(P)) = \pi(\mu(P')) = \pi(M).$$

Let $A \in \pi(M)$ be an arbitrary object. By hypothesis, there exists a morphism $\alpha' : X \to A$ in $\mu(P') = M$ and $X \in P'$. This gives that A is projective with respect to α'. By the definition of projectives, there exists a morphism $\gamma' : A \to X$ such that $\alpha'\gamma' = I_A$. This implies that α' is a retraction where

$\alpha' \in M$ and $X \in P'$. By Proposition 2.3, $A \in P$. Since $A \in \pi(M)$ is arbitrary, we can see that $\pi(M) \subseteq P$. Hence $\pi(M) = P$, proving our Theorem. □

Theorem 5.5 *If the category $\mathcal{C}$ has coproducts, then any class P'' of objects of $\mathcal{C}$ is a sub-base of projectives of a projective structure of $\mathcal{C}$.*

Proof Let P' be the class of all objects which are coproducts of the objects in P''. For any object $A \in \mathcal{C}$, consider the family of morphism $\{f_i : X_i \to A\}_{i \in I}$ which consists the morphisms in $\mathcal{C}$ from the objects X_i in P'' to A. Let $X = \sum_{i \in I} X_i$ and $u_i : X_i \to X$ be injections. By definition of coproducts, there exists a morphism $\alpha : X \to A$ such that $\alpha u_i = f_i$, for all, $i \in I$. It is obvious that $\alpha \in \mu(P'')$. Note that by Proposition 5.3, we have $\mu(P'') \subseteq \mu(P')$. This implies that $\alpha \in \mu(P')$, where $dom(\alpha) = X \in P'$. By Theorem 5.4, P' is a base of projectives of a projective structure of $\mathcal{C}$. Hence, P'' is a sub-base of projective of a projectives structure. □

6. Regular and finer projective structures

Definition 6.1 A projective structure (M, P) of $\mathcal{C}$ is called *regular* if for every object $A \in \mathcal{C}$, there is a morphism $\alpha : X \to A$ in M with $X \in P$ such that for any other morphism $\alpha' : X' \to A$ in $\mathcal{C}$ with $X' \in P$, there exists a unique morphism $\eta : X' \to X$ such that $\alpha\eta = \alpha'$.

Example 6.1 The exact projective structure of the category Ab of abelian groups is regular.

Theorem 6.2 *A class P of objects of the category $\mathcal{C}$ is the class of projectives of a regular projective structure of $\mathcal{C}$ if and only if for any object $A \in \mathcal{C}$, there is a morphism $\alpha : X \to A$ with $X \in P'$ such that for any $\alpha' : X' \to A$ with $X' \in P$ there exists a unique morphism $\eta : X' \to X$ such that $\alpha\eta = \alpha'$.*

Proof The direct part of the theorem is immediately followed from the Definition 6.1.

Conversely, we assume that the condition given in the theorem holds. By Theorem 5.4, the class P can be considered as a base of projectives of a projective structure of $\mathcal{C}$, (M', P') says. Let $A \in P'$ be arbitrary. By assumption there exists a morphism $\alpha : X \to A$ with $X \in P \subseteq P'$. By Theorem 1, α is a retraction. Further, by Proposition 3.2, $\alpha \in M'$. This shows that A is projective with respect to α, and therefore there exists a morphism $\gamma : A \to X$ such that $\alpha\gamma = I_A$. This implies that

$$\alpha(\gamma\alpha) = (\alpha\gamma)\alpha = 1_A\alpha = \alpha = \alpha.1_X,$$

and by the uniqueness, $\gamma\alpha = I_X$. Thus α is an isomorphism. Since $A \in P'$ is arbitrary, this shows that every object in P' is isomorphic to some object in P. Hence, $\mathcal{P}(M', P)$ is a also a projective structure of $\mathcal{C}$ which is regular. □

Definition 6.3 Let (M, P) and (M', P') be two projectives structures of $\mathcal{C}$. We say that (M', P') is *finer* than (M, P) if $M \supseteq M'$, or equivalently $P \subseteq P'$.

Remark 6.1 The relation "finer than" on projective structures of the category $\mathcal{C}$ is partially ordered.

Example 6.2 For any category $\mathcal{C}$, the projective structure of $\mathcal{C}$ whose proper morphisms are just the retraction in $\mathcal{C}$ and the class of projectives consists all the objects of $\mathcal{C}$, is finer than every other projective structure of $\mathcal{C}$.

Example 6.3 If the category $\mathcal{C}$ has zero object, then every projective structure of $\mathcal{C}$ is finer than the projective structure of $\mathcal{C}$ whose class of proper morphisms consists all the morphisms of $\mathcal{C}$ and the class of projectives consists only the zero object.

Theorem 6.4 *Let $\mathcal{C}$ be a category with coproducts. If $\{(M_i, P_i)\}_{i\in I}$ is a family of projectives structures of $\mathcal{C}$, then this family has a least upper bound $(M, P) = sup(M_i, P_i)$, where $M = \cap M_i$. Furthermore, if for each $i \in I$, $P''_{i\in I}$ is a sub-base of projectives of $(M_i, \mathcal{P}_i)$, then $\cup_{i\in I} P''_i$ is a sub-base of projectives of $(M, \mathcal{P})$.*

Proof For each $i \in I$, let P'_i be the class of all coproducts of the objects in P''_i. By the associativity of coproducts, the class P' of all coproducts of objects in $\cup_{i\in I} P'_i$ is also the class of all coproducts of the objects in $\cup_{i\in I} P''_i$. Let $A \in \mathcal{C}$ be arbitrary. For each $i \in I$, there exists a morphism $\alpha_i : X_i \to A$ in M_i with $X_i \in P_i$ (By Theorem 5.4). Suppose that $X = \sum_{i\in I} X_i$ and $u_i : X_i \to X$ are injections. By the definition of coproducts, there exists a morphism $\eta : X \to A$ such that $\eta u_i = \alpha_i$, for all $i \in I$. For any morphism $\alpha'_i : X'_i \to A$, where $X'_i \in P'_i \subseteq P_i$, there exists a morphism $\gamma : X'_i \to X_i$ such that $\alpha_i \gamma = \alpha'_i$. This implies that $\alpha'_i = (\eta u_i)\gamma = \eta(u_i \gamma)$. This shows that every morphism $\alpha'_i : X'_i \to A$, where $X'_i \in \cup_{i\in I} P'_i$ can be factored through $\eta : X \to A$, where $X \in P'$, i.e., X is the coproduct of $X_i \in \cup_{i\in I} P'_i$. Therefore, by Proposition 5.3, we get $\eta \in \mu(\cup_{i\in I} P'_i) \subseteq \mu(P')$.

Using Theorem 5.4 we can see that P' is a base of projectives and therefore $\cup_{i\in I} P''_i$ is a sub-base of projectives of a projective structure (M, P) of $\mathcal{C}$. By Corollary 5.1 and Proposition 3.6, we get

$$M = \mu(\cup_{i\in I} P''_i) = \cap_{i\in I}\, \mu(P''_i) = \cap_{i\in I}\, M_i$$

It is clear that (M, P) is a least upper bound of the family $\{(M_i, P_i)\}_{i\in I}$. □

Corollary 6.1 *For every family of projective structures, there exists the finest projective structure.*

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