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A VARIETY CONTAINING JORDAN AND PSEUDO-COMPOSITION ALGEBRAS

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Abstract

We consider *3-Jordan algebras*, i.e., the nonassociative commutative algebras satisfying $(x^3y)x = x^3(yx)$. The variety of 3-Jordan algebras contains all Jordan algebras and all pseudo-composition algebras. We prove that a simple 3-Jordan algebra with idempotent is either a Jordan algebra or a pseudo-composition algebra.

1 Introduction

All algebras considered in this paper are nonassociative algebras over a commutative associative ring R containing scalars $\frac{1}{2}$ and $\frac{1}{3}$. If R is a field, we denote it by K. By our assumptions on scalars K is of characteristic not 2 and 3.

As usual, if a, b, c are elements of an algebra, we denote the *associator* $(ab)c - a(bc)$ by (a, b, c). If B, C and D are subsets of an algebra, we denote by (B, C, D) the set of all (b, c, d) with $b \in B, c \in C$ and $d \in D$.

We say that an algebra A over K with idempotent e is of *e-quadratic type* if there are a linear form $\gamma : A \to K$ and a symmetric bilinear form $\phi : A \times A \to K$

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such that $x^2 + \gamma(x)x + \phi(x,x)e = 0$ for all $x \in A$. When A has an identity element 1 and A is of 1-quadratic type, we say that A is a *quadratic algebra*.

An algebra is called *alternative* if it satisfies the *alternative identities* $(x,x,y) = 0$ and $(y,x,x) = 0$. A *Jordan algebra* is a commutative algebra satisfying the *Jordan identity* $(x^2,y,x) = 0$. For a classification of alternative and Jordan algebras see Kuz'min and Shestakov [7].

A commutative algebra A over K is said to be a *pseudo-composition algebra* if there is a nonzero symmetric bilinear form $\phi : A \times A \to K$ such that $x^3 = \phi(x,x)x$ for all $x \in A$. When K is algebraically closed, such an algebra necessarily has an idempotent element e. In [9] Meyberg and Osborn characterized pseudo-composition algebras with idempotent: the algebra is of e-quadratic type, or modulo the radical of its bilinear form is of e-quadratic type, or can be constructed starting with an alternative quadratic algebra. Any proper ideal of a pseudo-composition algebra is contained in the radical of its bilinear form. Therefore, if this bilinear form is nondegenerate then the pseudo-composition algebra is simple. More results on pseudo-composition algebras can be found in the papers by Elduque and Okubo [4, 5].

When studying a variety determined by a class of algebras, one often asks if the variety is finitely based. An effective way to start looking for the basis is to construct the polynomial identities in the variety of smallest degree. For the variety determined by pseudo-composition algebras, the identities of smallest degree were determined by Giuliani and Peresi [6]. These authors showed that $(x^3,y,x) = 0$ is an identity that holds in all pseudo-composition algebras. They also showed that all polynomial identities of degree five or less of pseudo-composition algebras are consequences of commutativity and $(x^3,y,x) = 0$.

We call the identity $(x^3,y,x) = 0$ the *3-Jordan identity*. We say that a commutative algebra satisfying $(x^3,y,x) = 0$ is a *3-Jordan algebra*.

The variety of 3-Jordan algebras contains all Jordan algebras and all pseudo-composition algebras. Since the variety of 3-Jordan algebras arises naturally as a generalization of both Jordan and pseudo-composition algebras, the variety of 3-Jordan algebras deserves to be studied.

The structure of the simple algebras is one of the main questions in the theory of algebras. In this paper, we prove that a simple 3-Jordan algebra with idempotent is either a Jordan algebra or a pseudo-composition algebra.

2 Peirce Decomposition

We denote by $f(a,b,c,y,d) = 0$, the complete linearization of $(x^3,y,x) = 0$.

Let A be a 3-Jordan algebra and assume that A has an idempotent e. Let L_e denote the left multiplication operator: $L_e(a) = ea$ $(\forall a \in A)$. From $f(a,e,e,e,e) = 0$ we obtain $(2L_e^4 - L_e^3 - 2L_e^2 + L_e)(a) = 0$ for any $a \in A$. This means that L_e satisfies the polynomial $2t^4 - t^3 - 2t^2 + t$ and L_e has eigenvalues $0, 1, -1, \frac{1}{2}$. Therefore A has the *Peirce decomposition* $A = A_1 \oplus A_{\frac{1}{2}} \oplus A_{-1} \oplus A_0$

with $A_\lambda = \{z \in A \mid ez = \lambda z\}$. We indicate the unique *decomposition* of $x \in A$ by $x = [x]_1 + [x]_{\frac{1}{2}} + [x]_{-1} + [x]_0$. We denote by $a_\lambda, b_\lambda, c_\lambda, d_\lambda$ arbitrary elements of A_λ ($\lambda = 1, \frac{1}{2}, -1, 0$).

We now derive the relations among the subspaces A_λ.

Let $a \in A_1$. For $b \in A_1$ we obtain from $f(a,e,e,b,e) = 0$ that $[ab]_{\frac{1}{2}} + 4[ab]_{-1} + 2[ab]_0 = 0$. Therefore $ab = [ab]_1$, and we have that $A_1^2 \subset A_1$. If $b \in A_{\frac{1}{2}}$ then we get from $f(a,b,e,e,e) = 0$ that $[ab]_1 + 3[ab]_{-1} + [ab]_0 = 0$ and thus $A_1 A_{\frac{1}{2}} \subset A_{\frac{1}{2}}$. If $b \in A_{-1}$, $f(a,e,e,b,e) = 0$ gives $4[ab]_1 + 3[ab]_{\frac{1}{2}} + 2[ab]_0 = 0$. Therefore $A_1 A_{-1} \subset A_{-1}$. If $b \in A_0$, $f(a,b,e,e,e) = 0$ gives $4[ab]_1 + 3[ab]_{\frac{1}{2}} + 4[ab]_{-1} + 4[ab]_0 = 0$. Therefore $A_1 A_0 = 0$.

Now, let $a \in A_{\frac{1}{2}}$. For $b \in A_{\frac{1}{2}}$ we obtain from $f(a,b,e,e,e) = 0$ that $[ab]_{\frac{1}{2}} = 0$; it follows that $A_{\frac{1}{2}}^2 \subset A_1 + A_{-1} + A_0$. If $b \in A_{-1}$ the identity $f(a,b,e,e,e) = 0$ yields $[ab]_{-1} = 0$ and we get $A_{\frac{1}{2}} A_{-1} \subset A_1 + A_{\frac{1}{2}} + A_0$. If $b \in A_0$ the identity $f(a,b,e,e,e) = 0$ gives $[a,b]_1 - 3[ab]_{-1} + [ab]_0 = 0$. Therefore $A_{\frac{1}{2}} A_0 \subset A_{\frac{1}{2}}$.

Next, let $a \in A_{-1}$. For $b \in A_{-1}$, $f(a,b,e,e,e) = 0$ gives $[ab]_{\frac{1}{2}} - 16[ab]_{-1} = 0$. Therefore $A_{-1}^2 \subset A_1 + A_0$. If $b \in A_0$, using $f(a,b,e,e,e) = 0$ we get $4[a_{-1}b_0]_1 + 3[a_{-1}b_0]_{\frac{1}{2}} + 36[a_{-1}b_0]_{-1} + 4[a_{-1}b_0]_0 = 0$. Therefore $A_{-1}A_0 = 0$.

Finally, for $a, b \in A_0$ we use $f(a,e,e,b,e) = 0$ to obtain $2[ab]_1 + [ab]_{\frac{1}{2}} - 2[ab]_{-1} = 0$ and then $A_0^2 \subset A_0$.

We summarize these results in the following proposition.

Proposition 1 *Let A be a 3-Jordan algebra. If A contains an idempotent e, then A has the Peirce decomposition*

$$A = A_1 \oplus A_{\frac{1}{2}} \oplus A_{-1} \oplus A_0.$$

Furthermore,

$$\begin{aligned} A_1^2 \subset A_1,\ A_1 A_{\frac{1}{2}} \subset A_{\frac{1}{2}},\ A_1 A_{-1} \subset A_{-1},\ A_1 A_0 = 0, \\ A_{\frac{1}{2}}^2 \subset A_1 + A_{-1} + A_0,\ A_{\frac{1}{2}} A_{-1} \subset A_1 + A_{\frac{1}{2}} + A_0,\ A_{\frac{1}{2}} A_0 \subset A_{\frac{1}{2}}, \\ A_{-1}^2 \subset A_1 + A_0,\ A_{-1}A_0 = 0, \\ A_0^2 \subset A_0. \end{aligned}$$

3 Examples and First Results

An algebra A is called *power-associative* if each $x \in A$ generates an associative subalgebra. Any alternative algebra and any Jordan algebra is power-associative.

Example 1 Any commutative algebra $\mathcal{A}_1$ which is nilpotent of index five is a 3-Jordan algebra which has no idempotent. The algebra $\mathcal{A}_1$ need not be power-associative or Jordan.

Bernstein algebras. A commutative algebra A over K is called a *Bernstein algebra* if there is a nonzero algebra homomorphism $\omega : A \to K$ such that $x^2x^2 = \omega(x)^2x^2$ for any $x \in A$. Bernstein algebras always have idempotents (for example, a^2 with $w(a) = 1$). An idempotent e with $\omega(e) = 1$ of A determines the Peirce decomposition $A = A_1 \oplus A_{\frac{1}{2}} \oplus A_0$, and $A_1 = Ke$, $A_{\frac{1}{2}}^2 \subset A_0$, $A_{\frac{1}{2}}A_0 \subset A_{\frac{1}{2}}$, $A_0^2 \subset A_{\frac{1}{2}}$ and $A_{\frac{1}{2}}A_0^2 = 0$. The Bernstein algebra A is a Jordan algebra if and only if $x^3 = \omega(x)x^2$, or equivalently $A_0^2 = 0$ and $(uz)z = 0$ $(\forall u \in A_{\frac{1}{2}}, z \in A_0)$. See Lyubich [8], Alcalde, Baeza and Burgueño [1], and Walcher [11].

Proposition 2 *Let A be a Bernstein algebra. Then A is a Jordan algebra if and only if it is a 3-Jordan algebra.*

Proof If A is a Jordan algebra then A is a 3-Jordan algebra. Conversely, assume that A is a 3-Jordan algebra. By Proposition 1 $A_0^2 \subset A_0$. But we have also that $A_0^2 \subset A_{\frac{1}{2}}$. Hence $A_0^2 = 0$. For all $u \in A_{\frac{1}{2}}, z \in A_0$, from $f(e, u, z, z, e) = 0$ we obtain $(uz)z = 0$. It follows that A is a Jordan algebra.

From the classification of Bernstein-Jordan algebras of dimension 5 over the real field given by Correa and Peresi [2], we take the following example.

Example 2 Let $\mathcal{A}_2$ be the commutative algebra over the real field $\mathcal{R}$ with basis $\{e, e_1, e_2, e_3, e_4\}$ and nonzero products $e^2 = e, ee_1 = \frac{1}{2}e_1, ee_4 = \frac{1}{2}e_4, e_1^2 = e_3$ and $e_1e_2 = e_4$. This algebra has Peirce decomposition $\mathcal{A}_2 = \mathcal{R}e \oplus A_{\frac{1}{2}} \oplus A_0$, where $A_{\frac{1}{2}} =< e_1, e_4 >$ and $A_0 =< e_2, e_3 >$. For any $x \in \mathcal{A}_2$, $x^3 = \omega(x)x^2$, where $\omega : \mathcal{A}_2 \to \mathcal{R}$ is the algebra homomorphism defined by $\omega(e) = 1$, $\omega(e_i) = 0$ $(1 \le i \le 4)$.

Train algebras of rank three. A commutative algebra A over K with a nonzero algebra homomorphism $\omega : A \to K$ is a *train algebra of rank three* if it satisfies an identity $x^3 = (1+\delta)\omega(x)x^2 - \delta\omega(x)^2x$ (for some $\delta \in K$). When $\delta \neq \frac{1}{2}$, such an algebra A always has an idempotent e and a Peirce decomposition $A = Ke \oplus A_{\frac{1}{2}} \oplus A_\delta$, where $ker(\omega) = A_{\frac{1}{2}} \oplus A_\delta$ and $A_{\frac{1}{2}}^2 \subset A_\delta$, $A_{\frac{1}{2}}A_\delta \subset A_{\frac{1}{2}}$, $A_\delta^2 = 0$ (see Costa [3]). There are train algebras of rank three with $\delta = \frac{1}{2}$ which have no idempotent.

Proposition 3 *Let A be a train algebra of rank three. Then A is a 3-Jordan algebra if and only if it is a Jordan algebra or a pseudo-composition algebra. Furthermore, if $\delta \neq 1, -1$ and A is a 3-Jordan algebra then A is a Bernstein-Jordan algebra.*

Proof Assume that A is a 3-Jordan algebra. If $\delta = -1$ then $x^3 = \omega(x)^2x$ and A is a pseudo-composition algebra. Assume that $\delta \neq -1$. Given any x and y in A, $0 = (x^3, y, x) = (1 + \delta)\omega(x)(x^2, y, x)$. Therefore $\omega(x)(x^2, y, x) = 0$. If $\omega(x) \neq 0$ then $(x^2, y, x) = 0$. Since ω is nonzero there is an $a \in A$ such that $\omega(a) \neq 0$. If $\omega(x) = 0$ then, for any $\lambda \in K$, since $\omega(a + \lambda x) = \omega(a) \neq 0$, we have

$((a+\lambda x)^2, y, a+\lambda x) = 0$. By characteristic not 2 and 3 we get $(x^2, y, x) = 0$. In both cases we get $(x^2, y, x) = 0$. Therefore A is a Jordan algebra. The converse implication is clear.

Assume now that $\delta \neq 1, -1$ and A is a 3-Jordan algebra. We know that A is Jordan and so $x^2x^2 = x^3x$. We will prove that A is a Bernstein algebra. By linearization of $x^3 = (1+\delta)\omega(x)x^2 - \delta\omega(x)^2x$ we get $3x^4 = (3+4\delta+2\delta^2)\omega(x)^2x^2 + (-4\delta-2\delta^2)\omega(x)^3x$. By multiplying $x^3 = (1+\delta)\omega(x)x^2 - \delta\omega(x)^2x$ by x we obtain $x^4 = (\delta^2+\delta+1)\omega(x)^2x^2 - (\delta+\delta^2)\omega(x)^3x$. Adding $(1+\delta)$ times the first equation to $-(4+2\delta)$ times the second equation gives $(\delta-1)x^4 = (\delta-1)\omega(x)^2x^2$. Since $\delta \neq 1$, $x^2x^2 = \omega(x)^2x^2$.

Costa [3] classified the train algebras of rank three having dimension 5. From this classification we take the following class of examples.

Example 3 Let $\mathcal{A}_3$ be the commutative algebra with basis $\{e, e_1, e_2, e_3, e_4\}$ and nonzero products $e^2 = e, ee_1 = \frac{1}{2}e_1, ee_2 = -e_2, ee_3 = -e_3, ee_4 = \frac{1}{2}e_4, e_1^2 = e_3$ and $e_1e_2 = ke_4$ ($k \in K, k \neq 0$). The Peirce decomposition is $\mathcal{A}_3 = Ke \oplus A_{\frac{1}{2}} \oplus A_{-1}$, where $A_{\frac{1}{2}} =< e_1, e_4 >$ and $A_{-1} =< e_2, e_3 >$. If $\omega : \mathcal{A}_3 \to K$ is the algebra homomorphism defined by $w(e) = 1,\ w(e_i) = 0\ (1 \leq i \leq 4)$ then $x^3 = \omega(x)^2x$. Therefore $\mathcal{A}_3$ is a pseudo-composition algebra.

Algebras of rank three. A commutative algebra A over K is an *algebra of rank three* if there are a linear form $\gamma : A \to K$ and a symmetric bilinear form $\phi : A \times A \to K$ such that $x^3 = \gamma(x)x^2 + \phi(x,x)x$ for all $x \in A$. Walcher [12] proved that (with the exception of one class) finite dimensional algebras of rank three can be constructed either from a quadratic alternative algebra or from a representation of the Clifford algebra, and characterized the semisimple and simple rank three algebras.

Proposition 4 *Let A be an algebra of rank three. Then A is a 3-Jordan algebra if and only if A is a Jordan algebra or a pseudo-composition algebra.*

Proof As noticed before, Jordan and pseudo-composition algebras are 3-Jordan algebras. Assume now that A is a 3-Jordan algebra. By hypothesis we have $x^3 = \gamma(x)x^2 + \phi(x,x)x$ for all $x \in A$. If $\gamma = 0$ and $\phi = 0$ we have $x^3 = 0$ and it follows that A is a Jordan algebra. When $\gamma = 0$ and $\phi \neq 0$ we have $x^3 = \phi(x,x)x$ and A is a pseudo-composition algebra. Finally, assume that $\gamma \neq 0$. Then for some $a \in A$ we have $\gamma(a) \neq 0$. Given any x and y in A, $0 = (x^3, y, x) = \gamma(x)(x^2, y, x)$. If $\gamma(x) \neq 0$ we have $(x^2, y, x) = 0$. If $\gamma(x) = 0$ then, for any $\lambda \in K$, since $\gamma(a+\lambda x) = \gamma(a) \neq 0$, we have $((a+\lambda x)^2, y, a+\lambda x) = 0$. By characteristic not 2 and 3 we get $(x^2, y, x) = 0$. In both cases we get $(x^2, y, x) = 0$. Therefore A is a Jordan algebra.

A class of examples. Let V be a vector space of dimension ≥ 1 over K and $\phi : V \times V \to K$ a symmetric bilinear form. We define an algebra $J(\phi, V)$ on the vector space $K \oplus V$ by defining multiplication: $(\alpha + x)(\beta + y) = (\alpha\beta + \phi(x,y)) + (\alpha y + \beta x)$. The algebra $J(\phi, V)$ is a Jordan algebra called the *Jordan algebra* *of* ϕ. If J is any Jordan algebra over K and P is any pseudo-composition algebra over K, then the direct sum $J \oplus P$ is a 3-Jordan algebra. We denote by $J(\phi, V, P)$ the algebra $J(\phi, V) \oplus P$ and call it the *3-Jordan algebra of* ϕ *and* P.

Proposition 5 *Let* $J(\phi, V)$ *be the Jordan algebra of a symmetric bilinear form* $\phi : V \times V \to K$. *Let* P *be a pseudo-composition algebra over* K. *Then the idempotents of* $J(\phi, V, P)$ *are* $1,\ \frac{1}{2} + a,\ c,\ 1 + c,\ \frac{1}{2} + a + c$, *where* c *is an idempotent of* P *and* $a \in V$ *such that* $\phi(a, a) = \frac{1}{4}$. *Moreover, if* $P = Kc \oplus A^{(c)}_{\frac{1}{2}} \oplus A^{(c)}_{-1}$ *is the Peirce decomposition of* P *determined by* c, *then the Peirce subspaces of* $J(\phi, V, P)$ *with respect to an idempotent* e *are:*

(i) $A_1 = K \oplus V,\ A_{\frac{1}{2}} = 0,\ A_{-1} = 0,\ A_0 = P$ *if* $e = 1$;

(ii) $A_1 = K(1 + 2a),\ A_{\frac{1}{2}} = \{x \in V : \phi(a, x) = 0\},\ A_{-1} = 0,\ A_0 = K(1 - 2a) \oplus P$ *if* $e = \frac{1}{2} + a$;

(iii) $A_1 = Ke,\ A_{\frac{1}{2}} = A^{(c)}_{\frac{1}{2}},\ A_{-1} = A^{(c)}_{-1},\ A_0 = K \oplus V$ *if* $e = c$;

(iv) $A_1 = K \oplus V \oplus Kc,\ A_{\frac{1}{2}} = A^{(c)}_{\frac{1}{2}},\ A_{-1} = A^{(c)}_{-1},\ A_0 = 0$ *if* $e = 1 + c$;

(v) $A_1 = K(1 + 2a) \oplus Kc,\ A_{\frac{1}{2}} = \{x \in V : \phi(a, x) = 0\} \oplus A^{(c)}_{\frac{1}{2}}$,
$A_{-1} = A^{(c)}_{-1},\ A_0 = K(1 - 2a)$ *if* $e = \frac{1}{2} + a + c$.

Proof Let $e = \alpha + a + c$ $(\alpha \in K, a \in V, c \in P)$. We have $e^2 = \alpha^2 + \phi(a, a) + 2\alpha a + c^2$ and $e^2 = e$ gives $\alpha^2 + \phi(a, a) = \alpha,\ 2\alpha a = a,\ c^2 = c$. If $a = 0$ and $\alpha = 0$ then $e = c$. If $a = 0$ and $\alpha \neq 0$ then $\alpha = 1$ and $e = 1 + c$. If $a \neq 0$ then $\alpha = \frac{1}{2}$ and $e = \frac{1}{2} + a + c$ with $\phi(a, a) = \frac{1}{4}$. Hence the idempotents of $J(\phi, V, P)$ are as stated.

We give a proof for (v). The proofs of (i), (ii), (iii) and (iv) are analogous. Let $e = \frac{1}{2} + a + c$ and $y = \beta + x + d$ $(\beta \in K, x \in V, d \in P)$. If $ey = y$ we obtain $d \in A^{(c)}_1 = Kc$ and $x = 2\beta a$, hence $A_1 = K(1 + 2a) \oplus Kc$. When $ey = \frac{1}{2} y$ we have $\beta = 0, \phi(a, x) = 0$ and $d \in A^{(c)}_{\frac{1}{2}}$. For $ey = -y$ we have $3\beta = -2\phi(a, x),\ 3x = -2\beta a$ and $cd = -d$; it follows that $9\beta = -2\phi(a, 3x) = -2\phi(a, -2\beta a) = 4\beta\phi(a, a) = \beta$, i.e., $\beta = 0$, and thus $x = 0$; hence $A_{-1} = A^{(c)}_{-1}$. Finally, for $ey = 0y$ we obtain $x = -2\beta a$ and $d = 0$, hence $A_0 = K(1 - 2a)$.

4 Further Identities

In this section we obtain all degree three identities involving elements in the Peirce spaces $A_1, A_{\frac{1}{2}}, A_{-1}$ and A_0. We do this by making all possible substitutions in $f(a, b, c, y, d) = 0$.

Lemma 1 *Let* A *be a 3-Jordan algebra and assume that* A *has a Peirce decomposition* $A = A_1 \oplus A_{\frac{1}{2}} \oplus A_{-1} \oplus A_0$. *Then the following identities hold:*

$$(a_1b_1)c_{\frac12} = a_1(b_1c_{\frac12}) + b_1(a_1c_{\frac12}), \tag{1}$$
$$-(a_1b_1)c_{-1} = a_1(b_1c_{-1}) = b_1(a_1c_{-1}), \tag{2}$$
$$a_1[b_{\frac12}c_{\frac12}]_1 = [(a_1b_{\frac12})c_{\frac12}]_1 + [(a_1c_{\frac12})b_{\frac12}]_1, \tag{3}$$
$$a_1[b_{\frac12}c_{\frac12}]_{-1} = -2\ [(a_1b_{\frac12})c_{\frac12}]_{-1} = -2\ [b_{\frac12}(a_1c_{\frac12})]_{-1}, \tag{4}$$
$$[(a_1b_{\frac12})c_{\frac12}]_0 = [b_{\frac12}(a_1c_{\frac12})]_0, \tag{5}$$
$$a_1[b_{\frac12}c_{-1}]_1 = 2\ [(a_1b_{\frac12})c_{-1}]_1 = -[b_{\frac12}(a_1c_{-1})]_1, \tag{6}$$
$$2\ a_1[b_{\frac12}c_{-1}]_{\frac12} = 2\ [(a_1b_{\frac12})c_{-1}]_{\frac12} = -[b_{\frac12}(a_1c_{-1})]_{\frac12}, \tag{7}$$
$$2\ [(a_1b_{\frac12})c_{-1}]_0 = -[b_{\frac12}(a_1c_{-1})]_0, \tag{8}$$
$$(a_1, b_{\frac12}, c_0) = 0, \tag{9}$$
$$a_1[b_{-1}c_{-1}]_1 = -[(a_1b_{-1})c_{-1}]_1 = -[b_{-1}(a_1c_{-1})]_1, \tag{10}$$
$$(b_{-1}, a_1, c_{-1}) = 0, \tag{11}$$
$$3\ [a_{\frac12}^2]_1 a_{\frac12} = 3\ [a_{\frac12}^2]_0 a_{\frac12} + [[a_{\frac12}^2]_{-1} a_{\frac12}]_{\frac12}, \tag{12}$$
$$3[a_{\frac12}b_{\frac12}]_1 c_{-1} = -[[a_{\frac12}c_{-1}]_{\frac12} b_{\frac12}]_{-1} - [a_{\frac12}[b_{\frac12}c_{-1}]_{\frac12}]_{-1}, \tag{13}$$
$$[(a_{\frac12}, c_0, b_{\frac12})]_1 = 0, \tag{14}$$
$$[(a_{\frac12}c_0)b_{\frac12}]_{-1} = 0, \tag{15}$$
$$c_0[a_{\frac12}b_{\frac12}]_0 = [(c_0a_{\frac12})b_{\frac12}]_0 + [a_{\frac12}(c_0b_{\frac12})]_0, \tag{16}$$
$$-3a_{\frac12}[b_{-1}c_{-1}]_1 + 3a_{\frac12}[b_{-1}c_{-1}]_0 = [[a_{\frac12}b_{-1}]_{\frac12}c_{-1}]_{\frac12} + [[a_{\frac12}c_{-1}]_{\frac12}b_{-1}]_{\frac12}, \tag{17}$$
$$[a_{\frac12}b_{-1}]_1 c_{-1} = -[a_{\frac12}c_{-1}]_1 b_{-1}, \tag{18}$$
$$(a_{\frac12}c_0)b_{-1} = 0, \tag{19}$$
$$(a_{\frac12}b_{-1})c_0 = 0, \tag{20}$$
$$(b_0c_0)a_{\frac12} = c_0(b_0a_{\frac12}) + b_0(c_0a_{\frac12}), \tag{21}$$
$$[a_{-1}b_{-1}]_0 c_0 = 0. \tag{22}$$

Proof Identity $f(e, b_1, c_{\frac12}, a_1, e) = 0$ reduces to (1).

Identity $f(a_1, e, c_{-1}, b_1, e)$ simplifies to $(a_1b_1)c_{-1} + 2(a_1c_{-1})b_1 - (b_1c_{-1})a_1 = 0$, and $f(e, b_1, c_{-1}, a_1, e) = 0$ to $(a_1b_1)c_{-1} - (a_1c_{-1})b_1 + 2(b_1c_{-1})a_1$; from these two equations we obtain (2).

Identity $f(a_1, b_{\frac12}, e, c_{\frac12}, e) = 0$ gives

$$-[b_{\frac12}c_{\frac12}]_1 a_1 + [(a_1b_{\frac12})c_{\frac12}]_1 + [(a_1c_{\frac12})b_{\frac12}]_1 = 0,$$
$$-3[(a_1b_{\frac12})c_{\frac12}]_{-1} + [(a_1c_{\frac12})b_{\frac12}]_{-1} - [b_{\frac12}c_{\frac12}]_{-1} a_1 = 0,$$
$$-[(a_1b_{\frac12})c_{\frac12}]_0 + [(a_1c_{\frac12})b_{\frac12}]_0 = 0,$$

and $f(a_1, e, c_{\frac{1}{2}}, b_{\frac{1}{2}}, e) = 0$ gives

$$[(a_1 b_{\frac{1}{2}}) c_{\frac{1}{2}}]_{-1} - 3[(a_1 c_{\frac{1}{2}}) b_{\frac{1}{2}}]_{-1} - [b_{\frac{1}{2}} c_{\frac{1}{2}}]_{-1} a_1 = 0.$$

From these four equations we obtain (3), (4) and (5).

Identity $f(a_1, b_{\frac{1}{2}}, c_{-1}, e, e) = 0$ yields

$$2[(a_1 b_{\frac{1}{2}}) c_{-1}]_1 + [(a_1 c_{-1}) b_{\frac{1}{2}}]_1 = 0,$$
$$2[(a_1 b_{\frac{1}{2}}) c_{-1}]_0 + [(a_1 c_{-1}) b_{\frac{1}{2}}]_0 = 0.$$

From identity $f(a_1, b_{\frac{1}{2}}, e, c_{-1}, e) = 0$ we obtain

$$-[b_{\frac{1}{2}} c_{-1}]_1 a_1 + 4[(a_1 b_{\frac{1}{2}}) c_{-1}]_1 + [(a_1 c_{-1}) b_{\frac{1}{2}}]_1 = 0,$$
$$3[(a_1 b_{\frac{1}{2}}) c_{-1}]_{\frac{1}{2}} - [b_{\frac{1}{2}} c_{-1}]_{\frac{1}{2}} a_1 + [(a_1 c_{-1}) b_{\frac{1}{2}}]_{\frac{1}{2}} = 0.$$

We use identity $f(a_1, e, c_{-1}, b_{\frac{1}{2}}, e) = 0$ to get

$$[(a_1 b_{\frac{1}{2}}) c_{-1}]_{\frac{1}{2}} - [b_{\frac{1}{2}} c_{-1}]_{\frac{1}{2}} a_1 = 0.$$

These five equations give (6), (7) and (8).

Identity $f(e, b_{\frac{1}{2}}, c_0, a_1, e) = 0$ reduces to (9).

From $f(a_1, b_{-1}, e, c_{-1}, e) = 0$ we get

$$-[b_{-1} c_{-1}]_1 a_1 - 2[(a_1 b_{-1}) c_{-1}]_1 + [(a_1 c_{-1}) b_{-1}]_1 = 0,$$
$$-[(a_1 b_{-1}) c_{-1}]_0 + [(a_1 c_{-1}) b_{-1}]_0 = 0.$$

Identity $f(a_1, e, c_{-1}, b_{-1}, e) = 0$ gives

$$-[b_{-1} c_{-1}]_1 a_1 + [(a_1 b_{-1}) c_{-1}]_1 - 2[(a_1 c_{-1}) b_{-1}]_1 = 0.$$

And these three equations yield (10) and (11).

Identity $f(a_{\frac{1}{2}}, a_{\frac{1}{2}}, a_{\frac{1}{2}}, e, e) = 0$ simplifies to (12).

Identity (13) is the simplified form of $f(a_{\frac{1}{2}}, b_{\frac{1}{2}}, c_{-1}, e, e) = 0$.

Identity $f(a_{\frac{1}{2}}, b_{\frac{1}{2}}, c_0, e, e) = 0$ reduces to

$$[[a_{\frac{1}{2}} c_0]_{\frac{1}{2}} b_{\frac{1}{2}}]_{-1} + [[b_{\frac{1}{2}} c_0]_{\frac{1}{2}} a_{\frac{1}{2}}]_{-1} = 0.$$

Identity $f(a_{\frac{1}{2}}, e, c_0, b_{\frac{1}{2}}, e) = 0$ implies

$$[[a_{\frac{1}{2}} c_0]_{\frac{1}{2}} b_{\frac{1}{2}}]_1 - [[b_{\frac{1}{2}} c_0]_{\frac{1}{2}} a_{\frac{1}{2}}]_1 = 0,$$
$$3[[a_{\frac{1}{2}} c_0]_{\frac{1}{2}} b_{\frac{1}{2}}]_{-1} + [[b_{\frac{1}{2}} c_0]_{\frac{1}{2}} a_{\frac{1}{2}}]_{-1} = 0,$$
$$-[(a_{\frac{1}{2}} c_0) b_{\frac{1}{2}}]_0 - [(b_{\frac{1}{2}} c_0) a_{\frac{1}{2}}]_0 + [a_{\frac{1}{2}} b_{\frac{1}{2}}]_0 c_0 = 0.$$

From these four equations we obtain (14), (15) and (16).

Identity $f(a_{\frac{1}{2}}, b_{-1}, c_{-1}, e, e) = 0$ yields

$$-3[b_{-1}c_{-1}]_1 a_{\frac{1}{2}} - [[a_{\frac{1}{2}}b_{-1}]_{\frac{1}{2}}c_{-1}]_{\frac{1}{2}} - [[a_{\frac{1}{2}}c_{-1}]_{\frac{1}{2}}b_{-1}]_{\frac{1}{2}} + 3[b_{-1}c_{-1}]_0 a_{\frac{1}{2}} = 0,$$
$$[a_{\frac{1}{2}}b_{-1}]_1 c_{-1} + [a_{\frac{1}{2}}c_{-1}]_1 b_{-1} = 0,$$

and these two equations give (17) and (18).

We use $f(a_{\frac{1}{2}}, e, c_0, b_{-1}, e) = 0$ to obtain

$$[(a_{\frac{1}{2}}c_0)b_{-1}]_1 = 0,$$
$$[a_{\frac{1}{2}}b_{-1}]_{\frac{1}{2}}c_0 + 3[[a_{\frac{1}{2}}c_0]_{\frac{1}{2}}b_{-1}]_{\frac{1}{2}} = 0,$$
$$2[(a_{\frac{1}{2}}c_0)b_{-1}]_0 + [a_{\frac{1}{2}}b_{-1}]_0 c_0,$$

and $f(e, b_{-1}, c_0, a_{\frac{1}{2}}, e) = 0$ to get

$$[a_{\frac{1}{2}}b_{-1}]_{\frac{1}{2}}c_0 - [(a_{\frac{1}{2}}c_0)b_{-1}]_{\frac{1}{2}} = 0,$$
$$-[(a_{\frac{1}{2}}c_0)b_{-1}]_0 + [a_{\frac{1}{2}}b_{-1}]_0 c_0 = 0.$$

These five identities imply (19) and (20).

Identity $f(a_{\frac{1}{2}}, b_0, e, c_0, e) = 0$ reduces to (21) and $f(a_{-1}, e, c_0, b_{-1}, e) = 0$ simplifies to (22).

5 Annihilators of A_{-1}

This section contains results which will be used in Section 7.

Lemma 2 *Let A be a 3-Jordan algebra and assume that A has a Peirce decomposition $A = A_1 \oplus A_{\frac{1}{2}} \oplus A_{-1} \oplus A_0$. Then we have the following equations:*

$$(A_1, A_1, A_1)A_{-1} = 0, \tag{23}$$
$$[(A_1, A_{\frac{1}{2}}, A_1)A_{-1}]_1 = 0, \quad [(A_1, A_{\frac{1}{2}}, A_1)A_{-1}]_{\frac{1}{2}} = 0, \tag{24}$$
$$[(A_{\frac{1}{2}}, A_1, A_{\frac{1}{2}})]_1 A_{-1} = 0. \tag{25}$$

Proof Using (2) we obtain

$$(a_1, b_1, c_1)d_{-1} = ((a_1b_1)c_1)d_{-1} - (a_1(b_1c_1))d_{-1} =$$
$$-(a_1b_1)(c_1d_{-1}) + a_1((b_1c_1)d_{-1}) = a_1(b_1(c_1d_{-1})) - a_1(b_1(c_1d_{-1})) = 0,$$

and this proves (23).

By (6) we obtain

$$[(a_1(b_1c_{\frac{1}{2}}))d_{-1}]_1 = \frac{1}{2}a_1[(b_1c_{\frac{1}{2}})d_{-1}]_1 = -\frac{1}{4}a_1[c_{\frac{1}{2}}(b_1d_{-1})]_1 = -\frac{1}{2}[(a_1c_{\frac{1}{2}})(b_1d_{-1})]_1 = [(b_1(a_1c_{\frac{1}{2}}))d_{-1}]_1,$$

i.e., $[(a_1, c_{\frac{1}{2}}, b_1)d_{-1}]_1 = 0$. By (7) we have

$$[(a_1(b_1c_{\frac{1}{2}}))d_{-1}]_{\frac{1}{2}} = a_1[(b_1c_{\frac{1}{2}})d_{-1}]_{\frac{1}{2}} = -\frac{1}{2}a_1[c_{\frac{1}{2}}(b_1d_{-1})]_{\frac{1}{2}} = -\frac{1}{2}[(a_1c_{\frac{1}{2}})(b_1d_{-1})]_{\frac{1}{2}} = [(b_1(a_1c_{\frac{1}{2}}))d_{-1}]_{\frac{1}{2}},$$

i.e., $[(a_1, c_{\frac{1}{2}}, b_1)d_{-1}]_{\frac{1}{2}} = 0$. This proves (24).

Identity (13) gives

$$[(a_1b_{\frac{1}{2}})c_{\frac{1}{2}}]_1d_{-1} = -\frac{1}{3}[[(a_1b_{\frac{1}{2}})d_{-1}]_{\frac{1}{2}}c_{\frac{1}{2}}]_{-1} - \frac{1}{3}[(a_1b_{\frac{1}{2}})[c_{\frac{1}{2}}d_{-1}]_{\frac{1}{2}}]_{-1}.$$

Then it follows by (7) and (4) that

$$[(a_1b_{\frac{1}{2}})c_{\frac{1}{2}}]_1d_{-1} = -\frac{1}{3}[(a_1[b_{\frac{1}{2}}d_{-1}]_{\frac{1}{2}})c_{\frac{1}{2}}]_{-1} - \frac{1}{3}[b_{\frac{1}{2}}(a_1[c_{\frac{1}{2}}d_{-1}]_{\frac{1}{2}})]_{-1}.$$

Using commutativity

$$[(a_1b_{\frac{1}{2}})c_{\frac{1}{2}}]_1d_{-1} = -\frac{1}{3}[(a_1[b_{\frac{1}{2}}d_{-1}]_{\frac{1}{2}})c_{\frac{1}{2}}]_{-1} - \frac{1}{3}[(a_1[c_{\frac{1}{2}}d_{-1}]_{\frac{1}{2}})b_{\frac{1}{2}}]_{-1}.$$

Since the right-hand side of this last equation is symmetric in $b_{\frac{1}{2}}$ and $c_{\frac{1}{2}}$ we get $[(b_{\frac{1}{2}}, a_1, c_{\frac{1}{2}})]_1d_{-1} = 0$. We have proved (25).

6 Annihilator Ideals

Let A be a 3-Jordan algebra and assume that A has a Peirce decomposition $A = A_1 \oplus A_{\frac{1}{2}} \oplus A_{-1} \oplus A_0$. We let

$$I = \{x \in A_1 \mid xA_{-1} = 0\}, \quad J = I + A_{\frac{1}{2}}I + [A_{\frac{1}{2}}(A_{\frac{1}{2}}I)]_0, \quad M = \{y \in A_{\frac{1}{2}} \mid yA_{\frac{1}{2}} \subset A_1, yA_{-1} = 0\}.$$

Throughout the rest of the paper, the letters I, J and M are reserved for these sets.

Lemma 3 *The set J is an ideal of A.*

Proof We have to prove that $A_iJ \subset J$ $(i = 1, \frac{1}{2}, -1, 0)$. Throughout the whole proof we use the notation that the letter x represents an element of I.

We first show that $A_1 J \subset J$. By (2) we get $(a_1x)b_{-1} = -a_1(xb_{-1}) = 0$. Therefore $A_1 I \subset I$. We obtain by (1) that $a_1(xa_{\frac{1}{2}}) = (a_1x)a_{\frac{1}{2}} - x(a_1a_{\frac{1}{2}}) \in A_{\frac{1}{2}}I$. Therefore $A_1(A_{\frac{1}{2}}I) \subset A_{\frac{1}{2}}I$. We have that $A_1[A_{\frac{1}{2}}(A_{\frac{1}{2}}I)]_0 \subset A_1A_0 = 0$.

We next show that $A_{-1}J \subset J$. By definition of I, $A_{-1}I = 0$. Using (6), (7) and (8) we obtain $(xa_{\frac{1}{2}})b_{-1} = -\frac{1}{2}a_{\frac{1}{2}}(xb_{-1}) = 0$. Therefore $A_{-1}(A_{\frac{1}{2}}I) = 0$. We have $A_{-1}[A_{\frac{1}{2}}(A_{\frac{1}{2}}I)]_0 \subset A_{-1}A_0 = 0$.

We now show that $A_0 J \subset J$. We have that $A_0 I \subset A_0 A_1 = 0$. By (9) $A_0(A_{\frac{1}{2}}I) \subset (A_0A_{\frac{1}{2}})I \subset A_{\frac{1}{2}}I$. Using this and (16) we get

$$A_0[A_{\frac{1}{2}}(A_{\frac{1}{2}}I)]_0 \subset [(A_0A_{\frac{1}{2}})(A_{\frac{1}{2}}I)]_0 + [A_{\frac{1}{2}}(A_0(A_{\frac{1}{2}}I))]_0 \subset [A_{\frac{1}{2}}(A_{\frac{1}{2}}I)]_0.$$

We finally show that $A_{\frac{1}{2}}J \subset J$. It is clear that $A_{\frac{1}{2}}I \subset J$. We now consider the product $A_{\frac{1}{2}}(A_{\frac{1}{2}}I)$. Using identity (4) we get $[(xa_{\frac{1}{2}})b_{\frac{1}{2}}]_{-1} = -\frac{1}{2}x[a_{\frac{1}{2}}b_{\frac{1}{2}}]_{-1} = 0$ so that we know that $A_{\frac{1}{2}}(A_{\frac{1}{2}}I) \subset A_1 + A_0$. We now show that $[A_{\frac{1}{2}}(A_{\frac{1}{2}}I)]_1 \subset I$. Identity (13), (7) and (4) yield

$$3[(xa_{\frac{1}{2}})b_{\frac{1}{2}}]_1 c_{-1} = -[[(xa_{\frac{1}{2}})c_{-1}]_{\frac{1}{2}} b_{\frac{1}{2}}]_{-1} - [(xa_{\frac{1}{2}})[b_{\frac{1}{2}}c_{-1}]_{\frac{1}{2}}]_{-1} = \frac{1}{2}[[a_{\frac{1}{2}}(xc_{-1})]_{\frac{1}{2}} b_{\frac{1}{2}}]_{-1} + \frac{1}{2}x[a_{\frac{1}{2}}[b_{\frac{1}{2}}c_{-1}]_{\frac{1}{2}}]_{-1} = 0.$$

We have shown that $A_{\frac{1}{2}}(A_{\frac{1}{2}}I) \subset [A_{\frac{1}{2}}(A_{\frac{1}{2}}I)]_1 + [A_{\frac{1}{2}}(A_{\frac{1}{2}}I)]_0 \subset I + [A_{\frac{1}{2}}(A_{\frac{1}{2}}I)]_0 \subset J$. It remains to show that $[A_{\frac{1}{2}}(A_{\frac{1}{2}}I)]_0 A_{\frac{1}{2}} \subset A_{\frac{1}{2}}I$. The linearized form of (12) gives

$$\begin{aligned} 3[(xa_{\frac{1}{2}})b_{\frac{1}{2}}]_0 c_{\frac{1}{2}} + 3[b_{\frac{1}{2}}c_{\frac{1}{2}}]_0(xa_{\frac{1}{2}}) + 3[c_{\frac{1}{2}}(xa_{\frac{1}{2}})]_0 b_{\frac{1}{2}} = \\ 3[(xa_{\frac{1}{2}})b_{\frac{1}{2}}]_1 c_{\frac{1}{2}} + 3[b_{\frac{1}{2}}c_{\frac{1}{2}}]_1(xa_{\frac{1}{2}}) + 3[c_{\frac{1}{2}}(xa_{\frac{1}{2}})]_1 b_{\frac{1}{2}} \\ -[(xa_{\frac{1}{2}})b_{\frac{1}{2}}]_{-1} c_{\frac{1}{2}} - [b_{\frac{1}{2}}c_{\frac{1}{2}}]_{-1}(xa_{\frac{1}{2}}) - [c_{\frac{1}{2}}(xa_{\frac{1}{2}})]_{-1} b_{\frac{1}{2}}. \end{aligned}$$

By what we have already proved each summand of the right-hand side of this equation is zero or is an element of $A_{\frac{1}{2}}I$. Also, by (9), $[b_{\frac{1}{2}}c_{\frac{1}{2}}]_0(xa_{\frac{1}{2}}) = ([b_{\frac{1}{2}}c_{\frac{1}{2}}]_0 a_{\frac{1}{2}})x \in A_{\frac{1}{2}}I$. Thus

$$[(xa_{\frac{1}{2}})b_{\frac{1}{2}}]_0 c_{\frac{1}{2}} + [(xa_{\frac{1}{2}})c_{\frac{1}{2}}]_0 b_{\frac{1}{2}} \in A_{\frac{1}{2}}I.$$

Then we obtain also that

$$\begin{aligned} -[(xb_{\frac{1}{2}})c_{\frac{1}{2}}]_0 a_{\frac{1}{2}} - [(xb_{\frac{1}{2}})a_{\frac{1}{2}}]_0 c_{\frac{1}{2}} \in A_{\frac{1}{2}}I. \\ [(xc_{\frac{1}{2}})a_{\frac{1}{2}}]_0 b_{\frac{1}{2}} + [(xc_{\frac{1}{2}})b_{\frac{1}{2}}]_0 a_{\frac{1}{2}} \in A_{\frac{1}{2}}I. \end{aligned}$$

Adding these last three equations and using (5) we obtain that $2[(xa_{\frac{1}{2}})c_{\frac{1}{2}}]_0 b_{\frac{1}{2}} \in A_{\frac{1}{2}}I$. Hence $A_{\frac{1}{2}}[A_{\frac{1}{2}}(A_{\frac{1}{2}}I)]_0 \subset A_{\frac{1}{2}}I$. Therefore $A_{\frac{1}{2}}J \subset J$.

Lemma 4 *If $A_0 = 0$ and $I = 0$, then M is an ideal of A.*

Proof Let $y \in M$. First, we prove that $A_1 M \subset M$. We have $(a_1 y)b_{\frac{1}{2}} = [(a_1 y)b_{\frac{1}{2}}]_1 + [(a_1 y)b_{\frac{1}{2}}]_{-1}$. Then, since $[(a_1 y)b_{\frac{1}{2}}]_{-1} = -\frac{1}{2}a_1[yb_{\frac{1}{2}}]_{-1} = 0$ by (4), it follows that $(a_1 y)b_{\frac{1}{2}} = [(a_1 y)b_{\frac{1}{2}}]_1 \in A_1$. Thus $(A_1 M)A_{\frac{1}{2}} \subset A_1$. Using (6) and (7) we get $(a_1 y)b_{-1} = -\frac{1}{2}y(a_1 b_{-1}) = 0$ and this means that $(A_1 M)A_{-1} = 0$.

Next, we prove that $A_{\frac{1}{2}} M \subset M$. More precisely, we establish that $A_{\frac{1}{2}} M \subset I = 0$. Identity (13) gives $(a_{\frac{1}{2}} y)b_{-1} = [a_{\frac{1}{2}} y]_1 b_{-1} = -\frac{1}{3}[[a_{\frac{1}{2}} b_{-1}]_{\frac{1}{2}} y]_{-1} - \frac{1}{3}[a_{\frac{1}{2}}[yb_{-1}]_{\frac{1}{2}}]_{-1} = 0$.

Finally, from the definition of M we obtain $A_{-1} M = 0$.

7 Simple Algebras

We are now ready to prove our main result.

Theorem 1 *Let A be a 3-Jordan algebra and assume that A has an idempotent element. If A is simple then A is either a Jordan algebra or a pseudo-composition algebra.*

Proof By Proposition 1, A has the Peirce decomposition $A = A_1 \oplus A_{\frac{1}{2}} \oplus A_{-1} \oplus A_0$. Let J be the ideal of A established by Lemma 3. Since A is simple, we have two possibilities: $J = A$ or $J = 0$.

First case: $J = A = I + A_{\frac{1}{2}} I + [A_{\frac{1}{2}}(A_{\frac{1}{2}} I)]_0$. Here, $A = A_1 \oplus A_{\frac{1}{2}} \oplus A_0$, where $A_1 = I$, $A_{\frac{1}{2}} = A_{\frac{1}{2}} I$, and $A_0 = [A_{\frac{1}{2}}(A_{\frac{1}{2}} I)]_0$. Notice that $A_{-1} = 0$. By Lemma 1, A satisfies (1), (3), (5), (9), (12), (14), (16) and (21). Since all conditions of Proposition 6.7 of Osborn [10] are satisfied, we may conclude that A is a Jordan algebra.

Second case: $J = 0$. In particular, we have $I = 0$. Using (23), $(A_1, A_1, A_1) \subset I = 0$. Therefore A_1 is a commutative and associative algebra.

In this second case we will show that A is a pseudo-composition algebra. The proof is long. We present it in a series of lemmas.

Lemma 5 *The Peirce subspace A_0 is zero. Therefore $A = A_1 \oplus A_{\frac{1}{2}} \oplus A_{-1}$.*

Proof We claim that $L = A_0 + A_0 A_{\frac{1}{2}}$ is an ideal of A. We have $A_1 A_0 = 0$ and by (9) $A_1(A_0 A_{\frac{1}{2}}) \subset (A_1 A_{\frac{1}{2}})A_0 \subset A_0 A_{\frac{1}{2}}$; hence $A_1 L \subset L$. We know that

$$A_{\frac{1}{2}}(A_0 A_{\frac{1}{2}}) = [A_{\frac{1}{2}}(A_0 A_{\frac{1}{2}})]_1 + [A_{\frac{1}{2}}(A_0 A_{\frac{1}{2}})]_{-1} + [A_{\frac{1}{2}}(A_0 A_{\frac{1}{2}})]_0.$$

Using (13) we get

$$3[(a_0 b_{\frac{1}{2}})c_{\frac{1}{2}}]_1 d_{-1} = -[[(a_0 b_{\frac{1}{2}})d_{-1}]_{\frac{1}{2}} c_{\frac{1}{2}}]_{-1} - [(a_0 b_{\frac{1}{2}})[c_{\frac{1}{2}} d_{-1}]_{\frac{1}{2}}]_{-1}.$$

But by (19) $[(a_0b_{\frac{1}{2}})d_{-1}]_{\frac{1}{2}} = 0$ and by (15) $[(a_0b_{\frac{1}{2}})[c_{\frac{1}{2}}d_{-1}]_{\frac{1}{2}}]_{-1} = 0$. Thus $[(a_0b_{\frac{1}{2}})c_{\frac{1}{2}}]_1 d_{-1} = 0$ and this means that $[(a_0b_{\frac{1}{2}})c_{\frac{1}{2}}]_1 \in I$. It follows that $[A_{\frac{1}{2}}(A_0A_{\frac{1}{2}})]_1 \subset I$ and since $I = 0$ we obtain $[A_{\frac{1}{2}}(A_0A_{\frac{1}{2}})]_1 = 0$. By (15) $[A_{\frac{1}{2}}(A_0A_{\frac{1}{2}})]_{-1} = 0$. Therefore $A_{\frac{1}{2}}(A_0A_{\frac{1}{2}}) = [A_{\frac{1}{2}}(A_0A_{\frac{1}{2}})]_0 \subset A_0 \subset L$ and, since $A_{\frac{1}{2}}A_0 \subset L$ we have that $A_{\frac{1}{2}}L \subset L$. We know that $A_{-1}A_0 = 0$ and that $A_{-1}(A_0A_{\frac{1}{2}}) = 0$ by (19), hence $A_{-1}L \subset L$. Therefore $AL \subset L$, i.e., L is an ideal of A.

Since A is simple $L = 0$ or $L = A$. Since $e \in A$, but $e \notin L$ it follows that $L = 0$ and also that $A_0 = 0$.

We let $\overline{A_1}$ be an isomorphic image of A_1. If a is an element of A_1, we use $\overline{a}$ to denote the image of a in $\overline{A_1}$. We define a multiplication between $\overline{a} \in \overline{A_1}$ and $x \in A$ as follows:

$$\overline{a}x = a[x]_1 + 2a[x]_{\frac{1}{2}} - a[x]_{-1}.$$

Lemma 6 *The algebra A is an algebra over the commutative associative ring $\overline{A_1}$.*

Proof To prove the lemma it is enough to verify that $\overline{a}(xy) = (\overline{a}x)y = x(\overline{a}y)$, for any $\overline{a} \in \overline{A_1}$ and $x, y \in A$. We will show that $\overline{a}([x]_i[y]_j) = (\overline{a}[x]_i)[y]_j = [x]_i(\overline{a}[y]_j)$ $(i, j = 1, \frac{1}{2}, -1)$. Since A_1 is associative we get

$$\overline{a}([x]_1[y]_1) = (\overline{a}[x]_1)[y]_1 = [x]_1(\overline{a}[y]_1).$$

We now will show that $(A_1, A_{\frac{1}{2}}, A_1)A_{\frac{1}{2}} \subset A_1$. We know from Lemma 4 that M is an ideal of A. Since A is simple and $e \notin M$ we must have $M = 0$. Using (4) and (2) we obtain

$$[(a_1(b_1c_{\frac{1}{2}}))d_{\frac{1}{2}}]_{-1} = -\frac{1}{2}a_1[(b_1c_{\frac{1}{2}})d_{\frac{1}{2}}]_{-1} = \frac{1}{4}a_1(b_1[c_{\frac{1}{2}}d_{\frac{1}{2}}]_{-1}) = \frac{1}{4}b_1(a_1[c_{\frac{1}{2}}d_{\frac{1}{2}}]_{-1}) = -\frac{1}{2}b_1[(a_1c_{\frac{1}{2}})d_{\frac{1}{2}}]_{-1} = [(b_1(a_1c_{\frac{1}{2}}))d_{\frac{1}{2}}]_{-1}.$$

This implies that $[(a_1, c_{\frac{1}{2}}, b_1)d_{\frac{1}{2}}]_{-1} = 0$. It follows that $(a_1, c_{\frac{1}{2}}, b_1)d_{\frac{1}{2}} = [(a_1, c_{\frac{1}{2}}, b_1)d_{\frac{1}{2}}]_1 \in A_1$. We will now show that $(A_1, A_{\frac{1}{2}}, A_1)A_{-1} = 0$. Since $A_{\frac{1}{2}}A_{-1} \subset A_1 + A_{\frac{1}{2}}$ we have by (24) that

$$(A_1, A_{\frac{1}{2}}, A_1)A_{-1} = [(A_1, A_{\frac{1}{2}}, A_1)A_{-1}]_1 + [(A_1, A_{\frac{1}{2}}, A_1)A_{-1}]_{\frac{1}{2}} = 0.$$

In the two previous results we have shown that $(A_1, A_{\frac{1}{2}}, A_1) \subset M$. Since $M = 0$, we get $(A_1, A_{\frac{1}{2}}, A_1) = 0$. Using this last fact and (1) we obtain

$$\overline{a}([x]_1[y]_{\frac{1}{2}}) = (\overline{a}[x]_1)[y]_{\frac{1}{2}} = [x]_1(\overline{a}[y]_{\frac{1}{2}}).$$

By (2) we have

$$\overline{a}([x]_1[y]_{-1}) = (\overline{a}[x]_1)[y]_{-1} = [x]_1(\overline{a}[y]_{-1}).$$

We know that $A_{\frac{1}{2}}A_{\frac{1}{2}} \subset A_1 + A_{-1}$. Thus $(A_{\frac{1}{2}}, A_1, A_{\frac{1}{2}}) = [(A_{\frac{1}{2}}, A_1, A_{\frac{1}{2}})]_1 + [(A_{\frac{1}{2}}, A_1, A_{\frac{1}{2}})]_{-1} = [(A_{\frac{1}{2}}, A_1, A_{\frac{1}{2}})]_1$ by (4). By (25) we get $[(A_{\frac{1}{2}}, A_1, A_{\frac{1}{2}})]_1 A_{-1} = 0$. This says that $(A_{\frac{1}{2}}, A_1, A_{\frac{1}{2}}) \subset I$. Since $I \subset J$ and we are in the case $J = 0$, we get $(A_{\frac{1}{2}}, A_1, A_{\frac{1}{2}}) = 0$. This last fact and identities (3) and (4) gives

$$\overline{a}([x]_{\frac{1}{2}}[y]_{\frac{1}{2}}) = (\overline{a}[x]_{\frac{1}{2}})[y]_{\frac{1}{2}} = [x]_{\frac{1}{2}}(\overline{a}[y]_{\frac{1}{2}}).$$

Identities (6) and (7) yield

$$\overline{a}([x]_{\frac{1}{2}}[y]_{-1}) = (\overline{a}[x]_{\frac{1}{2}})[y]_{-1} = [x]_{\frac{1}{2}}(\overline{a}[y]_{-1}).$$

Finally, by (10) we obtain

$$\overline{a}([x]_{-1}[y]_{-1}) = (\overline{a}[x]_{-1})[y]_{-1} = [x]_{-1}(\overline{a}[y]_{-1}).$$

Lemma 7 *The algebra $\overline{A_1}$ is a field.*

Proof Let N be a nonzero ideal of A_1. Since $(\overline{n}x)y = \overline{n}(xy)$ for any $n \in N$ and $x, y \in A$, $\overline{N}A$ is an ideal of A. Since $\overline{N}e = N \subset \overline{N}A$, $\overline{N}A \neq 0$. Since A is simple, we must have $\overline{N}A = A$. It follows that $\overline{N}A_1 = A_1$. The fact that N is an ideal of A_1 implies that $\overline{N}A_1 \subset N$. Thus $A_1 = \overline{N}A_1 \subset N$ and so $N = A_1$. We have shown that A_1 is a simple algebra. Since we already know that A_1 is commutative and associative, we conclude that A_1 is a field. Since $\overline{A_1}$ is an isomorphic image of A_1 we have that $\overline{A_1}$ is also a field.

We define the symmetric bilinear function $\phi : A \times A \to A_1$ by setting

$$\phi(x, x) = [x]_1^2 + 2[[x]_{\frac{1}{2}}[x]_{\frac{1}{2}}]_1 + [[x]_{\frac{1}{2}}[x]_{-1}]_1 - [x]_{-1}^2.$$

The next step is to prove that for any $x \in A$ we have $x^3 = \overline{\phi(x, x)}\ x$.

To make the notation readable in the next lemmas, we let $a = [x]_1,\ b = [x]_{\frac{1}{2}}$ and $c = [x]_{-1}$. Notice that

$$\phi(x, x) = a^2 + 2[b^2]_1 + [bc]_1 - c^2.$$

Lemma 8 *We have $(b + c)^3 = \overline{\phi(b + c, b + c)}\ (b + c)$.*

Proof We first prove that $(b + c)^3 = \overline{\phi(b + c, b + c)}\ (b + c) + d$, where

$$d = [[b^2]_{-1}b]_1 + [b^2]_{-1}c + 2[[bc]_{\frac{1}{2}}b]_1 + 2[[bc]_{\frac{1}{2}}c]_1.$$

We have

$$(b + c)^3 = (b + c)^2(b + c) = (b^2 + 2bc + c^2)(b + c) =$$
$$b^2b + 2(bc)b + c^2b + b^2c + 2(bc)c + c^2c.$$

We have the following equations:

$$b^2b = [b^2]_1 b + [b^2]_{-1} b = [b^2]_1 b + \left[[b^2]_{-1} b\right]_1 + \left[[b^2]_{-1} b\right]_{\frac{1}{2}} = [b^2]_1 b + \left[[b^2]_{-1} b\right]_1 + 3[b^2]_1 b$$

by (12);

$$2(bc)b = 2[bc]_1 b + 2[bc]_{\frac{1}{2}} b = 2[bc]_1 b + 2\left[[bc]_{\frac{1}{2}} b\right]_1 + 2\left[[bc]_{\frac{1}{2}} b\right]_{-1} = 2[bc]_1 b + 2\left[[bc]_{\frac{1}{2}} b\right]_1 - 3[b^2]_1 c$$

by (13);

$$b^2 c = [b^2]_1 c + [b^2]_{-1} c;$$

$$2(bc)c = 2[bc]_1 c + 2[bc]_{\frac{1}{2}} c = 2[bc]_1 c + 2\left[[bc]_{\frac{1}{2}} c\right]_1 + 2\left[[bc]_{\frac{1}{2}} c\right]_{\frac{1}{2}} = -[bc]_1 c + 2\left[[bc]_{\frac{1}{2}} c\right]_1 - 3bc^2$$

since $[bc]_1 c = 0$ by (18), and by (17). Therefore

$$(b+c)^3 = (4[b^2]_1 + 2[bc]_1 - 2c^2)\ b + (-2[b^2]_1 - [bc]_1 + c^2)\ c + d = (2\ \overline{[b^2]_1} + \overline{[bc]_1} - \overline{c^2})\ (b+c) + d = \overline{\phi(b+c, b+c)}\ (b+c) + d.$$

We now prove that $d = 0$. Since A is an algebra over the field $\overline{A_1}$ (Lemmas 6 and 7), we have

$$((b+c)^3, e, b+c) = (\overline{\phi(b+c, b+c)}\ (b+c) + d, e, b+c) = \overline{\phi(b+c, b+c)}(b+c, e, b+c) + (d, e, b+c).$$

Since $((b+c)^3, e, b+c) = 0$ and $(b+c, e, b+c) = 0$ we get $(d, e, b+c) = 0$. Then $\frac{1}{2}db + 2dc = 0$. This implies $\overline{d}b = 0$. Therefore $\overline{d} = 0$ or $b = 0$. But if $b = 0$ then $d = 0$. In either case, we have $d = 0$.

Lemma 9 *For any $x \in A$ we have $x^3 = \overline{\phi(x,x)}\ x$. Therefore A is a pseudo-composition algebra over $\overline{A_1}$.*

Proof We have $x = a + p$, where $p = b + c$. Therefore

$$x^3 = (a+p)^2(a+p) = (a^2 + 2ap + p^2)(a+p) = a^2a + a^2p + 2(ap)a + 2(ap)p + p^2a + p^2p.$$

Since $a^2a = \overline{a}^2 a$, $a^2 p = \overline{a}^2(\frac{1}{2}b - c)$ and $2(ap)a = 2a(ap) = 2a(\overline{a}(\frac{1}{2}b - c)) = \overline{a}^2(\frac{1}{2}b + 2c)$ we have

$$a^2a + a^2p + 2(ap)a = \overline{a}^2(a + b + c). \tag{26}$$

We have

$$2(ap)p = 2\ (\overline{a}(\frac{1}{2}b - c))\ (b+c) = 2\ \overline{a}(\frac{1}{2}b^2 - \frac{1}{2}bc - c^2) = \overline{a}(b^2 - bc - 2c^2) = a([b^2]_1 - [b^2]_{-1} - [bc]_1 - 2[bc]_{\frac{1}{2}} - 2c^2)$$

and

$$p^2a = ap^2 = a(b^2 + 2bc + c^2) = a([b^2]_1 + [b^2]_{-1} + 2[bc]_1 + 2[bc]_{\frac{1}{2}} + c^2).$$

By Lemma 8 we get

$$p^2p = \overline{\phi(p,p)}\ p = \overline{(2[b^2]_1 + [bc]_1 - c^2)}\ (b+c) = \overline{(2[b^2]_1 + [bc]_1 - c^2)}\ (a+b+c) + (-2[b^2]_1 - [bc]_1 + c^2)a.$$

Thus

$$2(ap)p + p^2a + p^2p = \overline{(2[b^2]_1 + [bc]_1 - c^2)}\ (a+b+c). \tag{27}$$

Adding equations (26) and (27) we obtain

$$x^3 = \overline{(a^2 + 2[b^2]_1 + [bc]_1 - c^2)}\ (a+b+c) = \overline{\phi(x,x)}\ x.$$

Therefore $x^3 = \overline{\phi(x,x)}\ x$ for any $x \in A$. We have proved that A is a pseudo-composition algebra over $\overline{A_1}$.

8 Further Remarks

In this section we determine necessary and sufficient conditions for a 3-Jordan algebra with idempotent to be a Jordan algebra, and for a 3-Jordan algebra with idempotent to be pseudo-composition algebra.

Theorem 2 *Let A be a 3-Jordan algebra. Suppose that A contains an idempotent e with Peirce decomposition $A = A_1 \oplus A_{\frac{1}{2}} \oplus A_{-1} \oplus A_0$. Then A is a Jordan algebra if and only $A_{-1} = 0$ and A_0 is a Jordan subalgebra of A.*

Proof If A is a Jordan algebra then its Peirce decomposition is $A = A_1 \oplus A_{\frac{1}{2}} \oplus A_0$ and the relations

$$A_1^2 \subset A_1,\quad A_1A_{\frac{1}{2}} \subset A_{\frac{1}{2}},\quad A_1A_0 = 0,\quad A_{\frac{1}{2}}^2 \subset A_1 + A_0,\quad A_{\frac{1}{2}}A_0 \subset A_{\frac{1}{2}},\quad A_0^2 \subset A_0 \tag{28}$$

hold (see Osborn [10], p. 219). In particular, $A_{-1} = 0$ and A_0 is a Jordan subalgebra of A.

Conversely, assume that $A_{-1} = 0$ and A_0 is a Jordan subalgebra of A. Then by Proposition 1 relations (28) are satisfied and identities (1), (3), (5), (9), (12), (14), (16) and (21) hold. Therefore, by Proposition 6.7 in Osborn [10], A is a Jordan algebra if and only if A_1 and A_0 are Jordan subalgebras of A. By our hypothesis, A_0 is a Jordan subalgebra of A. Therefore, to conclude that A is a Jordan algebra, it remains to prove that A_1 is a Jordan subalgebra of A. Since $A_1^2 \subset A_1$ we have that A_1 is a subalgebra of A. Let $x, y \in A_1$. From $f(x, e, x, y, x) = 0$ we obtain $3(x^2, y, x) + (x^3, y, e) = 0$. Since $(x^3, y, e) = 0$, we get $(x^2, y, x) = 0$. Therefore A_1 is a Jordan algebra.

As a consequence of our Proposition 5 and Theorem 2, and Proposition 3.2 of Meyberg and Osborn [9] we obtain the following result.

Corollary 1 *The algebra $J(\phi, V, P)$ is a Jordan algebra if and only if P is of e-quadratic type satisfying $x^2 + \gamma(x)x = 0$ $(\gamma \neq 0)$.*

Theorem 3 *Let A be a 3-Jordan algebra over K. Suppose that A contains an idempotent e with Peirce decomposition $A = A_1 \oplus A_{\frac{1}{2}} \oplus A_{-1} \oplus A_0$. Then A is a pseudo-composition algebra if and only if $A_1 = Ke$ and $A_0 = 0$.*

Proof If A is a pseudo-composition algebra then it has a Peirce decomposition $A = Ke \oplus A_{\frac{1}{2}} \oplus A_{-1}$ (see Meyberg and Osborn [9]). Therefore $A_1 = Ke$ and $A_0 = 0$.

Conversely, let A be a 3-Jordan algebra and suppose it has a Peirce decomposition $A = Ke \oplus A_{\frac{1}{2}} \oplus A_{-1}$. In our present case, $\overline{\phi(x,x)} \in K$ and $\phi(x,x) = \overline{\phi(x,x)}e$. As in the Lemma 9, for any $x \in A$, we have that $x^3 = \overline{\phi(x,x)}\ x$. Therefore A is a pseudo-composition algebra over K.

It follows from Proposition 5 and Theorem 3 that $J(\phi, V, P)$ is a 3-Jordan algebra which is not a pseudo-composition algebra.

Corollary 2 *The following assertions are equivalent.*
(i) The algebra A is a 3-Jordan algebra and contains an idempotent e with Peirce decomposition $A = Ke \oplus A_{\frac{1}{2}} \oplus A_{-1} \oplus A_0$ satisfying $A_{-1} = 0$ and $A_0 = 0$ (respectively, $A_{\frac{1}{2}} = 0$ and $A_0 = 0$).
(ii) The algebra A is a pseudo-composition algebra with Peirce decomposition $A = Ke \oplus A_{-1} \oplus A_{\frac{1}{2}}$ satisfying $A_{-1} = 0$ (respectively, $A_{\frac{1}{2}} = 0$).
(iii) The algebra A is of e-quadratic type satisfying $x^2 + \beta(x)x = 0$, $\beta \neq 0$ (respectively, $x^2 + \beta(x)x + \gamma(x,x)e = 0$, $\gamma(e,e) = -3$, $\beta(x) = -\frac{2}{3}\gamma(x,e)$.)

Proof This is an immediate consequence of our Theorem 3 and Proposition 3.2 of Meyberg and Osborn [9].

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