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EXISTENCE OF 2-EXCEPTIONAL BERNSTEIN ALGEBRAS

N. Bezerra[†] , **J. Picanco**[†] and **R. Costa**[‡]

[†]*Centro de Ciências Exatas e Naturais*
Universidade Federal do Pará
Campus do Guamá 66075-000 – Belém – Brazil
e-mail: nbezerra@ufpa.br and jps@ufpa.br

[‡]*Instituto de Matemática e Estatística*
Universidade de São Paulo
Caixa Postal 66281 – Agência Cidade de São Paulo
05315-970 – São Paulo – Brazil
e-mail: rcosta@ime.usp.br

Abstract

Given a Bernstein algebra $\mathbf{A} = Fe \oplus U \oplus V$, the two ordered pairs of integers $(1 + \dim U, \dim V)$ and $(\dim(UV + V^2), \dim U^2)$ are called, respectively, the type and the subtype of $\mathbf{A}$. In this paper we determine the minimum and the maximum dimension of the subspace $UV + V^2$ in 2-exceptional Bernstein algebras (those satisfying $U(UV) \neq 0$ and $U((UV)V) = 0$) and we introduce an algorithm to construct 2-exceptional Bernstein algebras for some types and subtypes.

Introduction

This paper is a natural continuation of [2], where the authors investigate under which conditions, given a quadruple of non negative integers (r, s, t, z), there exists a n-exceptional Bernstein algebra of type $(1 + r, s)$ and subtype (t, z), for $n = 0, 1$. In this article, we study some properties of n-exceptional algebras for $n = 2$ and construct 2-exceptional algebras for some particular types and subtypes.

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We recall that a baric algebra over a field F is a pair $(\mathbf{A}, \omega)$, where $\mathbf{A}$ is not necessarily associative algebra over F and $\omega : \mathbf{A} \to F$ is nonzero homomorphism. A baric algebra $(\mathbf{A}, \omega)$ is Bernstein if $\mathbf{A}$ is commutative and $(x^2)^2 = \omega(x)^2 x^2$, for every $x \in \mathbf{A}$.

Every Bernstein algebra possesses at least one nonzero idempotent. If F is a field of characteristic not 2, then for every nonzero idempotent e, $\mathbf{A}$ has a Peirce decomposition $\mathbf{A} = Fe \oplus U_e \oplus V_e$ relative to e, where $U_e = \{x \in \mathbf{A} \mid 2ex = x\}$, $V_e = \{x \in \mathbf{A} \mid ex = 0\}$ and $Ker\omega = U_e \oplus V_e$. Unless necessary, we omit the subscript e in U_e and V_e.

In a Peirce decomposition $\mathbf{A} = Fe \oplus U \oplus V$, the subspaces U and V verify the relations $U^2 \subseteq V$, $UV \subseteq U$ and $V^2 \subseteq U$ and the following identities hold for all $u \in U$ and $v \in V$:

$$u^3 = 0, \quad uv^2 = 0, \quad u(uv) = 0, \quad (uv)^2 = 0, \quad (u^2)^2 = 0. \tag{1}$$

All the linearizations of the equations (1) are also identities in $\mathbf{A}$. In particular, $u_1(u_2v) + u_2(u_1v) = 0$, for all $u_1, u_2 \in U$, $v \in V$.

In this paper, we consider only finite dimensional Bernstein algebras over fields of characteristic not 2. Let $\mathbf{A} = Fe \oplus U \oplus V$ be a Bernstein algebra. It is known that the dimensions of U, V, $UV + V^2$ and U^2 are invariant under change of idempotent. The ordered pairs $(1+\dim U, \dim V)$ and $(\dim(UV+V^2), \dim U^2)$, which are well defined, are called the *type* and the *subtype* of $\mathbf{A}$, respectively.

A Bernstein algebra $\mathbf{A}$ is said Jordan-Bernstein if it is also Jordan. In [4] it was proved that $\mathbf{A} = Fe \oplus U_e \oplus V_e$ is Jordan-Bernstein if and only if $V_e^2 = 0$ and $(uv)v = 0$, for all $u \in U_e$, $v \in V_e$. If $\mathbf{A} = Fe \oplus U_e \oplus V_e$ is a Peirce decomposition of a Bernstein algebra $\mathbf{A}$, then the set $L = \{x \in U_e \mid xu = 0 \text{ for all } u \in U_e\}$ is an ideal of $\mathbf{A}$ contained in U_e, which is independent on the idempotent. The quotient algebra $(\overline{\mathbf{A}}, \overline{\omega})$, where $\overline{\mathbf{A}} = \mathbf{A}/L$ and $\overline{\omega}(a+L) = \omega(a)$, for all $a \in \mathbf{A}$, is Jordan-Bernstein. In the Peirce decomposition $\overline{\mathbf{A}} = F\overline{e} \oplus U_{\overline{e}} \oplus V_{\overline{e}}$ relative to the idempotent $\overline{e} = e + L$, we have $U_{\overline{e}} = \overline{U}_e := U_e/L$ and $V_{\overline{e}} = \overline{V}_e := (V_e + L)/L$. For a subspace X of a Bernstein $\mathbf{A}$, we will denote by $\overline{X}$ the quotient $(X+L)/L$. All these facts are well known and can be found in [5], [6], [7] and [8].

If X and Y are subspaces of a Bernstein algebra $\mathbf{A}$, we define $XY^{(0)} = X$ and $XY^{(k)} = (XY^{(k-1)})Y$, k integer ≥ 1, where $XY = \langle xy \mid x \in X, y \in Y \rangle$.

A Bernstein algebra $\mathbf{A} = Fe \oplus U_e \oplus V_e$ is called *exceptional of degree* n, or n-*exceptional*, if n is the least non negative integer such that the subspace $U_e(U_eV_e^{(n)}) = 0$, for some idempotent e. The integer n is called the degree of exceptionality of $\mathbf{A}$. If $\mathbf{A}$ satisfies $U_e^2 = V_e$ then $\mathbf{A}$ is said to be *nuclear*. These definitions do not depend on the choice of the idempotent element. It was proved in [2] that every Bernstein algebra of type $(1+r, s)$ is n-exceptional for some integer n, with $0 \leq n \leq s+1$.

For Bernstein algebras with degree of exceptionality greater than or equal to 1, we have the following result.

Proposition 1. *If $\mathbf{A} = Fe \oplus U_e \oplus V_e$ is a n-exceptional Bernstein algebra of finite dimension with $n \geq 1$, then for every idempotent element of $\mathbf{A}$ the chain of subspaces*

$$\overline{U_e} \supseteq \overline{U_eV_e} \supseteq \overline{(U_eV_e)V_e} \supseteq ... \supseteq \overline{U_eV_e^{(k)}}$$

is strictly decreasing for every integer $k < n$ and $\overline{U_eV_e^{(n)}} = \overline{0}$.

Proof Let e be an arbitrary idempotent of $\mathbf{A}$. As $\mathbf{A}$ is n-exceptional, $U_e(U_eV_e^{(k)}) \neq 0$, for every $k < n$ and $U_e(U_eV_e^{(n)}) = 0$, thus $\overline{U_eV_e^{(n)}} = \overline{0}$ and $\overline{U_eV_e^{(k)}} \neq \overline{0}$, for every $k < n$. Therefore the chain $\overline{U_e} \supseteq \overline{U_eV_e} \supseteq \overline{(U_eV_e)V_e} \supseteq ... \supseteq \overline{U_eV_e^{(n-1)}} \neq \overline{0}$ is strictly decreasing, because if $\overline{U_eV_e^{(k)}} = \overline{U_eV_e^{(k+1)}}$, for some integer k, $0 \leq k \leq n-1$, then $\overline{0} \neq \overline{U_eV_e^{(k)}} = \overline{U_eV_e^{(k+1)}} = ... = \overline{U_eV_e^{(n)}} = \overline{0}$, a contradiction. □

2-exceptional Bernstein algebras

The aim of this section is to study some general properties of 2-exceptional Bernstein algebras.

Proposition 2. *If $\mathbf{A} = Fe \oplus U \oplus V$ is a n-exceptional Bernstein algebra of type $(1+r, s)$ with $n \geq 2$, then*
(i) $r \geq 4$ and $s \geq 2$;
(ii) $\dim L \leq r-4$;
(iii) $\dim \overline{UV} \geq 2$.

Proof Note that $\dim \overline{U(UV)} = \dim U(UV) \neq 0$, since $n \geq 2$. Then there exist $\bar{u}_1, \bar{u}_2 \in \overline{U}$, $\bar{v} \in \overline{V}$, such that $\bar{u}_1(\bar{u}_2\bar{v}) = -\bar{u}_2(\bar{u}_1\bar{v}) \neq \overline{0}$. It was proved in [3, Prop. 9] that both sets $\{\bar{u}_1, \bar{u}_2, \bar{u}_1\bar{v}, \bar{u}_2\bar{v}\}$ and $\{\bar{u}_1(\bar{u}_2\bar{v}), \bar{v}\}$ are linearly independent. This establishes (i), (ii) and (iii). □

Now we investigate bounds to the dimension of the subspace $UV + V^2$ in 2-exceptional algebras.

Let $\mathbf{A} = Fe \oplus U_e \oplus V_e$ be the Peirce decomposition of a Bernstein algebra with respect to an idempotent e. For a basis $B = \{\overline{u}_1, \overline{u}_2, ..., \overline{u}_t\}$ of $\overline{U_eV_e}$, the set $\{u_1, u_2, ..., u_t\}$, where u_i $(i = 1, 2, ..., t)$ is a representative of the equivalence class $\overline{u}_i$, is a linearly independent set of U_e. Let $M = \langle u_1, u_2, ..., u_t \rangle$ be the subspace of $\mathbf{A}$ generated by this set. It is clear that $\dim M = t$, independent of the basis and of the representatives u_i chosen. Moreover, $\dim M = \dim \overline{U_eV_e}$, which is invariant under change of idempotent. Therefore $\dim M$ is an invariant of the algebra. By construction, $M \subset U_eV_e + L$ and $M \cap L = 0$. If $\mathbf{A}$ is a n-exceptional Bernstein algebra with $n \leq 2$, then, for every idempotent, $M^2 = 0$, independent of the basis of $\overline{U_eV_e}$ and of the representatives u_i.

Proposition 3. *If* $\mathbf{A} = Fe \oplus U \oplus V$ *is a 2-exceptional Bernstein algebra of type* $(1 + r, s)$*, then* $2 \leq \dim(UV + V^2) \leq r - 2$.

Proof As $\mathbf{A}$ is 2-exceptional, $\dim \overline{UV} \geq 2$, according to Proposition 2. We consider $\dim L = r - k$, with $4 \leq k \leq r$. By Proposition 1, $\dim \overline{UV} \leq \dim \overline{U} - 1 = k - 1$. Suppose, by contradiction, that $\dim \overline{UV} = k - 1$. Let $\{\bar{u}_1, \bar{u}_2, ..., \bar{u}_{k-1}\}$ be a basis of $\overline{UV}$ and let $M = \langle u_1, u_2, ..., u_{k-1} \rangle$ be the subspace as previously defined. We have $M^2 = 0$, then $u_1 M = 0 = u_1 L$. As $u_1 \notin L$, since $\overline{u}_1 \neq \overline{0}$, there exists $u \in U$ such that $u_1 u \neq 0$. If $\alpha \in F$, $m \in M$ and $l \in L$ are such that $\alpha u = m + l$, then $\alpha u_1 u = u_1 m + u_1 l = 0$ and so $\alpha = 0$. Therefore, $U = \langle u \rangle \oplus M \oplus L$. Thus, $M \subseteq UV + L = (\langle u \rangle + M + L)V + L \subseteq uV + L$. Then there exist $v \in V$, $l \in L$ such that $u_1 = uv + l$. This implies that $0 \neq uu_1 = u(uv) + ul = 0$, a contradiction. Then $2 \leq \dim \overline{UV} \leq k - 2$ and consequently $2 \leq \dim(UV + V^2) \leq r - 2$. □

Some 2-exceptional Bernstein algebras

A natural question is to try to improve the bounds given in Proposition 3 or to investigate wether these values can be reached. Let us show, by construction, that there exist 2-exceptional Bernstein algebras with $\dim(UV + V^2) = t$, for $2 \leq t \leq \dim U - 2$. An estimate for the maximum dimension of the subspace U^2 is a hard problem. Thus we construct 2-exceptional Bernstein algebras where the dimension of the subspace $UV + V^2$ reaches the minimum and maximum values given in Proposition 3 for some values of the dimension of U^2.

In what follows, we denote by $\lceil\ \rceil : \mathbb{R} \to \mathbb{Z}$ the mapping defined by $\lceil x \rceil = n$, where $n - 1 < x \leq n$ and n is an integer.

Theorem 1. *Let* r, s, t *and* z *be integers satisfying any one of the following conditions:*

(i) $r \geq 4$, $s \geq 2$, $2 \leq t \leq r - 2$ *and* $\frac{1}{2}(r - t)(r - t + 1) < z \leq min\{\frac{1}{2}[(r - t)(r - t + 1) + (r - t - 1)t - r_1(r - t - r_1)], s - \lceil \frac{t}{r - t} \rceil\}$*, where* r_1 *is the remainder of division of the integer* t *by* $r - t$*;*

(ii) $t = \frac{1}{3}p(p^2 - 1)$*, for some integer* $p \geq 3$, $r = t + p$ *and* $s = z = \frac{1}{8}p(p + 1)(p^2 - 3p + 6)$.

Then there exists a 2-exceptional Bernstein algebra of type $(1 + r, s)$ *and subtype* (t, z).

Proof The proof is an algorithm to construct such algebra. Initially we make the following attribution of values for m, n and p, integers used as bound to the dimension of the some subspace in algorithm: if the integers satisfy the conditions of item (i), take $m = n = 0$ and $p = r - t$. Otherwise, take $m = t$ and $n = \frac{1}{8}(p - 2)(p - 1)p(p + 1)$. Furthermore, let $m_1 = \frac{1}{2}p(p + 1)$, q and r_1, respectively, the quotient and remainder of division of $(t - m)$ by p. The letters i, j, k and l, used as indexes, represent always integer values.

Let $\mathbf{A}$ be the vector space over F having $\{e, u_1, u_2, ..., u_r, v_1, v_2, ..., v_s\}$ as a basis. We define in $\mathbf{A}$ the commutative product given by:

(1). $e^2 = e; \quad eu_i = \frac{1}{2}u_i; \quad ev_j = 0 \quad (i = 1, 2, ..., r \quad and \quad j = 1, ..., s);$

(2). $u_i u_j = v_{\epsilon(i,j)}$, if $1 \leq i \leq j \leq p$, where $\epsilon(i,j) = \frac{1}{2}(2p - i)(i - 1) + j;$

(3). $u_i^2 u_j = -2u_i(u_i u_j) = 2u_{\rho_1(i,j)};$
$u_j^2 u_i = -2u_j(u_i u_j) = 2u_{\rho_2(i,j)};$
if $1 \leq i < j \leq p$ and $\rho_w(i,j) \leq p + m$ for $w = 1, 2$, where
$\rho_w(i,j) = p + \sum_{x=1}^{i-1}[(p - x)(p - (x - 1)] + (w - 1)(p - i) + (j - i);$

(4). $(u_i u_j)u_k = u_{\tau_1(i,j,k)};$
$(u_j u_k)u_i = u_{\tau_2(i,j,k)};$
if $1 \leq i < j < k \leq p$ and $\tau_w(i,j,k) \leq p + m$ for $w = 1, 2$, where
$\tau_w(i,j,k) = p + \sum_{x=1}^{i-1}[(p - x)(p - (x - 1)] + 2(p - i) + (2p - (i + j))(j - i - 1) + 2(k - j) - 2 + w;$

(5). $(u_i u_k)u_j = -[(u_i u_j)u_k + (u_j u_k)u_i]$, if $1 \leq i < j < k \leq p;$

(6). $(u_i^2 u_j)u_k = -(u_i^2 u_k)u_j = 2(u_i u_j . u_k)u_i = 2v_{\delta_1(i,j,k)};$
$(u_j^2 u_i)u_k = -(u_j^2 u_k)u_i = 2(u_i u_j . u_k)u_j = -2(u_j u_k . u_i)u_j = 2v_{\delta_2(i,j,k)};$
$(u_k^2 u_i)u_j = -(u_k^2 u_j)u_i = 2(u_i u_k . u_j)u_k = -2(u_j u_k . u_i)u_k = 2v_{\delta_3(i,j,k)};$
if $1 \leq i < j < k \leq p$ and $\delta_w(i,j,k) \leq min\{z, m_1 + n\}$ for $w = 1, 2, 3$, where
$\delta_w(i,j,k) = m_1 + \sum_{x=1}^{i-1} \frac{3}{2}[(p - x)(p - (x + 1))] + \frac{3}{2}(2p - (i + j))(j - i - 1) + 3(k - j) - 3 + w;$

(7). $(u_i u_j . u_k)u_l = -(u_i u_j . u_l)u_k = v_{\gamma_1(i,j,k,l)};$
$(u_j u_k . u_i)u_l = -(u_j u_k . u_l)u_i = v_{\gamma_2(i,j,k,l)};$
$(u_k u_l . u_i)u_j = -(u_k u_l . u_j)u_i = v_{\gamma_3(i,j,k,l)};$
if $1 \leq i < j < k < l \leq p$ and $\gamma_w(i,j,k,l) \leq min\{z, m_1 + n\}$ for $w = 1, 2, 3$, where
$\gamma_w(i,j,k,l) = m_1 + \frac{1}{2}p(p - 1)(p - 2) + \sum_{x=1}^{i-1} \frac{1}{2}[(p - x)(p - (x + 1))(p - (x + 2))] +$
$\sum_{x=i+1}^{j-1} \frac{3}{2}[(p - x)(p - (x + 1))] + 3(2p - (k + j))(k - j - 1) + 3(l - k) - 3 + w;$

(8). $(u_i u_k . u_l)u_j = (u_i u_j . u_k)u_l + (u_j u_k . u_i)u_l;$
$(u_j u_l . u_i)u_k = (u_j u_k . u_l)u_i + (u_k u_l . u_j)u_i$, if $1 \leq i < j < k < l \leq p;$

(9). $u_i v_{z+j} = u_{p+m+(i-1)q+j}$, if $1 \leq i \leq p$ and $1 \leq j \leq q;$

(10). $u_i v_{z+q+1} = u_{m+p(q+1)+i}$, if $1 \leq i \leq r_1;$

(11). $u_i u_{p+m+(j-1)q+k} = -u_j u_{p+m+(i-1)q+k} = v_{\sigma_1(i,j,k)};$
if $1 \leq i < j \leq p$, $1 \leq k \leq q$ e $\sigma_1(i,j,k) \leq z$, where
$\sigma_1(i,j,k) = m_1 + n + [\frac{1}{2}(2p - i)(i - 1) + (j - i - 1)]q + k;$

(12). $u_i u_{m+p(q+1)+j} = -u_j u_{m+p(q+1)+i} = v_{\sigma_2(i,j)}$,
if $1 \leq i < j \leq r_1$ and $\sigma_2(i,j) \leq z$, where
$\sigma_2(i,j) = m_1 + n + \frac{1}{2}p(p - 1)q + \frac{1}{2}(2r_1 - i)(i - 1) + (j - i);$

(13). Other products are zero.

Let $\omega : \mathbf{A} \to F$, defined by $\omega(e) = 1$; $\omega(u_i) = \omega(v_j) = 0$, for $i = 1, 2, .., r$ and $j = 1, 2, ..., s$. In order to show that $(\mathbf{A}, \omega)$ is Bernstein, we will use Theorem 3.4.8 of [5]. Consider $U = \langle u_1, u_2, ..., u_r \rangle$ and $V = \langle v_1, v_2, ..., v_s \rangle$, then $\mathbf{A} = Fe \oplus$

$U \oplus V$. As $p+m \leq r$, we may write $U = \langle u_1, u_2, ..., u_p, u_{p+1}, ..., u_{p+m}, ..., u_r\rangle$. ¿From commutativity of the product it follows that $U^2 = \langle u_iu_j \mid 1 \leq i \leq j \leq r\rangle = \langle u_iu_j \mid 1 \leq i \leq j \leq p\rangle + \langle u_iu_j \mid 1 \leq i \leq p,\ p+1 \leq j \leq p+m\rangle + \langle u_iu_j \mid 1 \leq i \leq p,\ p+m+1 \leq j \leq r\rangle + \langle u_iu_j \mid p+1 \leq i \leq j \leq r\rangle$. ¿From items (3), (4) and (5) above it follows that $\langle u_{p+1}, u_{p+2}, ..., u_{p+m}\rangle = \langle u_i^2u_j, u_j^2u_i \mid 1 \leq i < j \leq p\rangle + \langle (u_iu_j)u_k, (u_ju_k)u_i \mid 1 \leq i < j < k \leq p\rangle$. Thus $U^2 = \langle u_iu_j \mid 1 \leq i \leq j \leq p\rangle + \langle (u_i^2u_j)u_k, (u_j^2u_i)u_k \mid 1 \leq i < j \leq p,\ 1 \leq k \leq p\rangle + \langle (u_iu_j.u_k)u_l, (u_ju_k.u_i)u_l \mid 1 \leq i < j < k \leq p,\ 1 \leq l \leq p\rangle + \langle u_iu_j \mid 1 \leq i \leq p,\ p+m+1 \leq j \leq r\rangle + \langle u_iu_j \mid p+1 \leq i \leq j \leq r\rangle$. Using the products given in items (2), (6), (7), (8), (11), (12) and (13) we obtain $U^2 = \langle v_1, v_2, ..., v_{m_1}, v_{m_1+1}, ..., v_{m_1+n}, v_{m_1+n+1}, ..., v_z\rangle \subseteq V$. Let $W = \langle v_{z+1}, v_{z+2}, ..., v_s\rangle$, then $V = U^2 \oplus W$ e $UV = U^3 + UW$. From (3), (4) and (5), it follows that $U^3 = \langle u_i^2u_j, u_j^2u_i \mid 1 \leq i < j \leq p\rangle + \langle (u_iu_j)u_k, (u_ju_k)u_i \mid 1 \leq i < j \leq p\rangle = \langle u_{p+1}, u_{p+2}, ..., u_{p+m}\rangle$ and from (9) and (10) we obtain $UW = \langle u_iv_{z+j} \mid 1 \leq i \leq p, 1 \leq j \leq q\rangle + \langle u_iv_{z+q+1} \mid 1 \leq i \leq r_1\rangle = \langle u_{p+m+1}, u_{p+m+2}, ..., u_{p+t}\rangle$. Hence $UV = \langle u_{p+1}, u_{p+2}, ..., u_{p+t}\rangle \subseteq U$. Moreover, we have $V^2 = 0$. Let $x = \omega(x)e + u + v \in \mathbf{A}$, where $u = \sum_{i=1}^r \alpha_iu_i$, $v = \sum_{j=1}^s \beta_jv_j$, with $\alpha_1, ..., \alpha_r, \beta_1, ..., \beta_s \in F$. As the product is commutative, from (1) it follows that $x^2 = \omega(x)^2e + (\omega(x)u + 2uv + v^2) + u^2$, with $\omega(x)u + 2uv + v^2 \in U + UV + V^2 = U$ and $u^2 \in U^2 \subseteq V$. It remains to show the identities:

$$uv^2 \;=\; (u^2)^2 \;=\; u^3 \;=\; u(uv) \;=\; (uv)^2 \;=\; 0$$

for all $u = \sum_{i=1}^r \alpha_iu_i \in U$ and $v = \sum_{j=1}^s \beta_jv_j \in V$.

(i) $uv^2 \in UV^2 = 0$, because $V^2 = 0$;

(ii) $(u^2)^2 \in (U^2)^2 \subseteq V^2 = 0$;

(iii) $u^3 = 0$:

From commutativity of the product we have

$$u^2 = \sum_{i=1}^r \alpha_i^2u_i^2 + 2\sum_{i=1}^{r-1}\sum_{j=i+1}^r \alpha_i\alpha_ju_iu_j \text{ and}$$

$$u^3 = \sum_{i=1}^r\sum_{j=1}^r \alpha_i^2\alpha_ju_i^2u_j + 2\sum_{i=1}^{r-1}\sum_{j=i+1}^r\sum_{k=1}^r \alpha_i\alpha_j\alpha_k(u_iu_j)u_k.$$

Consider a product of the form $(u_iu_j)u_k$, with $1 \leq i, j, k \leq r$. If i or $j \geq p+1$, then the product u_iu_j lies in $\langle v_{m_1+n+1}, v_{m_1+n+2}, ..., v_z\rangle$, according to items (11) and (12). Thus $(u_iu_j)u_k = 0$, for every k, because by (9), (10) and (13), $u_kv_\sigma = 0$, for any $\sigma \leq z$. If $i, j \leq p$, then $u_iu_j \in \langle v_1, v_2, ..., v_{m_1}\rangle$, according to (2) and in this case, if $k \geq p+1$, then $(u_iu_j)u_k = 0$, by the same reason. Hence, it is enough to consider $i, j, k \leq p$. Then $u^3 = \sum\limits_{i=1}^p\sum\limits_{j=1}^p \alpha_i^2\alpha_ju_i^2u_j + 2\sum\limits_{i=1}^{p-1}\sum\limits_{j=i+1}^p\sum\limits_{k=1}^p \alpha_i\alpha_j\alpha_k(u_iu_j)u_k = \sum\limits_{1\leq i<j\leq p} \alpha_i^2\alpha_ju_i^2u_j + \sum\limits_{i=1}^p \alpha_i^3u_i^3 + \sum\limits_{1\leq i<j\leq p} \alpha_i\alpha_j^2u_iu_j^2 +$

$2\Big(\sum_{1\le i<j\le p}\alpha_i^2\alpha_j(u_iu_j)u_i+\sum_{1\le i<j\le p}\alpha_i\alpha_j^2(u_iu_j)u_j+\sum_{1\le i<j<k\le p}\alpha_i\alpha_j\alpha_k(u_ju_k)u_i+$
$\sum_{1\le i<j<k\le p}\alpha_i\alpha_j\alpha_k(u_iu_k)u_j+\sum_{1\le i<j<k\le p}\alpha_i\alpha_j\alpha_k(u_iu_j)u_k\Big).$

Using (3), (4), (5) and (13) we get:

$u^3=\sum_{1\le i<j\le p}\Big(2\alpha_i^2\alpha_ju_{\rho_1(i,j)}+2\alpha_i\alpha_j^2u_{\rho_2(i,j)}+2(-\alpha_i^2\alpha_ju_{\rho_1(i,j)}-\alpha_i\alpha_j^2u_{\rho_2(i,j)})\Big)+$
$\sum_{1\le i<j<k\le p}2\alpha_i\alpha_j\alpha_k\Big(u_{\tau_2(i,j,k)}-(u_{\tau_1(i,j,k)}+u_{\tau_2(i,j,k)})+u_{\tau_1(i,j,k)}\Big)=0.$

(iv) $u(uv)=0$:

Initially we calculate the product uv. Let $W_1=\langle v_1,v_2,...,v_{m_1}\rangle$, $W_2=\langle v_{m_1+1},v_{m_1+2},...,v_z\rangle$ and $W_3=\langle v_{z+1},v_{z+2},...,v_s\rangle$. Then $V=W_1\oplus W_2\oplus W_3$ and thus $v=w_1+w_2+w_3$ with $w_i\in W_i$, for $i=1,2,3$. By the rules of the product it follows that $UW_2=0$, therefore $uv=uw_1+uw_3$. By (2), $W_1=\langle u_iu_j\mid 1\le i\le j\le p\rangle$, then there exist $\beta_{ij}\in F$ $(1\le i\le j\le p)$ such that $w_1=\sum_{i=1}^p\sum_{j=i}^p\beta_{ij}u_iu_j$. Thus $uw_1=\sum_{1\le i\le j\le p}\sum_{k=1}^{r}\alpha_k\beta_{ij}(u_iu_j)u_k=$
$\sum_{i=1}^{p}\sum_{k=1}^{r}\alpha_k\beta_{ii}u_i^2u_k+\sum_{1\le i<j\le p}\sum_{k=1}^{p}\alpha_k\beta_{ij}(u_iu_j)u_k=\sum_{1\le i<j\le p}\Big(\alpha_i\beta_{jj}u_iu_j^2+\alpha_j\beta_{ii}u_i^2u_j+$
$\alpha_i\beta_{ij}(u_iu_j)u_i+\alpha_j\beta_{ij}(u_iu_j)u_j\Big)+\sum_{1\le i<j<k\le p}\Big(\alpha_i\beta_{jk}(u_ju_k)u_i+\alpha_j\beta_{ik}(u_iu_k)u_j+$
$\alpha_k\beta_{ij}(u_iu_j)u_k\Big).$

Using (3) and (5) we get:

$$uw_1=\sum_{1\le i<j\le p}\Big((\alpha_i\beta_{jj}-\frac{1}{2}\alpha_j\beta_{ij})u_iu_j^2+(\alpha_j\beta_{ii}-\frac{1}{2}\alpha_i\beta_{ij})u_i^2u_j\Big)+$$
$$+\sum_{1\le i<j<k\le p}\Big((\alpha_i\beta_{jk}-\alpha_j\beta_{ik})(u_ju_k)u_i+(\alpha_k\beta_{ij}-\alpha_j\beta_{ik})(u_iu_j)u_k\Big).\quad(2)$$

As $z\le s-\lceil\frac{t-m}{p}\rceil$, then $z+q\le s$, thus $W_3=<v_{z+1},v_{z+2},...,v_{z+q},...,v_s>$. Let $\lambda_{z+j}\in F$ with $j=1,2,...,s-z$ such that $w_3=\sum_{j=1}^{s-z}\lambda_{z+j}v_{z+j}$. Then $uw_3=\sum_{j=1}^{s-z}\lambda_{z+j}uv_{z+j}=\sum_{j=1}^{q}\lambda_{z+j}uv_{z+j}+\lambda_{z+q+1}uv_{z+q+1}+\sum_{j=q+2}^{s-(z+1)}\lambda_{z+j}uv_{z+j}+$
λ_suv_s. Using (9) and (10), it follows:

$$uw_3=\sum_{i=1}^{p}\sum_{j=1}^{q}\alpha_i\lambda_{z+j}u_{p+m+(i-1)q+j}+\sum_{i=1}^{r_1}\alpha_i\lambda_{z+q+1}u_{m+p(q+1)+i}.\quad(3)$$

From the identities given in (2) and (3) we get:

$$\begin{aligned}uv=&\sum_{1\le i<j\le p}\Big((\alpha_i\beta_{jj}-\tfrac{1}{2}\alpha_j\beta_{ij})u_iu_j^2+(\alpha_j\beta_{ii}-\tfrac{1}{2}\alpha_i\beta_{ij})u_i^2u_j\Big)\\&+\sum_{1\le i<j<k\le p}\Big((\alpha_i\beta_{jk}-\alpha_j\beta_{ik})(u_ju_k)u_i+(\alpha_k\beta_{ij}-\alpha_j\beta_{ik})(u_iu_j)u_k\Big)\\&+\sum_{i=1}^{p}\sum_{j=1}^{q}\alpha_i\lambda_{z+j}u_{p+m+(i-1)q+j}+\sum_{i=1}^{r_1}\alpha_i\lambda_{z+q+1}u_{m+p(q+1)+i}\ .\end{aligned}\quad(4)$$

In order to establish the product $u(uv)$, we calculate in separate forms, the products of u by each of the parts given in (4).

1) $\sum_{1\le i<j\le p}(\alpha_i\beta_{jj}-\frac{1}{2}\alpha_j\beta_{ij})(u_iu_j^2)u=\sum_{1\le i<j\le p}\sum_{k=1}^{r}\alpha_k(\alpha_i\beta_{jj}-\frac{1}{2}\alpha_j\beta_{ij})(u_iu_j^2)u_k$ $=\sum_{1\le i<j<k\le p}\Big(\alpha_i(\alpha_j\beta_{kk}-\frac{1}{2}\alpha_k\beta_{jk})(u_ju_k^2)u_i+\alpha_j(\alpha_i\beta_{kk}-\frac{1}{2}\alpha_k\beta_{ik})(u_iu_k^2)u_j$ $+\alpha_k(\alpha_i\beta_{jj}-\frac{1}{2}\alpha_j\beta_{ij})(u_iu_j^2)u_k\Big)+\sum_{1\le i<j\le p}\Big((\alpha_i^2\beta_{jj}-\frac{1}{2}\alpha_i\alpha_j\beta_{ij})(u_iu_j^2)u_i+(\alpha_i\alpha_j\beta_{jj}$ $-\frac{1}{2}\alpha_j^2\beta_{ij})(u_iu_j^2)u_j)\Big)$. Using the items (6), (7), (8) and (14) we get:

$$\sum_{1\le i<j\le p}(\alpha_i\beta_{jj}-\frac{1}{2}\alpha_j\beta_{ij})(u_iu_j^2)u= \tag{5}$$

$$=\sum_{1\le i<j<k\le p}\Big(\alpha_i\alpha_k\beta_{jk}-\alpha_j\alpha_k\beta_{ik})v_{\delta_3(i,j,k)}+(2\alpha_i\alpha_k\beta_{jj}-\alpha_j\alpha_k\beta_{ij})v_{\delta_2(i,j,k)}\Big);$$

2) $\sum_{1\le i<j\le p}(\alpha_j\beta_{ii}-\frac{1}{2}\alpha_i\beta_{ij})(u_i^2u_j)u=\sum_{1\le i<j\le p}\sum_{k=1}^{r}\alpha_k(\alpha_j\beta_{ii}-\frac{1}{2}\alpha_i\beta_{ij})(u_i^2u_j)u_k$ $=\sum_{1\le i<j<k\le p}\Big((\alpha_i\alpha_k\beta_{jj}-\frac{1}{2}\alpha_i\alpha_j\beta_{jk})(u_j^2u_k)u_i+(\alpha_j\alpha_k\beta_{ii}-\frac{1}{2}\alpha_i\alpha_j\beta_{ik})(u_i^2u_k)u_j$ $+(\alpha_j\alpha_k\beta_{ii}-\frac{1}{2}\alpha_i\alpha_k\beta_{ij})(u_i^2u_j)u_k\Big)$. Using the items (6), (7) and (8) of the product we get:

$$\sum_{1\le i<j\le p}(\alpha_j\beta_{ii}-\frac{1}{2}\alpha_i\beta_{ij})(u_i^2u_j)u= \tag{6}$$

$$\sum_{1\le i<j<k\le p}\Big((\alpha_i\alpha_j\beta_{jk}-2\alpha_i\alpha_k\beta_{jj})v_{\delta_2(i,j,k)}+(\alpha_i\alpha_j\beta_{ik}-\alpha_i\alpha_k\beta_{ij})v_{\delta_1(i,j,k)}\Big);$$

3) $\sum_{1\le i<j<k\le p}(\alpha_i\beta_{jk}-\alpha_j\beta_{ik})(u_ju_k.u_i)u=\sum_{1\le i<j<k\le p}\sum_{l=1}^{r}\alpha_l(\alpha_i\beta_{jk}-\alpha_j\beta_{ik})(u_ju_k.u_i)u_l$ $=\sum_{1\le i<j<k<l\le p}\Big(\alpha_i\alpha_j\beta_{kl}-\alpha_i\alpha_k\beta_{jl})(u_ku_l.u_j)u_i+(\alpha_i\alpha_j\beta_{kl}-\alpha_j\alpha_k\beta_{il})(u_ku_l.u_i)u_j$ $+(\alpha_i\alpha_k\beta_{jl}-\alpha_j\alpha_k\beta_{il})(u_ju_l.u_i)u_k+(\alpha_i\alpha_l\beta_{jk}-\alpha_j\alpha_l\beta_{ik})(u_ju_k.u_i)u_l\Big)$ $+\sum_{1\le i<j<k\le p}\Big((\alpha_i\alpha_j\beta_{jk}-\alpha_j^2\beta_{ik})(u_ju_k.u_i)u_j+(\alpha_i\alpha_k\beta_{jk}-\alpha_k\alpha_j\beta_{ik})(u_ju_k.u_i)u_k\Big)$.

Using now the items (6), (7) and (8) we have:

$$\sum_{1\le i<j<k\le p} (\alpha_i\beta_{jk}-\alpha_j\beta_{ik})(u_ju_k.u_i)u = \\ =\sum_{1\le i<j<k<l\le p}\Big(\alpha_k\alpha_j\beta_{il}-\alpha_i\alpha_k\beta_{jl}+\alpha_i\alpha_l\beta_{jk}-\alpha_j\alpha_l\beta_{ik}\Big)v_{\gamma_2(i,j,k,l)}+ \\ \textstyle\sum_{1\le i<j<k\le p}\Big((\alpha_j^2\beta_{ik}-\alpha_i\alpha_j\beta_{jk})v_{\delta_2(i,j,k)}+(\alpha_j\alpha_k\beta_{ik}-\alpha_i\alpha_k\beta_{jk})v_{\delta_3(i,j,k)}\Big); \tag{7}$$

4) $\sum_{1\le i<j<k\le p}(\alpha_k\beta_{ij}-\alpha_j\beta_{ik})(u_iu_j.u_k)u=\sum_{1\le i<j<k\le p}\sum_{l=1}^{r}\alpha_l(\alpha_k\beta_{ij}-\alpha_j\beta_{ik})(u_iu_j.u_k)u_l$
$=\sum_{1\le i<j<k<l\le p}\Big((\alpha_i\alpha_l\beta_{jk}-\alpha_k\alpha_i\beta_{jl})(u_ju_k.u_l)u_i+(\alpha_j\alpha_l\beta_{ik}-\alpha_k\alpha_j\beta_{il})(u_iu_k.u_l)u_j$
$+(\alpha_k\alpha_l\beta_{ij}-\alpha_j\alpha_k\beta_{il})(u_iu_j.u_l)u_k+(\alpha_k\alpha_l\beta_{ij}-\alpha_j\alpha_l\beta_{ik})(u_iu_j.u_k)u_l\Big)$
$+\sum_{1\le i<j<k\le p}\Big((\alpha_i\alpha_k\beta_{ij}-\alpha_i\alpha_j\beta_{ik})(u_iu_j.u_k)u_i+(\alpha_j\alpha_k\beta_{ij}-\alpha_j^2\beta_{ik})(u_iu_j.u_k)u_j\Big).$

From items (6), (7), (8) and (13) it follows that:

$$\sum_{1\le i<j<k\le p} (\alpha_k\beta_{ij}-\alpha_j\beta_{ik})(u_iu_j.u_k)u = \\ =\sum_{1\le i<j<k<l\le p}\Big(\alpha_i\alpha_k\beta_{jl}+\alpha_j\alpha_l\beta_{ik}-\alpha_i\alpha_l\beta_{jk}-\alpha_j\alpha_k\beta_{il}\Big)v_{\gamma_2(i,j,k,l)} \\ +\sum_{1\le i<j<k\le p}\Big((\alpha_i\alpha_k\beta_{ij}-\alpha_i\alpha_j\beta_{ik})v_{\delta_1(i,j,k)}+(\alpha_j\alpha_k\beta_{ij}-\alpha_j^2\beta_{ik})v_{\delta_2(i,j,k)}\Big); \tag{8}$$

5) $\sum_{i=1}^{p}\sum_{j=1}^{q}\alpha_i\lambda_{z+j}u_{p+m+(i-1)q+j}u=\sum_{i=1}^{r}\sum_{j=1}^{p}\sum_{k=1}^{q}\alpha_i\alpha_j\lambda_{z+k}u_iu_{p+m+(j-1)q+k}$
$=\sum_{1\le i<j\le p}\sum_{k=1}^{q}\Big(\alpha_i\alpha_j\lambda_{z+k}u_iu_{p+m+(j-1)q+k}+\alpha_j\alpha_i\lambda_{z+k}u_ju_{p+m+(i-1)q+k}\Big)$. Using the item (11):

$$\sum_{i=1}^{p}\sum_{j=1}^{q}\alpha_i\lambda_{z+j}u_{p+m+(i-1)q+j}u=\sum_{1\le i<j\le p}\sum_{k=1}^{q}\alpha_i\alpha_j\lambda_{z+k}\Big(v_{\sigma_1(i,j,k)}-v_{\sigma_1(i,j,k)}\Big)=0; \tag{9}$$

6) $\sum_{i=1}^{r_1}\alpha_i\lambda_{z+q+1}u_{m+p(q+1)+i}u=\sum_{i=1}^{r}\sum_{j=1}^{r_1}\alpha_i\alpha_j\lambda_{z+q+1}u_iu_{p+m+p(q+1)+j}$
$=\sum_{1\le i<j\le r_1}\Big(\alpha_i\alpha_j\lambda_{z+q+1}u_iu_{p+m+p(q+1)+j}+\alpha_j\alpha_i\lambda_{z+q+1}u_ju_{m+p(q+1)+i}\Big)$. Using the item (12):

$$\sum_{i=1}^{r_1}\alpha_i\lambda_{z+q+1}u_{m+p(q+1)+i}u=\sum_{1\le i<j\le r_1}\alpha_i\alpha_j\lambda_{z+q+1}\Big(v_{\sigma_2(i,j)}-v_{\sigma_2(i,j)}\Big)=0; \tag{10}$$

From identities (5) to (10), it follows that $u(uv) = 0$.

(v) $(uv)^2 = 0$: As we have just seen, $UV = \langle u_{p+1}, u_{p+2}, ..., u_r\rangle$, then $(UV)^2 = \langle u_i u_j \mid p+1 \leq i \leq j \leq r\rangle = 0$, because the products $u_i u_j \neq 0$, only with at least one of the indexes less than or equal to p. Thus $(uv)^2 \in (UV)^2 = 0$.

In this way, we showed that $(\mathbf{A}, \omega)$ is a Bernstein algebra with $U_e = U$ and $V_e = V$. As $U_eV_e + V_e^2 = U_eV_e = \langle u_{p+1}, u_{p+2}, ..., u_r\rangle$ and $U_e^2 = \langle v_1, v_2, ..., v_z\rangle$, then $\mathbf{A}$ is of type $(1 + r, s)$ and subtype (t, z). Moreover, $U_e(U_eV_e) = \langle u_i u_j \mid 1 \leq i \leq r, p+1 \leq j \leq r\rangle = \langle v_{m_1+1}, v_{m_1+2}, ..., v_z\rangle \neq 0$, because $z \geq m_1 + 1$. If W_1, W_2 and W_3 are the previous defined subspaces, then $(U_eV_e)V_e = (U_eV_e)W_1 + (U_eV_e)W_2 + (U_eV_e)W_3 = (U_eV_e)W_1 + (U_eV_e)W_3$. By items (3), (4), (5) and (13) of the product, $(U_eV_e)W_1 = \langle u_{p+i}v_j \mid 1 \leq i \leq t,\ 1 \leq j \leq m_1\rangle = \langle (u_j u_k)u_{p+i} \mid 1 \leq i \leq t,\ 1 \leq j \leq k \leq p\rangle = 0$. By (9), (10) and (13) it follows that$(U_eV_e)W_3 = \langle u_{p+i}v_{z+j} \mid 1 \leq i \leq t,\ 1 \leq j \leq s - (z+1)\rangle = 0$. Thus $U_e((U_eV_e)V_e) = 0$. Therefore $\mathbf{A}$ is 2-exceptional. □

The next example exhibits 2-exceptional Bernstein algebras constructed according to Theorem 1. In Example 1, we have a non nuclear Bernstein algebra of dimension 65 for the quadruple $(r, s, t, z) = (30, 34, 24, 26)$ and in the second case, we have a nuclear Bernstein algebra of dimension 50 for quadruple $(r, s, t, z) = (24, 25, 20, 25)$.

Example 1. Let $\mathbf{A} = Fe \oplus U \oplus V$ be a Bernstein algebra with $U = \langle u_1, u_2, ..., u_{30}\rangle$, $V = \langle v_1, v_2, ..., v_{34}\rangle$ and multiplication table in $N = U \oplus V$ given by:

Table of U^2:

	u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8	u_9	u_{10}	u_{11}	u_{12}	u_{13}	u_{14}	u_{15}	...	u_{30}
u_1	v_1	v_2	v_3	v_4	v_5	v_6					v_{22}	v_{23}	v_{24}	v_{25}	v_{26}		
u_2	v_2	v_7	v_8	v_9	v_{10}	v_{11}	$-v_{22}$	$-v_{23}$	$-v_{24}$	$-v_{25}$							
u_3	v_3	v_8	v_{12}	v_{13}	v_{14}	v_{15}	$-v_{26}$										
u_4	v_4	v_9	v_{13}	v_{16}	v_{17}	v_{18}											
u_5	v_5	v_{10}	v_{14}	v_{17}	v_{19}	v_{20}											
u_6	v_6	v_{11}	v_{15}	v_{18}	v_{20}	v_{21}											
...																	
u_{29}																	
u_{30}																	

Table of $UV + V^2$:

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	...	v_{25}	v_{26}	v_{27}	v_{28}	v_{29}	v_{30}	v_{31}	v_{32}	v_{33}	v_{34}
u_1											u_7	u_8	u_9	u_{10}				
u_2											u_{11}	u_{12}	u_{13}	u_{14}				
u_3											u_{15}	u_{16}	u_{17}	u_{18}				
u_4											u_{19}	u_{20}	u_{21}	u_{22}				
u_5											u_{23}	u_{24}	u_{25}	u_{26}				
u_6											u_{27}	u_{28}	u_{29}	u_{30}				
u_7																		
u_8																		
...																		
u_{29}																		
u_{30}																		
v_1																		
v_2																		
...																		
v_{33}																		
v_{34}																		

Example 2. Let $\mathbf{A} = Fe \oplus U \oplus V$ be a Bernstein algebra, having $\{u_1, u_2, ..., u_{24}\}$ as basis of U, $\{v_1, v_2, ..., v_{25}\}$ as basis of V and multiplication table in $N = U \oplus V$ given by:

Table of U^2:

	u_1	u_2	u_3	u_4	u_5	u_6	.....	u_{20}	u_{21}	u_{22}	u_{23}	u_{24}
u_1	v_1	v_2	v_3	v_4				$-v_{16}$	$-v_{24}$	$-v_{25}$	$-v_{18}$	$-v_{19}$
u_2	v_2	v_5	v_6	v_7		$-v_{11}$			v_{20}		$-v_{21}$	$-v_{22}$
u_3	v_3	v_6	v_8	v_9	v_{11}			v_{22}	v_{21}	$-v_{21}$		
u_4	v_4	v_7	v_9	v_{10}	v_{14}	v_{17}				$-v_{22}$		
u_5			v_{11}	v_{14}								
u_6		$-v_{11}$		v_{17}								
u_7		$-v_{14}$	$-v_{17}$									
u_8			v_{12}	v_{15}								
u_9		v_{13}		v_{18}								
u_{10}		v_{16}	v_{19}									
u_{11}	v_{11}	v_{12}		v_{23}								
u_{12}		$-v_{12}$	$-v_{13}$	v_{24}								
u_{13}	v_{14}	v_{15}	$-v_{23}$									
u_{14}		$-v_{15}$	$-v_{24} - v_{25}$	$-v_{16}$								
u_{15}	v_{17}	$v_{23} + v_{24}$	v_{18}									
u_{16}		v_{25}	$-v_{18}$	$-v_{19}$								
u_{17}	$-v_{12}$			v_{20}								
u_{18}	$-v_{15}$		$-v_{20}$									

	u_1	u_2	u_3	u_4	u_5	u_6	.....	u_{20}	u_{21}	u_{22}	u_{23}	u_{24}
u_{19}	$-v_{13}$			v_{21}								
u_{20}	$-v_{16}$		v_{22}									
u_{21}	$-v_{24}$	v_{20}	v_{21}									
u_{22}	$-v_{25}$		$-v_{21}$	$-v_{22}$								
u_{23}	$-v_{18}$	$-v_{21}$										
u_{24}	$-v_{19}$	$-v_{22}$										

Table of $UV + V^2$:

	u_1	u_2	u_3	u_4	u_5	....	u_{24}	v_1	v_2	v_3	...	v_{25}
v_1		$2u_5$	$2u_6$	$2u_7$								
v_2	$-u_5$	$-u_8$	u_{11}	u_{13}								
v_3	$-u_6$	$-u_{11} - u_{12}$	$-u_9$	u_{15}								
v_4	$-u_7$	$-u_{13} - u_{14}$	$-u_{15} - u_{16}$	$-u_{10}$								
v_5	$2u_8$		$2u_{17}$	$2u_{18}$								
v_6	u_{12}	$-u_{17}$	$-u_{19}$	u_{21}								
v_7	u_{14}	$-u_{18}$	$-u_{21} - u_{22}$	$-u_{20}$								
v_8	$2u_9$	$2u_{19}$		$2u_{23}$								
v_9	u_{16}	u_{22}	$-u_{23}$	$-u_{24}$								
v_{10}	$2u_{10}$	$2u_{20}$	$2u_{24}$									
v_{11}												
...												
v_{25}												

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