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SOME FINITE PROPERTIES OF GENERALIZED LOCAL COHOMOLOGY MODULES

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Abstract

Let (R, m) be a commutative Noetherian local ring, I an ideal of R and M, N finitely generated $R-$modules. In this paper we prove some finite properties of generalized local cohomology modules $H_I^i(M, N)$. Set $I_M = \operatorname{ann}(M/IM)$ and $r = \operatorname{depth}(I_M, N)$. We show that $\operatorname{Ass} H_I^r(M, N) = \operatorname{Ass} \operatorname{Ext}_R^r(M/IM, N)$. We also characterize the least integer i such that $H_I^i(M, N)$ is not artinian by using the notion of filter regular sequences.

1 Introduction

The generalized local cohomology module of two $R-$modules M and N with respect to an ideal I of R is introduced by J. Herzog [6] and it is defined by

$$H_I^i(M, N) = \varinjlim_n \operatorname{Ext}_R^i(M/I^n M, N).$$

Then $H_I^i(R, N) = H_I^i(N)$ is just the i-th local cohomology module of N. Therefore the notion of generalized local cohomology module is an extension of the usual local cohomology modules. But, many basic properties of local cohomology modules can not extend to generalized local cohomology modules. Par example, the well-known vanishing and non-vanishing theorems for local cohomology modules over a local ring (R, m) state that $H_m^{\dim N}(N) \neq 0$ and

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$H^i_m(N) = 0$ for all $i > \dim N$, while one does not know about the last integer i such that $H^i_m(M, N) \neq 0$; or $H^i_I(N) = 0$ for all $i \geqslant 1$ provided N is $I-$torsion, while a similar property for generalized local cohomology modules is not appropriate. Concretely, if N is $I-$torsion then $H^i_I(M, N) \cong \mathrm{Ext}^i_R(M, N)$, and the later does not vanish in general. However, it seems to us that several finite properties of local cohomology can still be established for generalized local cohomology (cf. [1], [5] and [11]). The purpose of this paper is to study the finiteness of associated primes and the artinianness of generalized local cohomology modules.

Let M, N be finitely generated modules over a local ring (R, m) and $I_M = \mathrm{ann}_R(M/IM)$. We prove in Section 2 that $\mathrm{Ass}\, H^r_I(M, N) = \mathrm{Ass}\, \mathrm{Ext}^r_R(M/IM, N)$, where $r = \mathrm{depth}(I_M, N)$; therefore $\mathrm{Ass}\, H^r_I(M, N)$ is a finite set (Theorem 2.4). We also show in Theorem 2.5 that the sets $\mathrm{Ass}\, H^j_I(M, N)$ and $\mathrm{Ass}\, \mathrm{Ext}^j_R(M/IM, N)$ are different at most the maximal ideal for all $j \leqslant s$, where $s = \text{f-depth}(I_M, N)$ is the length of a maximal filter regular sequence of N in I_M, and therefore $\mathrm{Ass}\, H^j_I(M, N)$ is finite. Section 3 is devoted to study the artinianness of generalized local cohomology modules $H^j_I(M, N)$. The main result of this section is Theorem 3.1, which shows that

$$\text{f-depth}(I_M, N) = \inf\{i \mid H^i_I(M, N) \text{ is not artinian }\}.$$

It should be mentioned that this theorem is an extension to generalized local cohomology modules of a result on usual local cohomology modules of Melkersson [9, Theorem 3.1]. However, a basic property of local cohomology that used in Melkersson's proof can not be extended for generalized local cohomology modules. So our proof is base on the standard properties of generalized local cohomology and the vanishing of Bass numbers. Then we can derive from Theorem 3.1 many consequences for the artinanness of generalized local cohomology and local cohomology modules (Corollaries 3.2, 3.4, 3.5).

2 Associated primes of certain generalized local cohomology modules

Throughout this paper M, N are finitely generated modules over a Noetherian local ring (R, m). For any ideal I of R we denote by $I_M = \mathrm{ann}_R(M/IM)$ the annihilator of the module M/IM and by Γ_I the $I-$torsion functor.

The following lemma follows easily from the definition of generalized local cohomology modules.

Lemma 2.1. *The following statements are true.*
(i) Let $0 \to N \to E^\bullet$ *be an injective resolution of* N. *Then*

$$H^i_I(M, N) \cong H^i(\Gamma_I(\mathrm{Hom}(M, E^\bullet))) \cong H^i(\mathrm{Hom}(M, \Gamma_I(E^\bullet)))$$

for all $i \geqslant 0$. *Therefore,* $H_I^i(M,N)$ *is* $I-$*torsion and* $H_I^i(M,N) \cong H_J^i(M,N)$ *for any ideal* J *satisfying* $\operatorname{rad}(J) = \operatorname{rad}(I)$.
(ii) *If* N *is* $I-$*torsion then* $H_I^i(M,N) \cong \operatorname{Ext}_R^i(M,N)$ *for all* $i \geqslant 0$.

The next result was proved by M. H. Bijan-Zadeh in [1, Proposition 5.5].

Lemma 2.2. *The following equality is true.*

$$\operatorname{depth}(I_M, N) = \inf\{i \mid H_I^i(M,N) \neq 0\},$$

where we use the convention that $\inf(\emptyset) = \infty$.

Lemma 2.3. *Let* I, J *be ideals of* R. *If* $\operatorname{rad}(I_M) = \operatorname{rad}(J_M)$ *then* $H_I^i(M,N) \cong H_J^i(M,N)$ *for all* $i \geqslant 0$. *In particular,* $H_I^i(M,N) \cong H_{I_M}^i(M,N)$, *and therefore* $H_I^i(M,N)$ *is* I_M-*torsion.*

Proof Suppose $\operatorname{rad}(I_M) = \operatorname{rad}(J_M)$. Let

$$0 \to N \to E^0 \to E^1 \to \ldots \to E^i \to \ldots$$

be an injective resolution of N. For each $i \geqslant 0$, we have

$$\begin{aligned}\Gamma_I(\operatorname{Hom}(M,E^i)) &= \Gamma_{I+\operatorname{ann}(M)}(\operatorname{Hom}(M,E^i))\\ &= \Gamma_{\operatorname{rad}(I_M)}(\operatorname{Hom}(M,E^i)) = \Gamma_{\operatorname{rad}(J_M)}(\operatorname{Hom}(M,E^i))\\ &= \Gamma_J(\operatorname{Hom}(M,E^i)).\end{aligned}$$

Therefore we get by Lemma 2.1,(i) that $H_I^i(M,N) \cong H_J^i(M,N)$. □

Now we describe the set of associated primes of certain generalized local cohomology module.

Theorem 2.4. *Let* $r = \operatorname{depth}(I_M, N)$. *Then we have*

$$\operatorname{Ass} H_I^r(M,N) = \operatorname{Ass} \operatorname{Ext}_R^r(M/IM, N).$$

In particular, $\operatorname{Ass} H_I^r(M,N)$ *is a finite set.*

Proof First, we claim by induction on $r \geqslant 0$ that

$$\operatorname{Hom}(R/I_M, H_{I_M}^r(M,N)) \cong \operatorname{Ext}_R^r(M/IM, N).$$

Let $r = 0$. We have $H_{I_M}^0(M,N) = \Gamma_{I_M}(\operatorname{Hom}(M,N))$ by Lemma 2.1,(i). Since $\operatorname{Hom}(M,N)$ is finitely generated, there exists an integer $k > 0$ such that

$$\Gamma_{I_M}(\operatorname{Hom}(M,N)) = \big(0 : (I_M)^k\big)_{\operatorname{Hom}(M,N)}.$$

Therefore

$$\begin{aligned}\operatorname{Hom}(R/I_M, H^0_{I_M}(M,N)) &\cong (0:I_M)_{(0:(I_M)^k)_{\operatorname{Hom}(M,N)}}\\ &\cong (0:I_M)_{\operatorname{Hom}(M,N)}\\ &\cong \operatorname{Ext}^0_R(M/IM,N).\end{aligned}$$

Thus the claim is true for $r=0$. Let $r>0$. Let $x_1,\ldots,x_r$ be a regular sequence of N in I_M. From the exact sequence

$$0\to N \xrightarrow{x_1} N \to N/x_1N \to 0$$

we have the exact sequence

$$H^{r-1}_{I_M}(M,N)\to H^{r-1}_{I_M}(M,N/x_1N)\to H^r_{I_M}(M,N)\xrightarrow{x_1} H^r_{I_M}(M,N).$$

Since $H^{r-1}_{I_M}(M,N)=H^{r-1}_I(M,N)=0$ by Lemma 2.2, we get an isomorphism

$$H^{r-1}_{I_M}(M,N/x_1N)\cong (0:x_1)_{H^r_{I_M}(M,N)}.$$

Therefore

$$\begin{aligned}\operatorname{Hom}(R/I_M, H^r_{I_M}(M,N)) &\cong (0:I_M)_{(0:x_1)_{H^r_{I_M}(M,N)}}\\ &\cong \operatorname{Hom}(R/I_M, H^{r-1}_{I_M}(M,N/x_1N)).\end{aligned}$$

On the other hand, since $\operatorname{depth}(I_M, N/x_1N)=r-1$, we get by the induction assumption and that

$$\operatorname{Hom}(R/I_M, H^{r-1}_{I_M}(M,N/x_1N))\cong \operatorname{Ext}^{r-1}_R(M/IM,N/x_1N).$$

Thus $\operatorname{Hom}(R/I_M, H^r_{I_M}(M,N))\cong \operatorname{Ext}^r_R(M/IM,N)$ and the claim is proved.

Now, because $H^r_{I_M}(M,N)$ is I_M-torsion, we get by the claim that

$$\begin{aligned}\operatorname{Ass} H^r_{I_M}(M,N) &= \operatorname{Ass}\operatorname{Hom}(R/I_M, H^r_{I_M}(M,N))\\ &= \operatorname{Ass}\operatorname{Ext}^r_R(M/IM,N).\end{aligned}$$

Therefore, by Lemma 2.3 we get $\operatorname{Ass} H^r_I(M,N) = \operatorname{Ass}\operatorname{Ext}^r_R(M/IM,N)$ as required. □

Recall that a sequence $x_1,\ldots,x_s$ of elements in m is called a *filter regular sequence* of N if $x_i\notin p$ for all $p\in \operatorname{Ass}(N/(x_1,\ldots,x_{i-1})N)\setminus\{m\}$ for all $i=1,\ldots,s$ (cf. [4]). For an ideal J of R, if $\dim N/JN>0$ then any filter regular sequence of N in J is of finite length, and all maximal filter regular sequences of N in J have the same length. This common length is called *f-depth* of N in J and denoted by f-depth(J,N) (cf. [10]). Now, by virtue of Theorem 2.4 we can describe concretely the set of associated primes of $H^j_I(M,N)$ for all $j\leqslant$ f-depth(I_M,N).

Theorem 2.5. *Let* $s = \text{f-depth}(I_M, N)$*. For any* $j \leqslant s$*, we have*

$$\operatorname{Ass} H_I^j(M,N) \cup \{m\} = \operatorname{Ass} \operatorname{Ext}_R^j(M/IM, N) \cup \{m\}.$$

Therefore $\operatorname{Ass} H_I^j(M,N)$ *is finite for all* $j \leqslant s$*.*

Proof For each $j \leqslant s$, let $p \in \operatorname{Ass} H_I^j(M,N)$ with $\dim R/p \geqslant 1$. Then we have $H_{I_p}^j(M_p, N_p) \neq 0$. Hence $\operatorname{depth}((I_p)_{M_p}, N_p) \leqslant j$ by Lemma 2.2. Let $x_1, \ldots, x_j$ be a filter regular sequence of N in I_M. Since $I_M \subseteq p$, $x_1/1, \ldots, x_j/1$ is a regular sequence of N_p in $(I_M)_p = (I_p)_{M_p}$. Therefore $\operatorname{depth}((I_p)_{M_p}, N_p) \geqslant j$, and hence $\operatorname{depth}((I_p)_{M_p}, N_p) = j$. Now, by Theorem 2.4, we have

$$pR_p \in \operatorname{Ass} \operatorname{Ext}_{R_p}^j(M_p/I_pM_p, N_p) = \operatorname{Ass} \operatorname{Ext}_R^j(M/IM, N)_p.$$

Hence $p \in \operatorname{Ass} \operatorname{Ext}_R^j(M/IM, N)$. Conversely, let $p \in \operatorname{Ass} \operatorname{Ext}_R^j(M/IM, N)$ with $\dim R/p \geqslant 1$. Then, we can show as above that $\operatorname{depth}((I_p)_{M_p}, N_p) = j$. Therefore we get by Theorem 2.4 that

$$\operatorname{Ass} \operatorname{Ext}_{R_p}^j(M_p/I_pM_p, N_p) = \operatorname{Ass} H_{I_p}^j(M_p, N_p).$$

Hence $p \in \operatorname{Ass} H_I^j(M,N)$. □

As consequences of Theorem 2.4 and Theorem 2.5, we get the following results on the usual local cohomology modules.

Corollary 2.6. *Let* $r = \operatorname{depth}(I, N)$ *and* $s = \text{f-depth}(I, N)$*. Then we have*
(i) (cf. [8, Proposition 1.1]) $\operatorname{Ass} H_I^j(N) = \operatorname{Ass} \operatorname{Ext}_R^j(R/I, N)$ *for all* $j \leqslant r$*,*
(ii) $\operatorname{Ass} H_I^j(N) \cup \{m\} = \operatorname{Ass} \operatorname{Ext}_R^j(R/I, N) \cup \{m\}$ *for all* $j \leqslant s$*.*
In particular, $\operatorname{Ass} H_I^j(N)$ *is a finite set for all* $j \leqslant s$*.*

3 The artinianness of certain generalized local cohomology modules

The following characterization of $\text{f-depth}(I_M, N)$ by generalized local cohomology modules $H_I^i(M,N)$ is the main result of this section.

Theorem 3.1. *Let* s *be a positive integer. Then the following statements are equivalent:*
(i) I_M *contains a filter regular sequence of* N *of length* s*.*
(ii) $H_I^j(M,N)$ *is artinian for all* $j < s$*.*
Therefore we have

$$\text{f-depth}(I_M, N) = \inf\{i \mid H_I^i(M,N) \textit{ is not artinian }\}.$$

Proof $(i) \Rightarrow (ii)$. Suppose that f-depth$(I_M, N) \geqslant s$. Let $0 \to N \to E^\bullet$, where

$$E^\bullet : E^0 \to E^1 \to \ldots \to E^j \to \ldots,$$

be a minimal injective resolution of N. Then by [2, 10.1.10] we have

$$\Gamma_{I_M}(E^j) = \bigoplus_{I_M \subseteq q \in \operatorname{Ass} E^j} E(R/q)^{\mu^j(q,N)}$$

for all $j < s$, where $\mu^j(q, N) = \dim_{k(q)} \operatorname{Ext}^j_{R_q}(k(q), N_q)$ is the $j^{th}-$ Bass number of N with respect to q.

Note that, for any $q \in \operatorname{Ass} E^j \setminus \{m\}$ containing I_M, the sequence

$$0 \to N_q \to E^0_q \to E^1_q \to \ldots \to E^j_q \to \ldots$$

is a minimal injective resolution of N_q (cf. [2, 11.1.6]). Since $E^j_q \neq 0$, $N_q \neq 0$ and $q \in \operatorname{Supp} N/I_M N$. It follows by virtue of [9] that

$$\operatorname{depth}(N_q) \geqslant \operatorname{depth}((I_M)_q, N_q) \geqslant \text{f-depth}(I_M, N) \geqslant s.$$

Therefore we get $\operatorname{Ext}^j_{R_q}(k(q), N_q) = 0$ for all $j < s$ and all $q \in \operatorname{Ass} E^j \setminus \{m\}$. Hence

$$\mu^j(q, N) = \dim_{k(q)} \operatorname{Ext}^j_{R_q}(k(q), N_q) = 0$$

for all $j < s$ and all $q \in \operatorname{Ass} E^j \setminus \{m\}$. This implies that

$$\Gamma_{I_M}(E^j) = \bigoplus_{I_M \subseteq q \in \operatorname{Ass} E^j} E(R/q)^{\mu^j(q,N)} = E(R/m)^{\mu^j(m,N)}$$

is artinian for all $j < s$, and so $H^j_I(M, N) = H^j(\operatorname{Hom}(M, \Gamma_{I_M}(E^\bullet)))$ is artinian for all $j < s$ as required.

$(ii) \Rightarrow (i)$. We prove by induction on s that I_M contains a filter regular sequence of length s. Let $s = 1$. Then $H^0_I(M, N)$ is artinian. Therefore by Lemma 2.1,(i) we get

$$\{m\} \supseteq \operatorname{Ass} \Gamma_I(\operatorname{Hom}(M, N)) = \operatorname{Ass} N \cap \operatorname{Supp} R/I_M.$$

It follows by the Prime Advoidance Theorem that there always exists an $N-$filter regular element in I_M. Let $s > 1$. By the inductive hypothesis, there is an $N-$filter regular element $x_1 \in I_M$. Thus we obtain exact sequences

$$H^j_I(M, N) \longrightarrow H^j_I(M, N/(0 :_N x_1)) \longrightarrow H^{j+1}_I(M, (0 :_N x_1))$$

for all j. Since $\ell(0 :_N x_1) < \infty$, it follows by Lemma 2.1,(ii) that

$$H^j_I(M, (0 :_N x_1)) \cong \operatorname{Ext}^j_R(M, (0 :_N x_1))$$

is artinian for all j. So, from the above exact sequences, we get by (ii) that $H^j_I(M, N/(0 :_N x_1))$ is artinian for all $j < s$. On the other hand, from the short exact sequence

$$0 \to N/(0 :_N x_1) \xrightarrow{x_1} N \to N/x_1N \to 0$$

we obtain the following exact sequences

$$H^j_I(M, N) \to H^j_I(M, N/x_1N) \to H^{j+1}_I(M, N/(0 :_N x_1))$$

for all j. Therefore $H^j_I(M, N/x_1N)$ is artinian for all $j < s - 1$. Then by the inductive hypothesis there exists an N/x_1N-filter regular sequence of length $s - 1$ in I_M, and the conclusion follows.

For the last conclusion of the theorem, we need only to show that if $H^0_I(M, N)$ is not artinian then f-depth$(I_M, N) = 0$. Indeed, if $H^0_I(M, N)$ is not artinian then

$$\operatorname{Ass} H^0_I(M, N) = \operatorname{Ass} N \cap \operatorname{Supp} R/I_M \not\subseteq \{m\}.$$

Hence $I_M \subseteq p$ for some $p \in \operatorname{Ass} N \setminus \{m\}$. Therefore f-depth$(I_M, N) = 0$. □

As an immediate consequence of Theorem 3.1, we get a result on the artinianness of generalized local cohomology modules with respect to the maximal ideal.

Corollary 3.2. (cf. [5, Theorem 2.2]) *Assume that I_M is $m-$primary. Then $H^j_I(M, N)$ is artinian for all $j \geqslant 0$. In particular, $H^j_m(M, N)$ is artinian for all $j \geqslant 0$.*

Proof The result follows from the fact that f-depth$(I_M, N) = \infty$. □

The next corollary shows relation between the artinianness of local cohomology modules and of generalized local cohomology modules.

Corollary 3.3. *Let s is a positive integer. If $H^j_I(N)$ is artinian for all $j < s$ then $H^j_I(M, N)$ is artinian for all finitely generated $R-$module M and all $j < s$.*

Proof The result follows by Theorem 3.1 and the fact that $s \leqslant$ f-depth$(I, N) \leqslant$ f-depth(I_M, N). □

Corollary 3.4. *The following equality is true:*

$$\text{f-depth}(I_M, N) = \sup\{i \mid H^j_I(M, N) \cong H^j_m(M, N), \forall j < i\}.$$

Proof Set $s =$ f-depth(I_M, N). Let $0 \to N \to E^0 \to \ldots \to E^j \to \ldots$ be a minimal injective resolution of N. By a similar argument as in the proof of Theorem 3.1, we have

$$\Gamma_{I_M}(E^j) = E(R/m)^{\mu^j(m,N)} = \Gamma_m(E^j)$$

for all $j < s$. Therefore we get by Lemmas 2.1,(i), 2.3 that

$$H_I^j(M,N) \cong H_{I_M}^j(M,N) \cong H_m^j(M,N)$$

for all $j < s$. On the other hand, since $H_I^s(M,N)$ is not artinian by Theorem 3.1 and $H_m^s(M,N)$ is artinian by Corollary 3.2, $H_I^s(M,N) \ncong H_m^s(M,N)$ as required. □

It is known that the fact, which says that an $R-$module K is artinian if and only if $\operatorname{Supp} K \subseteq \{m\}$, is not true in general. However, the next consequence shows that this fact is true for generalized local cohomology modules.

Corollary 3.5. *The following equality is true:*

$$\text{f-depth}(I_M, N) = \inf\{i \mid \operatorname{Supp} H_I^i(M,N) \nsubseteq \{m\}\}.$$

Therefore $H_I^j(M,N)$ is artinian for all $j < i$ if and only if $\operatorname{Supp} H_I^j(M,N) \subseteq \{m\}$ *for all $j < i$.*

Proof Let $s = \text{f-depth}(I_M, N)$. It is enough to show that $\operatorname{Supp} H_I^s(M,N) \nsubseteq \{m\}$. Indeed, since

$$s = \inf\{\operatorname{depth}(I_M R_p, N_p) \mid p \in \operatorname{Supp} N/I_M N, \dim R/p \geqslant 1\},$$

we have $\operatorname{depth}(I_M R_p, N_p) = s$ for some $p \in \operatorname{Supp} N/I_M N$ with $\dim R/p \geqslant 1$. Therefore $p \in \operatorname{Supp} H_I^s(M,N)$ and the conclusion follows. □

Finally, the following characterizations of f-depth(I, N) in terms of local cohomology modules are immediate consequences of Theorem 3.1, Corollaries 3.2, 3.4, 3.5.

Corollary 3.6. (cf. [9, Theorem 3.1] and [7, Lemma 2.4, Theorem 2.5])

$$\begin{aligned}\text{f-depth}(I,N) &= \inf\{i \mid H_I^i(N) \textit{ is not artinian }\}\\ &= \inf\{i \mid \operatorname{Supp} H_I^i(N) \nsubseteq \{m\}\}\\ &= \sup\{i \mid H_I^j(N) \cong H_m^j(N), \forall j < i\}.\end{aligned}$$

Therefore $H_I^j(N)$ is artinian for all $j < i$ if and only if $\operatorname{Supp} H_I^j(N) \subseteq \{m\}$ *for all $j < i$.*

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