East-West J. of Mathematics: Vol. 7, No 1 (2005) pp. 99-106

THE LIFTING CONDITION AND FULLY INVARIANT SUBMODULES

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Abstract

A module M is lifting if for every submodule A of M, there exists a direct summand B of M such that $B \leq A$ and A/B small in M/B. Every non-cosingular lifting module has the summand sum property. We call any module M *FI-lifting* if for every fully invariant submodule A of M there exists a direct summand B of M such that $B \leq A$ and A/B is small in M/B. In contrast to lifting modules, any finite direct sum of FI-lifting modules is FI-lifting.

I. Introduction

Throughout this paper R denotes an associative ring with unity and all R-modules are unitial right R-modules.

A submodule N of a module M is called *small*, written $N << M$, if $M \neq N + L$ for every proper submodule L of M. Properties of small submodules are given in [9, Lemma 4.2] and [13, Proposition 19.3]. Let M be a module. M is called *lifting module (or $(D1)$)*, if for every submodule N of M, M has a decomposition $M = M_1 \oplus M_2$ with $M_1 \leq N$ and $M_2 \cap N$ small in M_2, equivalenty if for every submodule A of M there exists a direct summand B of M such that $B \leq A$ and A/B is small in M/B. Let M be a module. M has *summand sum property* if the sum of any two direct summands of M is a direct summand of M and denoted by SSP. M has *summand intersection property* if the intersection of any two direct summands of M is a direct summand of M and denoted by SIP (see [6,7,12]). Let M be an R-module. M is called *small*

Key words: Lifting module, (non-)cosingular module, fully invariant submodule.
2000 AMS Mathematics Subject Classification: 16S90

, if $M << E(M)$, where $E(M)$ is the injective hull of M. In [10], Talebi and Vanaja defined $\overline{Z}(M) = \cap\{Ker(g) : g \in Hom(M, N), N \ll E(N)\}$. They call M *cosingular (non-cosingular)* module if $\overline{Z}(M) = 0$ ($\overline{Z}(M) = M$). Cosingular and non-cosingular modules are studied in [10] and [11].

In Section 2, we prove that (1) for every fully invariant submodule Y of M, M/Y is lifting and (2) every non-cosingular lifting module has the summand sum property.

Following [3], M is called *FI-extending*, every fully invariant submodule of M is essential in a direct summand of M. In Section 3, dually, we called the module M is *FI-lifting* if for every fully invariant submodule A of M, there exists a direct summand B of M such that $B \subseteq A$ and A/B small in M/B, and shown that

Proposition Let M be a module and X a fully invariant submodule of M. If M is FI-lifting then M/X is FI-lifting.

Theorem Let $M = \oplus_{i=1}^{n} X_i$. If each X_i is FI-lifting, then M is FI- lifting.

We will refer to [1, 9, 13] for all undefined notions used in the text, and also for basic facts concerning coatomic and singular modules.

2. The lifting condition for a factor submodule

In this section we investigate conditions which ensure that a factor submodule of a lifting module will be a lifting module. The following theorem is dual of [2, Theorem 1.1].

Theorem 2.1 *Let M be an $R-$module.*

1. *Assume that M is a lifting module and X a submodule of M. If for every direct summand K of M, $(X + K)/X$ is a direct summand of M/X then M/X is lifting .*

2. *Let D be a submodule of M and X a direct summand of M. Assume that M/X is lifting. If $D/(D \cap X)$ is non-cosingular, then $D + X$ is a direct summand of M.*

3. *If M is non-cosingular and M/X is lifting with X a direct summand of M, then $(X+D)/X$ is a direct summand of M/X for all direct summands D of M.*

Proof (1) Let $A/X \leq M/X$. Since M is lifting, there exists a direct summand D of M such that $D \subseteq A$ and A/D is small in M/D. By hypothesis, $(D+X)/X$

is a direct summand of M/X. Clearly, $(D+X)/X \subseteq A/X$. Now we show that $A/(D+X)$ is small in $M/(D+X)$. Let $M/(D+X) = A/(D+X)+L/(D+X)$ for any submodule $L/(D+X)$ of $M/(D+X)$. Then $M = A + L$ implies that $M/D = A/D + L/D$. Since A/D is small in M/D, $M = L$. Therefore $A/(D+X)$ is small in $M/(D+X)$. Thus M/X is lifting.
(2) Let $D, X \leq M$ with X a direct summand of M. Consider the submodule $(D+X)/X \leq M/X$. Since M/X is lifting, there exists a direct summand C/X of M/X such that $C/X \subseteq (D+X)/X$ and $(D+X)/C$ is small in M/C. Hence $(D+X)/C$ is cosingular. On the other hand $(D+X)/X \cong D/(D\cap X)$ and so $(D+X)/X$ is non-cosingular. Therefore by [10, Proposition 2.4], $(D+X)/C$ is non-cosingular. Hence $D+X = C$.
(3) Let M be non-cosingular module and M/X lifting with X a direct summand of M. Let D be a direct summand of M. Then $D/(D\cap X)$ is non-cosingular by [10, Proposition. 2.4]. By (2) $D+X$ is a direct summand of M and hence $(D+X)/X$ is a direct summand of M/X. □

Let M be a module. A submodule X of M is called *fully invariant* if for every $h \in End_R(M)$, $h(X) \subseteq X$. Some properties of fully invariant submodules are given in Lemma 3.2.

A module M is called *distributive* if its lattice of submodules is a distributive lattice.

Corollary 2.2 *Let M be a lifting module.*

1. *If M is a distributive module, then M/X is lifting for every submodule X of M.*
2. *Let $X \leq M$ and $eX \subseteq X$ for all $e^2 = e \in End(M)$. Then M/X is lifting. In particular, for every fully invariant submodule Y of M, M/Y is lifting.*

Proof (1) Let D be a direct summand of M. Then $M = D \oplus D'$ for some submodule D' of M. Now $M/X = [(D+X)/X] + [(D'+X)/X]$ and $X = X+(D\cap D') = (X+D)\cap(X+D')$. So, $M/X = [(D+X)/X]\oplus[(D'+X)/X]$. By Theorem 2.1.(1), M/X is lifting.
(2) Let D be a direct summand of M. Consider the projection map $e : M \to D$. Then $e^2 = e \in End(M)$. By hypothesis, $eX \subseteq X$ and hence $eX = X \cap D$. There exists a direct summand D' of M such that $M = D \oplus D'$. Therefore $X = (X\cap D)\oplus(X\cap D')$. Now $(D+X)/X = (D\oplus(X\cap D'))/X$ and $(D'+X)/X = (D'\oplus(X\cap D))/X$. Hence $M = D\oplus D' = D+X+D'+X = [D\oplus(X\cap D')]+D'+X$ implies that $M/X = (D\oplus(X\cap D'))/X + (D'+X)/X$. Since $[D\oplus(X\cap D')]\cap(D'+X) = (X\cap D')\oplus(X\cap D)$, $M/X = (D\oplus(X\cap D'))/X \oplus (D'+X)/X$. Thus by Theorem 2.1.(1), M/X is lifting. □

Theorem 2.3 *Let R be a semiperfect ring.*

1. *If R has every idempotent central then, for every right ideal I of R, R/I is right lifting.*

2. *For every ideal I of R, R/I is semiperfect.*

Proof They follows from Corollary 2.2.(2) and [1]. □

Corollary 2.4 *Every non-cosingular lifting module has the summand sum property.*

Proof Let M be a non-cosingular lifting module. Let A and B be two direct summands of M. Let $M = A \oplus A' = B \oplus B'$ for some submodules A', B'. Note that A' and B' are lifting modules. Since $M/A \cong A'$ and $M/B \cong B'$, $(A+B)/A$ is a direct summand of M/A and $(A+B)/B$ is a direct summand of M/B by theorem 2.1.(3). Hence $A+B$ is a direct summand of M. □

We know that there are modules having the SSP and (D1) but not the SIP.

Example 2.5. Let F be a field and R the upper triangular matrix ring $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$. For submodules $A = \begin{pmatrix} 0 & F \\ 0 & F \end{pmatrix}$ and $B = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$, $A \oplus (R/B)$ has the SSP by [6] and (D1) by [9]. But has not the SIP.

We consider the following condition:

$(D3)$ If M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, then $M_1 \cap M_2$ is also a direct summand of M.

Lemma 2.6 *Assume that M is $(D3)$. If M has the SSP then M has the SIP.*

Proof Let M_1 and M_2 be direct summands of M. Since $M_1 + M_2$ is direct summand of M by assumption, we have $(M_1+M_2) \oplus X$ for some submodule X of M. Again by assumption, $M_1 + X$ and $M_2 + X$ are direct summands. Since M is $(D3)$, $M = [(M_1 + X) \cap (M_2 + X)] \oplus Y$ for some submodule Y of M. Now we have $M = (M_1 \cap M_2) \oplus X \oplus Y$. That is $M_1 \cap M_2$ is direct summand of M. □

Corollary 2.7 *Let M be a non-cosingular module with $(D3)$. Then*
M is lifting $\Rightarrow$ M has SSP $\Rightarrow$ M has SIP

Example 2.8 (1) Let $M_{\mathbb{Z}} = \mathbb{Z}_{\mathbb{Z}} \oplus \mathbb{Z}_{\mathbb{Z}}$. $M_{\mathbb{Z}}$ is not lifting. Since $\mathbb{Z}_{\mathbb{Z}} \oplus \mathbb{Z}_{\mathbb{Z}} << \mathbb{Q}_{\mathbb{Z}} \oplus \mathbb{Q}_{\mathbb{Z}}$, we have $M_{\mathbb{Z}}$ is co-singular. Furthermore, M has the SIP and so M has

$(D3)$. Let $N = \mathbb{Z}(2,3)$ and $K = \mathbb{Z}(3,2)$. Since $N \oplus K$ is not direct summand of M, M has not the SSP.
(2) Let $M_{\mathbb{Z}} = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^2\mathbb{Z}$, where p is any prime. $M_{\mathbb{Z}}$ is a lifting module and, since $\mathbb{Z}/p\mathbb{Z} << \mathbb{Q}/p\mathbb{Z}$, $\mathbb{Z}/p\mathbb{Z}$is co-singular and so M is cosingular module. Furthermore $M_{\mathbb{Z}}$ is not $(D3)$ and $M_{\mathbb{Z}}$ has neither the SIP nor the SSP.
(3) The $\mathbb{Z}$-module $\mathbb{Q}$, the set of all rational numbers, is non- cosingular module by [10, Remark 2.11]. We know that $\mathbb{Q}_{\mathbb{Z}}$ has the SIP and so $(D3)$ and has the SSP. But $\mathbb{Q}_{\mathbb{Z}}$ is not a lifting module.

3. FI-lifting modules

Let M be a lifting module. In Corollary 2.2 we proved that, for every fully invariant submodule Y of M, M/Y is lifting. In this section, we determine a generalization of the lifting modules. Let M be any module. Following [3], M is called *FI-extending*, every fully invariant submodule of M is essential in a direct summand of M. FI-extending modules are studied [3], [4] and [5]. Dually, we say the module M is *FI-lifting* if for every fully invariant submodule A of M, there exists a direct summand B of M such that $B \subseteq A$ and A/B small in M/B.

Clearly, M is FI-lifting if and only if for every fully invariant submodule A of M there is a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $M_2 \cap A$ is small in M_2.

Lemma 3.1 *Let M be a module.*
(1) *M is FI-lifting.*
(2) *For every fully invariant submodule A of M there is a decomposition $A = N \oplus S$ with N a direct summand of M and S small in M.*

Proof For the proof, we completely follow the proof of [9, Proposition 4.8].
$(i) \Rightarrow (ii)$ Let A be a fully invariant submodule of M. Since M is FI-lifting, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq A$ and $M_2 \cap A$ small in M_2. Therefore $A = M_1 \oplus (A \cap M_2)$, as required.
$(ii) \Rightarrow (i)$ Assume that every fully invariant submodule has the stated decomposition. Let A be a fully invariant submodule of M. By hypothesis, there exists a direct summand N of M and a small submodule S of M such that $A = N \oplus S$. Now $M = N \oplus N'$ for some submodule N' of M. Consider the natural epimorphism $\pi : M \longrightarrow M/N$. Then $\pi(S) = (S+N)/N = A/N$ small in M/N. Therefore M is FI-lifting. □

Lemma 3.2 *Let M be a module.*

1. *Any sum and intersection of fully invariant submodules of M is again a fully invariant submodule of M.*

2. *If $X \leq Y \leq M$ such that Y is a fully invariant submodule of M and X is a fully invariant submodule of Y, then X is a fully invariant submodule of M.*

3. *If $M = \oplus_{i\in I} X_i$ and S is a fully invariant submodule of M, then $S = \oplus_{i\in I}\pi_i(S) = \oplus_{i\in I}(X_i \cap S)$, where π is the $i-th$ projection homomorphism of M.*

4. *If $X \leq Y \leq M$ such that X is a fully invariant submodule of M and Y/X is a fully invariant submodule of M/X, then Y is a fully invariant submodule of M.*

Proof $(1), (2), (3)$ see [3, Lemma 1.1].
(4) Let $f : M \to M$ be any homomorphism. Then $f(X) \subseteq X$. Now, consider the homomorphism $g : M/X \to M/X$ defined by $g(m+X) = f(m)+X$, $(m \in M)$. Then $g(Y/X) \subseteq Y/X$. Clearly, $g(Y/X) = (f(Y) + X)/X$. Therefore $f(Y) \subseteq Y$. □

Proposition 3.3 *Let M be a module and X a fully invariant submodule of M. If M is FI-lifting then M/X is FI-lifting.*

Proof Let Y be a submodule of M with $X \subseteq Y$ and assume that Y/X is a fully invariant submodule of M/X. By Lemma 3.2, Y is a fully invariant submodule of M. Since M is FI-lifting, there exists a direct summand D of M such that $D \leq Y$ and Y/D is small in M/D. Assume $M = D \oplus D'$ for some submodule D' of M. Let π be the projection with the kernel D and $i : D' \to M$ the inclusion map. Now, $\alpha = i\pi : M \to M$ be a homomorphism of M. Since X and Y are fully invariant submodules of M, $\alpha(X) \subseteq X$ and $\alpha(Y) \subseteq Y$. It is easy to see that $Y = \alpha^{-1}(Y)$. Now, $\alpha^{-1}(X) \subseteq Y = \alpha^{-1}(Y)$. Let K be a submodule of M with $\alpha^{-1}(X) \subseteq K$ and $M/\alpha^{-1}(X) = (Y/\alpha^{-1}(X)) + (K/\alpha^{-1}(X))$. Then $M = Y + K$ and since Y/D is small in M/D, $M = K$. Therefore $Y/\alpha^{-1}(X)$ is small in $M/\alpha^{-1}(X)$, namely $(Y/X)/(\alpha^{-1}(X)/X) << (M/X)/(\alpha^{-1}(X)/X)$. Now, we want to show that $\alpha^{-1}(X)/X$ is a direct summand of M/X. Since $M = D \oplus D'$, then $M = \alpha^{-1}(X) + D'$. Therefore $M/X = (\alpha^{-1}(X)/X) + (D' + X)/X$. Since $\alpha^{-1}(X) \cap (D' + X) = X + (\alpha^{-1}(X) \cap D') = X$, $\alpha^{-1}(X)/X$ is a direct summand of M/X. □

Theorem 3.4 *Let $M = \oplus_{i=1}^{n} X_i$. If each X_i is FI-lifting, then M is FI- lifting.*

Proof Let S be a fully invariant submodule of M. It is easy to see that for every $1 \leq i \leq n$, $S \cap X_i$ is fully invariant in X_i. Since X_i is FI-lifting for every i, there exists a direct summand D_i of X_i such that $D_i \leq S \cap X_i$ and $(S \cap X_i)/D_i$ is small in X_i/D_i for every i. Clearly, $D = \oplus_{i=1}^{n} D_i$ is a direct summand of M and $D \subseteq \oplus_{i=1}^{n}(S \cap X_i)$. We know that $\oplus_{i=1}^{n}(S \cap X_i) = S$ by Lemma 3.2. Now consider the homomorphism $\beta : \oplus_{i=1}^{n}(X_i/D_i) \to (\oplus_{i=1}^{n} X_i)/D$ with $(x_1 + D_1, ..., x_n + D_n) \to (\Sigma_{i=1}^{n} x_i) + D_i$, where $x_i \in X_i$ for $1 \leq i \leq n$.

Then $\beta(\oplus_{i=1}^{n}((S \cap X_i)/D_i)) = (\oplus_{i=1}^{n}(S \cap X_i))/D$. Since any finite sum of small submodules again a small submodule, $\oplus_{i=1}^{n}((S \cap X_i)/D_i)$ is small in $\oplus_{i=1}^{n}(X_i/D_i)$. Then by [9, Lemma 4.2], $(\oplus_{i=1}^{n}(S \cap X_i))/D$ is small in M/D. $\square$

We don't know if any direct sum of FI-lifting module is an FI-lifting module.

Corollary 3.5 *If M is a finite direct sum of lifting (or hollow) modules, then M is FI-lifting.*

Corollary 3.6 *Let R be a PID. Then the torsion submodule of any finitely generated $R-$module M is FI-lifting.*

Proof Let M be a finitely generated $R-$module. Then the torsion submodule $Tor(M)$ of M is a finite direct sum of hollow R-modules. Therefore $Tor(M)$ is FI-lifting by Corollary 3.5. $\square$

Proposition 3.7 *Let M be an FI-lifting module. If M is indecomposable then every proper fully invariant submodule of M is small in M.*

Proof Clear. $\square$

Proposition 3.8 *Let R be any ring and let M be an FI-lifting R-module. Then every fully invariant submodule of the module $M/Rad(M)$ is a direct summand.*

Proof Let $N/Rad(M)$ be any fully invariant submodule of $M/Rad(M)$.Then N is fully invariant submodule of M by Lemma 3.2. By hypothesis, there exists a decomposition $M = M_1 \oplus M_2$ such that $M_1 \leq N$ and $N \cap M_2$ is small in M_2. Since $N \cap M_2$ is also small in M, $N \cap M_2 \leq Rad(M)$. Thus $M/Rad(M) = (N/Rad(M)) \oplus ((M_2 + Rad(M))/Rad(M))$, as required. $\square$

Example 3.9 (i) Let $M_{\mathbb{Z}} = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$. Then $M_{\mathbb{Z}}$ is FI-lifting by Corollary 3.5. We note that $M_{\mathbb{Z}}$ is not lifting by [8, Example 1] and not non-cosingular module. Furthermore $M_{\mathbb{Z}}$ has the SIP but it is not $(D3)$.
(ii) The $\mathbb{Z}$-module $\mathbb{Q}$, the set of all rational numbers, is non- cosingular module (see example 2.10). $\mathbb{Q}_{\mathbb{Z}}$ is not FI-lifting module.

Example 3.10 Take $R = \mathbb{Z}$ and $M = \mathbb{Z}$. Let $A_i = \mathbb{Z}/2^i\mathbb{Z}$, $foralli \in \mathbb{N}$ and $E = E(A_1)$. Consider $N = \oplus_{i=1}^{n} E_i$, where $E_i = E$ and $n \in \mathbb{N}$. By [11, Example 1.14], N is non-cosingular $\mathbb{Z}$-module and FI-lifting by Corollary 3.5.

Acknowledgments

The author wishes to express his sincere gratitude to his Ph. D. supervisor Prof. Abdullah Harmanci for his encouragement and direction. The author also thank to the Prof. Derya Keskin for valuable suggestions and comments.

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