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SELF-SIMILAR MEASURES AND HARMONIC ANALYSIS

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Abstract

This is a survey of some recent work on the spectral properties and the Fourier asymptotics of self-similar measures defined on $\mathbb{R}$. Given an IFS $\{F_1, ..., F_m\}$ and a set of probability weights $p_1, ..., p_m$, then there is a unique self-similar probability μ which satisfies

$$\mu = \sum_{j=1}^{m} p_j \mu \circ F_j^{-1}.$$

It is known that μ is either purely singular or absolutely continuous. We will explain how this question is closely related to the asymptotic properties of its Fourier transform. We also explore the existence of the orthonormal bases of exponential functions in the $L^2(\mu)$ space and its relation to tiling. Some open questions are listed.

1 Self-similar measures

We first review the concept of Hausdorff measure and Hausdorff dimension (see [5] and [M]). Let E be a subset of $\mathbb{R}^n$ and $s \geq 0$. For all $\delta > 0$ we define

$$\mathcal{H}^s_\delta(E) = \inf\{\sum_{i=1}^{\infty} |U_i|^s : E \subset \bigcup_{i=1}^{\infty} U_i \text{ and} |U_i| < \delta\}$$

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where $|U| = \sup\{|x-y| : x, y \in U\}$ is the diameter of U. As δ decreases, this infimum increases and approaches a limit as $\delta \to 0$, define

$$\mathcal{H}^s(E) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(E) = \sup_{\delta > 0} \mathcal{H}^s_\delta(E).$$

This limit exists, perhaps as 0 or ∞, for all $E \subset \mathbb{R}^n$. $\mathcal{H}^s(E)$ is called the $s-$dimensional Hausdorff measure of E. Hausdorff measure generalizes Lebesgue measure.

It is easy to see that for every set $E \subset \mathbb{R}^n$ there is a number $\dim_H E$, called the Hausdorff dimension of E, such that $\mathcal{H}^s(E) = \infty$ if $s < \dim_H E$ and $\mathcal{H}^s(E) = 0$ if $s > \dim_H E$. Thus

$$\dim_H E = \inf\{s : \mathcal{H}^s(E) = 0\} = \sup\{s : \mathcal{H}^s(E) = \infty\}.$$

So the Hausdorff dimension of a set E may be thought of as the number s at which $\mathcal{H}^s(E)$ 'jumps' from ∞ to 0.

Let D be a closed set in $\mathbb{R}^n$. A map $F : D \to D$ is called a *contraction* on D if there is a number c with $0 < c < 1$ such that $|F(x) - F(y)| \leq c|x - y|$ for all x, y in D.

If the equality holds, i.e., $|F(x) - F(y)| = c|x - y|$, then $F(x) = cRx + b$, where R is an orthogonal matrix and b is a vector in $\mathbb{R}^n$. Thus F transforms sets into geometrically similar ones, and we call F a *similarity.*

Let $\{F_1, ..., F_m\}$ be a finite set of contractions on D, which is also called an *iterated function system* (IFS), then there is a unique compact set E in D which is invariant under the IFS, i.e.,

$$E = \bigcup_{j=1}^{m} F_j(E). \tag{1}$$

To see this, we define a metric between subsets of D. Let Ψ denote the class of all non-empty compact subsets of D. For any $\delta > 0$ and any $A \in \Psi$, let $A_\delta = \{x \in D : |x - a| \leq \delta \text{ for some } a \in A\}$ be the $\delta-$parallel body of A. The *Hausdorff metric* on Ψ is defined by

$$d(A, B) = \inf\{\delta : A \subset B_\delta \text{ and} B \subset A_\delta\}.$$

It is easy to check that d is a complete distance on Ψ.

Let $F : \Psi \to \Psi$ be defined by $F(A) = \bigcup_{j=1}^{m} F_j(A)$. Let A be any compact set in Ψ such that $F_j(A) \subset A$ for all j, for example $D \cap B_r(0)$ for large r will do, where $B_r(0)$ is the closed ball centered at the origin. Then the $k-$th iterate of F, $F^k(A) = F \circ F \circ \cdots \circ F(A)$, is a decreasing sequence of non-empty compact sets and it has a non-empty compact intersection E satisfying $E = F(E) = \bigcup_{j=1}^{m} F_j(E)$. It can be checked that such set E is unique and it is called the *attractor* or the *invariant set* of the IFS $\{F_1, ..., F_m\}$.

When the union $E = \bigcup_{j=1}^m F_j(E)$ is disjoint we say that $\{F_1, ..., F_m\}$ satisfies the *strong separation condition*. A weaker version of the separation condition is the *open set condition*, it means that there is a non-empty bounded open set $U \subset D$ such that

$$\bigcup_{j=1}^{m} F_j(U) \subset U \tag{2}$$

with the union disjoint. If this holds, then $E \subset \overline{U}$, where $\overline{U}$ is the closure of U, indeed, since $F(U) \subset U$, so $F(\overline{U}) \subset \overline{F(U)} \subset \overline{U}$. Thus $F^k(\overline{U})$ is a decreasing sequence of compact sets convergent to the $F-$invariant set E in Hausdorff metric. We have

$$E = \bigcap_{k=1}^{\infty} F^k(\overline{U}) \subset \overline{U}.$$

Clearly, strong separation condition implies the open set condition.

If $F_1, ..., F_m$ are similarities, i.e., $F_j(x) = \rho_j R_j x + b_j$, where $0 < |\rho_j| < 1$, R_j is an orthogonal matrix and b_j is a vector in $\mathbb{R}^n$, for $j = 1, ..., m$, then E in (1.1) is called a self-similar set since it is made up of a finite copies of itself of reduced size. Furthermore, if in addition assume that the IFS satisfies the open set condition, then $0 < \mathcal{H}^\alpha(E) < \infty$, where α is the unique solution to the equation

$$\sum_{j=1}^{m} \rho_j^\alpha = 1. \tag{3}$$

α is called the *similarity dimension* of the IFS $\{F_1, ..., F_m\}$, which is also equal to the Hausdorff dimension of the set E [H].

Example For $0 < \rho < 1$, let $F_1(x) = \rho x$ and $F_2(x) = \rho x + (1 - \rho)$. If $\rho = 1/3$, then the attractor of the contractions F_1, F_2 is the standard middle-third Cantor set, and F_1, F_2 satisfy the strong separation condition. If $\rho = 1/2$, then the attractor of the contractions F_1, F_2 is [0,1], and F_1, F_2 satisfy the open set condition (1.2) with an open set $U = (0, 1)$. If $\rho > 1/2$, then the attractor of the contractions F_1, F_2 is [0,1], but F_1, F_2 do not satisfy the open set condition.

We will be interested in measures supported in E. Given an IFS $\{F_1, ..., F_m\}$ and a set of probability weights $p_1, ..., p_m$, where $0 \le p_j \le 1$ and

$$\sum_{j=1}^{m} p_j = 1.$$

Then there is a unique probability μ which satisfies

$$\mu(A) = \sum_{j=1}^{m} p_j \mu \circ F_j^{-1}(A) \tag{4}$$

for all Borel measurable sets $A \subset E$. We call μ a self-similar measure if all $F_j's$ are similarities.

Let I_A be the indicator function of a subset A. Then (1.4) is equivalent to

$$\int I_A(x)d\mu = \sum_{j=1}^{m} p_j \int I_A(F_j(x))d\mu.$$

By approximations by simple functions, it is also equivalent to

$$\int f(x)d\mu = \sum_{j=1}^{m} p_j \int f(F_j(x))d\mu \tag{5}$$

for every $\mu-$integrable function f.

The existence and the uniqueness of μ was proved by Hutchinson [H] using the contractive mapping principle. To do this we define a metric on the space of all probability measures supported on E. The Hutchinson metric is defined by

$$d(\mu,\nu) = \sup\{|\int f d\mu - \int f d\nu| : f \text{ satisfies} |f(x) - f(y)| \le |x-y|\}.$$

It is not hard to show that this defines a metric and the metric space is complete. It is also easy to show that the mapping $T : \mu \to \sum_{j=1}^{m} p_j \mu \circ F_j^{-1}$ is contractive in the Hutchinson metric. Thus we have existence and uniqueness, with μ given constructively as the limit (in the Hutchinson metric) of iterating T starting with any probability measure.

Example (Continued from Example 1) We assign equal probability $p_1 = p_2 = 1/2$ to F_1 and F_2, respectively, then the invariant measure μ is the Lebesgue measure on [0,1] if $\rho = 1/2$, and μ is the standard middle-third Cantor measure if $\rho = 1/3$.

Let μ be the self-similar measure associated with the probability weights $\{p_1, ..., p_m\}$ and the IFS $\{F_1, ..., F_m\}$, where $F_j(x) = \rho_j R_j x + b_j$, for $j = 1, ..., m$. Then there is a natural choice for these weights, under the open set condition which we call *natural weights*, given by the identity

$$p_j = \rho_j^\alpha$$

where α is the unique value that makes (1.3) hold. Under the open set condition, the value α coincides with the Hausdorff dimension of the attractor E, and we have $0 < \mathcal{H}^\alpha(E) < \infty$, and the self-similar measure μ is equal to a multiple of $\mathcal{H}^\alpha$ restricted on the attractor E [H]. If we choose probability weights p_j not equal to the natural weights, we obtain measures that are still supported in E, but in fact can be supported by a smaller sets. In this sense, the natural weights give rise to the biggest measure. It was shown in [St1] that the minimum Hausdorff dimension of a set that supports μ is given by

$$(\sum_{j=1}^{m} p_j \log p_j)/(\sum_{j=1}^{m} p_j \log \rho_j). \tag{6}$$

Example As in Example 2, we assign the natural weights $p_1 = p_2 = 1/2$ to $F_1(x) = x/2$ and $F_2(x) = x/2 + 1/2$, then we obtain the Lebesgue measure on [0,1]. If we choose different weights we obtain a measure supported on a proper subset of real numbers in [0,1] whose binary expansion has asymptotically a proportion of p_1 zeroes to p_2 ones. This set has Lebesgue measure zero (unless $p_1 = p_2$) and Hausdorff dimension given by (1.6).

For the rest of the paper, unless specified, we will only study the case where $\mathbb{R}^n = \mathbb{R}$.

When the IFS are similarities with equal contraction ratio ρ with $0 < |\rho| < 1$, i.e., $F_j(x) = \rho(x + b_j)$, then $|F_j(x) - F_j(y)| = |\rho||x - y|$ for all $j = 1, ..., m$, and the induced self-similar set and the self-similar measure can be viewed as generated by a sequence of i.i.d. random variables as follows.

Let $X_{1,}\ X_{2,}\ \ldots$ be a sequence of i.i.d. random variables each taking real values $b_{1,}\ \ldots,\ b_m$ with probability $p_{1,}\ \ldots, p_m$ respectively. For $0 < |\rho| < 1$, define a random variable

$$S = S_\rho = \sum_{n=1}^{\infty} \rho^n X_{n.} \tag{7}$$

Let μ_ρ be the probability measure induced by S, i.e.,

$$\mu_\rho(A) = \Pr ob\{\omega : S(\omega) \in A\}. \tag{8}$$

The range of S, or the support of μ_ρ, is given by

$$\begin{aligned} E &= \{\sum_{n=1}^{\infty} \rho^n d_n : d_n \in \{b_1, \ldots, b_m\}\} \\ &= \{\rho(d_1 + \sum_{n=1}^{\infty} \rho^n d_n) : d_n \in \{b_1, \ldots, b_m\}\} \\ &= \bigcup_{j=1}^{m} \rho(b_j + E) \\ &= \bigcup_{j=1}^{m} F_j(E) \end{aligned} \tag{9}$$

where $F_j(x) = \rho(x + b_j)$. Thus E is exactly the invariant compact set under the IFS $\{F_1, ..., F_m\}$. Note that F_j is bijective from E to $F_j(E)$ and E depends only on ρ and the set of digits $b_{1,}\ \ldots, b_m$.

It is easy to see that the random variable S in (1.7) satisfies the equation

$$S = \rho(X_1 + S') \tag{10}$$

where S' has the same distribution as S and it is independent of X_1.

It can be verified that the measure μ_ρ also satisfies equation (1.4). In fact, we use (1.10) along with the total probability formula to obtain

$$\begin{aligned}\mu_\rho(A) &= \Pr ob(S \in A) = \Pr ob(\rho(X_1 + S') \in A) \\ &= \sum_{j=1}^{m} \Pr ob(X_1 = b_j) \Pr ob(\rho(b_j + S') \in A) \\ &= \sum_{j=1}^{m} p_j \Pr ob(F_j(S') \in A) \\ &= \sum_{j=1}^{m} p_j \Pr ob(S' \in F_j^{-1}(A)) \\ &= \sum_{j=1}^{m} p_j \mu_\rho(F_j^{-1}(A)).\end{aligned}$$

By uniqueness we obtain $\mu = \mu_\rho$. In the following we will write μ for μ_ρ if no confusion will occur.

For any real number x, let

$$F_\mu(x) = \mu((-\infty, x])$$

be the distribution function of μ, then $F_\mu(x)$ is continuous by a theorem of Lévy [L]. Furthermore, by Jessen and Wintner's "the law of pure types" theorem, the measure μ is either purely singular or absolutely continuous [JW]. This can also be argued as follows.

Suppose that μ has a nonzero singular component μ_s with $0 < \mu_s(E) \leq 1$. Let $E_s \subset E$ be the support of μ_s,then $\mu_s(E_s) = \mu_s(E)$ and $Leb(\ E_s) = 0$.

Since F_j is bijective from E to $F_j(E)$, so for any subset $A \subseteq E$, the restriction of F_j on A is a bijection from A to its image, it follows that

$$F_j^{-1}(E_s) = F_j^{-1}(F_j(E_s)) = E_s.$$

For any measurable set A by (1.4) we have

$$\mu_s(A) = \mu_s(A \cap E_s) = \mu(A \cap E_s) = \sum_{j=1}^{m} p_j \mu(F_j^{-1}(A \cap E_s))$$

and

$$\begin{aligned}\mu_s(F_j^{-1}(A)) &= \mu_s(F_j^{-1}(A) \cap E_s) = \mu(F_j^{-1}(A) \cap E_s) \\ &= \mu(F_j^{-1}(A) \cap F_j^{-1}(E_s)) = \mu(F_j^{-1}(A \cap E_s)).\end{aligned}$$

It follows that

$$\mu_s(A) = \sum_{j=1}^{m} p_j \mu_s(F_j^{-1}(A)).$$

Hence the normalized probability measure $\mu_s/\mu_s(E)$ also satisfies the self-similar equation (1.4), by uniqueness we obtain $\mu = \mu_s/\mu_s(E)$ and thus μ is purely singular. If $\mu_s = 0$, then μ is absolutely continuous.

If $0 < \mid \rho \mid < 1/m$, then the support of μ is a set of Cantor type and has Lebesgue measure zero; hence μ is purely singular. If $1/m \leq \rho < 1$, then different choice of the values $b_1, \ldots, b_m$ and the probability weights $p, \ldots, p_m$ will produce different type of the measure μ. The determination of which type in general is very difficult, which we will explain in details in the following sections.

2 Fourier transforms

For convenience, we write $e_t(x) = e^{i2\pi tx}$ in the sequel. The study of the measure μ is closely related to the study of its Fourier transform

$$\hat{\mu}(t) = \int_{-\infty}^{\infty} e^{i2\pi tx} d\mu(x) = \int_{-\infty}^{\infty} e_t(x) d\mu(x).$$

It is known that $|\hat{\mu}(t)| \leq \hat{\mu}(0) = 1$, and $\hat{\mu}(t)$ is uniformly continuous on $\mathbb{R}$. Suppose that μ is a self-similar measure defined by (1.4) with $F_j(x) = \rho_j(x+b_j)$, $0 < |\rho_j| < 1$, associated with probability weights p_j for $j = 1, ..., m$. By replacing $f(x)$ by $e_t(x)$ in (1.5) it yields

$$\hat{\mu}(t) = \sum_{j=1}^{m} p_j e_t(\rho_j b_j) \hat{\mu}(\rho_j t).$$

In particular, if $\rho_j = \rho$ for all $j = 1, ..., m$, then

$$\begin{aligned}
\hat{\mu}(t) &= \hat{\mu}(\rho t) \sum_{j=1}^{m} p_j e_t(\rho b_j) \\
&= \hat{\mu}(\rho^2 t) \sum_{j=1}^{m} p_j e_t(\rho^2 b_j) \sum_{j=1}^{m} p_j e_t(\rho b_j) \\
&= \cdots \\
&= \hat{\mu}(\rho^n t) \prod_{k=1}^{n} \sum_{j=1}^{m} p_j e_t(\rho^k b_j)
\end{aligned}$$

Since $\hat{\mu}(t)$ is uniform continuous and $\hat{\mu}(0) = 1$, so

$$\hat{\mu}(t) = \prod_{k=1}^{\infty} \sum_{j=1}^{m} p_j e_t(\rho^k b_j). \tag{11}$$

Note that (2.1) can also be obtained from (1.7) by considering convolutions of i.i.d. random variables. Since $\rho^k X_k$ has a Fourier transform $\sum_{j=1}^m p_j e_t(\rho^k b_j)$, thus the Fourier transform of $S = \sum_{k=1}^\infty \rho^k X_k$ is $\hat{\mu}(t) = \prod_{k=1}^\infty \sum_{j=1}^m p_j e_t(\rho^k b_j)$.

Using (2.1) and denote $\hat{\mu}_\rho(t) = \hat{\mu}(t)$, it is easy to derive the following identity

$$\hat{\mu}_{\rho^{1/n}}(t) = \prod_{k=1}^\infty \sum_{j=1}^m p_j e_t(\rho^{k/n} b_j) = \prod_{l=1}^n \hat{\mu}_\rho(\rho^{\frac{l}{n}-1} t). \tag{12}$$

It is known that the asymptotic behavior of $\hat{\mu}(t)$ gives information on the distribution type of μ. For example, we have the following classical results (Chapter 11, [Ka]):

(1) If $\hat{\mu}(t) \in L^2(\mathbb{R})$ then μ is absolutely continuous and has a density function in L^2 (This ceases to be true if $\hat{\mu}(t) \in L^p(\mathbb{R})$ for $p > 2$. If $p > 2$, then there is a Fourier transform $\hat{\mu}(t) \in L^p(\mathbb{R})$ such that the corresponding probability measure is purely singular.)

(2) If $\hat{\mu}(t) \cdot t^p \in L^1(\mathbb{R})$, where $p \geq 0$ is an integer, then μ is absolutely continuous and its density function has a bounded continuous $(p+1)th$ derivative.

Thus (2.2) gives the asymptotic information of $\hat{\mu}_{\rho^{1/n}}(t)$ in terms of $\hat{\mu}_\rho(t)$. For example, if we know that $\hat{\mu}_\rho(t) \in L^4(\mathbb{R})$, then $\hat{\mu}_{\rho^{1/2}}(t) \in L^2(\mathbb{R})$.

3 Bernoulli convolutions and Fourier asymptotics

A very basic and nontrivial example is when $m = 2$ and each X_n takes two values -1 and 1 with equal probability in (1.7). Then μ is the so called *infinitely convolved Bernoulli measure*, or simply *Bernoulli convolution*. This measure has been studied for seventy years but is still not completely understood today (see [PSS] and the references there).

Let $m = 2$, $p_1 = p_2 = 1/2$, $0 < \rho < 1$ and $b_1 = 1$, $b_2 = -1$ in (2.1), we obtain

$$\hat{\mu}_\rho(t) = \prod_{n=1}^\infty \cos \rho^n 2\pi t. \tag{13}$$

We know that measure is singular for $0 < \rho < 1/2$. For $\rho = 1/2$, then

$$\widehat{\mu}_{1/2}(t) = \prod_{n=0}^\infty \cos(\pi t/2^n).$$

Using a double angle formula $\sin 2x = 2 \sin x \cos x$, for any k we have

$$\begin{aligned}
\prod_{n=0}^k \cos(\pi t/2^n) &= [2^{k+1} \sin(\pi t/2^k) \prod_{n=0}^k \cos(\pi t/2^n)]/[2^{k+1} \sin(\pi t/2^k)] \\
&= \frac{\sin 2\pi t}{2^{k+1} \sin(\pi t/2^k)} \\
&= \frac{\sin 2\pi t}{2\pi t} \frac{\pi t/2^k}{\sin(\pi t/2^k)} \to \frac{\sin 2\pi t}{2\pi t}, \text{ as} k \to \infty.
\end{aligned}$$

Hence $\widehat{\mu}_{1/2}(t) = \frac{\sin 2\pi t}{2\pi t} = \int_{-1}^{1} e^{i2\pi tx} d(\frac{x}{2})$ is the Fourier transform of the probability measure equals to $\frac{1}{2}Leb$ on $[-1,1]$, where Leb denotes the Lebesgue measure.

Using this, it is not hard to verify that for any fixed $k = 1, 2, \ldots, \mu_{2^{-1/k}}$ is absolutely continuous with a bounded density that has a k^{th} derivative. In fact, this can be obtained from (2.2) by letting $\rho = 1/2$. then

$$\hat{\mu}_{2^{-1/k}}(t) = \prod_{j=1}^{k} \frac{\sin(2^{-j/k}4\pi t)}{2^{-j/k}4\pi t}.$$

Hence $\hat{\mu}_{2^{-1/k}}(t) \cdot t^{k-1} \in L^1(\mathbb{R})$ and we obtain the result.

It was conjectured that μ ought to be absolutely continuous for $\rho \geq 1/2$. However, this conjecture is not true and the results are known to be in connection with the algebraic integers.

Definition 3.1. *An algebraic integer $\beta > 1$ is called a PV-number (Pisot-Vijayarahavan number) if all its conjugate roots (i.e., all other roots of its minimal polynomial), denoted by β_i, $i = 1, ..., m$, satisfy $|\beta_i| < 1$.*

For example, every integer greater than one is a PV-number. The golden ratio $(1+\sqrt{5})/2$ is an example of a nontrivial PV-number (It is a root of $x^2 - x - 1 = 0$, its conjugate root is $(1-\sqrt{5})/2$, which is strictly less than one in absolute value.) For any $n \geq 2$, the positive root of the equation $x^n - x^{n-1} - \cdots - x - 1 = 0$ is a PV-number, which is called a simple PV-number.

In 1939 Erdös [3] showed that

Theorem 3.1. *If $1 < \rho^{-1} < 2$ is a PV-number, then the Fourier transform $\hat{\mu}(t)$ does not tend to zero at infinity.*

By the Riemann-Lebesgue lemma (If μ is absolutely continuous then $\hat{\mu}(t)$ tend to zero at infinity) and the "pure theorem", hence μ is purely singular.

To see why this happens, we first understand an important property of a PV number.

A fundamental property of a PV number β_1 is that $dist(\beta_1^n, \mathbb{Z})$ tends to zero at a geometric rate as $n \to \infty$. Hence β_1^n is roughly an integer for all large n. In fact, let $\beta_2, ..., \beta_k$ be all conjugates of β_1 and let

$$\prod_{i=1}^{k}(x - \beta_i) = x^k - \sigma_1 x^{k-1} + \sigma_2 x^{k-2} + \cdots + (-1)^{k-1}\sigma_{k-1}x + (-1)^k \sigma_k$$

be it minimal integral polynomial, where

$$\sigma_1 = \sum_i \beta_i, \ \sigma_2 = \sum_{i<j} \beta_i\beta_j, \ ..., \ \sigma_k = \prod_i \beta_i$$

are all integers by the relation between roots and coefficients. Since $\beta_1^n + \beta_2^n + \cdots + \beta_k^n$ is a symmetric polynomial (a polynomial remains unchanged when all subindexes replaced by any of their permutations), by the Fundamental Theorem on Symmetric Polynomials, there is a unique integral polynomial $p(x_1, x_{2,}..., x_k)$ such that

$$\beta_1^n + \beta_2^n + \cdots + \beta_k^n = p(\sigma_1, \sigma_2, ..., \sigma_k).$$

Hence $\beta_1^n + \beta_2^n + \cdots + \beta_k^n$ is an integer for all n.

Since β_1 is a PV number, by definition, $\max_{2\le j\le k} |\beta_j| = \theta < 1$. Note that $\beta_1^n > 1$, we have for all positive integer n

$$dist(\beta_1^n, \mathbb{Z}) = dist(\beta_2^n + \cdots + \beta_k^n, \mathbb{Z}) \le (k-1)\theta^n.$$

Using this property we can prove the Erdös Theorem. Let $1 < \beta = \rho^{-1} < 2$, then for all $N \ge 1$

$$\hat{\mu}(\beta^N) = \prod_{n=1}^{\infty} \cos(\rho^{n-N} 2\pi) = \hat{\mu}(1) \prod_{j=1}^{N-1} \cos(\beta^j 2\pi).$$

Since β is an algebraic integer, $2\beta^j \not\equiv \frac{1}{2}(\text{mod}1)$ for all j, which implies that $\prod_{j=0}^{N-1} \cos(\beta^j 2\pi) \neq 0$ for all N. Note that $\hat{\mu}(1) \neq 0$, hence $\hat{\mu}(\beta^N) \neq 0$ for all N. Thus

$$|\hat{\mu}(\beta^N)| \ge |\hat{\mu}(1)| \prod_{j=0}^{\infty} \cos((k-1)\theta^j 2\pi) = \delta > 0$$

for some δ. This proves that $\hat{\mu}(t)$ does not tend to zero as $t \to \infty$.

Salem [S] showed that

Theorem 3.2. *For $0 < \rho < 1$ with $\rho \neq 1/2$, $\hat{\mu}(t) \not\to 0$ as $t \to \infty$ only if ρ^{-1} is a PV-number.*

Note that μ is purely singular for $0 < \rho < 1/2$, the Erdös-Salem's theorem gives a large family of singular measures whose Fourier transforms tend to zero at infinity, as is opposed to the Riemann-Lebesgue lemma.

It is known that the set of all PV-numbers is closed and bounded below, hence it has a least element. Siegel [Si] showed that the positive root of $\beta^3 - \beta - 1 = 0$, denoted by $\beta_0 \approx 1.3247179572...$ and $\beta_0^{-1} \approx 0.754877666...$, is this least element. This number is also called the silver number or plastic number. Pisot and Dufresnoy [PD] showed that the golden ratio $(1+\sqrt{5})/2$ is the smallest limit point of such numbers.

Thus by Erdös-Salem's theorem if $\beta_0^{-1} < \rho < 1$, then $\hat{\mu}(t) \to 0$ as $t \to \infty$. But it is unknown whether μ is absolutely continuous when ρ is in this interval. Nevertheless, Erdös (1940) showed that

Theorem 3.3. *There exists a sequence $\rho_k \to 1$ such that μ has k derivatives for almost all $\rho \in (\rho_k, 1)$.*

The proof ideas is as follows.

Erdös use a combinatorial argument to show that there exists $\gamma > 0$ such that

$$|\hat{\mu}_\rho(t)| = O(|t|^{-\gamma}) \text{ for} a.e.\ \rho \in (2^{-1}, 2^{-1/2}).$$

Replacing $\rho^{1/k}$ by ρ in (2.2) we get

$$|\hat{\mu}_\rho(t)| = \prod_{j=1}^{k} |\hat{\mu}_{\rho^k}(\rho^{j-1}t)| = O(|t|^{-k\gamma}) \text{ for} a.e.\ \rho \in (2^{-1/k}, 2^{-1/(2k)}). \tag{14}$$

If we let $\rho_k = 2^{-\gamma/(k+1)}$, then $\rho > \rho_k$ implies that $|\hat{\mu}_\rho(t)| = O(|t|^{-(k+1)})$ for $a.e.\ \rho \in (2^{-\gamma/(k+1)}, 2^{-\gamma/(2k+2)})$. Thus $\hat{\mu}_\rho(t) \cdot t^{k-1} \in L^1$, hence μ is absolutely continuous and its density function is bounded and has k derivative.

Garsia [7] conjectured that μ is absolutely continuous for almost all $1/2 < \rho < 1$. This conjecture has been confirmed recently by Solomyak [So]. He showed that μ is absolutely continuous and has a L^2-density for almost all $1/2 < \rho < 1$. By Plancherel's theorem, a function is in $L^2(\mathbb{R})$ if and only if its Fourier transform is in $L^2(\mathbb{R})$, thus the Fourier transform $\hat{\mu}(t)$ is in $L^2(\mathbb{R})$ for almost all $1/2 < \rho < 1$.

However, the only explicit values of ρ for which μ is known to be absolutely continuous are $\rho = 2^{-1/n}$, for $n = 1, 2, \ldots,$ and $\rho = \beta^{-1}$ satisfies $\beta \prod_{|\beta_i|>1} |\beta_i| = 2$ [7], i.e., this set consists of reciprocals of algebraic integers in (1,2) whose minimal polynomial has other roots outside the unit circle and the constant coefficients ± 2. For instance, the polynomials $x^{n+p} - x^n - 2$ where $p, n \geq 1$ and $\max\{p, m\} \geq 2$. Another of such examples is $x^3 - 2x - 2$.

On the other hand, all PV-numbers are the only numbers for which μ is known to be purely singular.

There was a renewed interest in Bernoulli convolutions since the 1980's, after the discoveries of their importance in various problems in dynamical system [1] and the calculation of the Hausdorff dimensions of some self-affine graphs and self-affine sets ([PU], [PSS], [HL1].)

Questions (The first two have been open for a long time):

(3-1) If ρ^{-1} is not a PV-numbers, can μ be purely singular?

Obviously, by the Erdös-Salem's theorem, we could not use the Fourier theory to prove or disprove this question.

(3-2) Other than the explicit values mentioned above, can we find another value of ρ so that the corresponding measure μ is absolutely continuous?

Recall that μ is absolutely continuous if and only if the lower derivative $\underline{D}(\mu, x) < \infty$ for $\mu-$almost $x \in \mathbb{R}$, where

$$\underline{D}(\mu, x) = \underline{\lim}_{r\to 0}(2r)^{-1}\mu(B_r(x))$$

and $B_r(x) = [x-r, x+r]$ [M]. It was shown [PS2] that $\underline{D}(\mu_\rho, x) < \infty$ for $\mu_\rho-$almost $x \in \mathbb{R}$ and for $Leb-$almost all $\rho \in (1/2, 1)$. Other than the examples provided by Garsia, can we produce a nontrivial example so that $\underline{D}(\mu_\rho, x) < \infty$ for $\mu_\rho-$almost $x \in \mathbb{R}$ for some $\rho \in (1/2, 1)$?

(3-3) By a result from [So], $\hat{\mu}_\rho(t) \in L^2(\mathbb{R})$ for $Leb-$almost all $\rho \in (1/2, 1)$. Can we produce a nontrivial example so that $\hat{\mu}_\rho(t) \in L^2(\mathbb{R})$ for some $\rho \in (1/2, 1)$?

(3-4) If $\beta = \rho^{-1} (\neq 2)$ is a PV-number, how to find $\limsup_{t\to\infty} \hat{\mu}(t)$ and $\liminf_{t\to\infty} \hat{\mu}(t)$ (it was proved that if $\rho^{-1} = n$, for $n = 3, 4, ...$, then $\limsup_{t\to\infty} |\hat{\mu}(t)| = \hat{\mu}(\pi)$ [HL2]).

Recently, Huang and Strichartz [HS] studied the limit

$$g(x) = \limsup_{n\to\infty} \hat{\mu}(R^n x) \tag{15}$$

where $R \geq 4$ is an even integer. They showed that $g(x)$ is different from zero when $x = 0$ or $x = p/R^m$ for integers $m \geq 0$ and even integers p not divisible by R, and 0 otherwise. Thus $g(x)$ is nowhere continuous.

(3-5) Study the properties of $g(x)$ (continuity, supremum, infmum, etc.) defined by (3.3) with R replaced by an odd integer, or by any PV number.

The properties of $\hat{\mu}(t)$ and its square average asymptotic rate $\int_{-T}^{T} |\hat{\mu}(t)|^2 dt$ as $T \to \infty$ were studied extensively by Strichartz [St1-St4], Lau [L1-L2], Lau and Wang [LWa]. It was proved that if the open set condition is assumed then $\int_{-T}^{T} |\hat{\mu}(t)|^2 dt \sim O(T^{1-\alpha})$ as $T \to \infty$, where $\alpha = (\log \sum_{k=1}^{m} p_k^2)/\log \rho$. In [JRS] the following questions are raised:

(3-6) What is the asymptotic rate of $\int_{-T}^{T} |\hat{\mu}(t)|^q dt$ as $T \to \infty$ for $q > 0$?

(3-7) If $\beta = \rho^{-1}$ is not a PV-number, what is the pointwise asymptotic rate of $\hat{\mu}_\rho(t) \to 0$ as $t \to \infty$?

It was shown [PSS] that if $\beta = \rho^{-1}$ is a Salem-number (An algebraic integer $\beta > 1$ is called a Salem-number if all its conjugate roots, denoted by β_i, $i = 1, ..., m$, satisfy $|\beta_i| \leq 1$ and at least one of the conjugates has absolute value equal to one. For example, the positive root of the polynomial $x^4 - x^3 - x^2 - x + 1$ is a Salem number), then

$$\limsup_{t\to\infty} |\hat{\mu}_\rho(t)||t|^\varepsilon = \infty \ \text{forall} \varepsilon > 0.$$

Thus as $t \to \infty$, the upper limit of $\hat{\mu}_\rho(t) \to 0$ at a speed slower than $1/t^\varepsilon$ for all $\varepsilon > 0$.

Note that if ρ^{-1} is a PV-number, then ρ must be a root of a polynomial with coefficients ± 1 [7]. Thus the points in the range of the finite sum of S will have multiple representations. This causes the weight distribution of μ to be extremely irregular [H1], [6]. It has been conjectured that the singularity of μ may only occur when ρ is an algebraic integer and satisfies a polynomial equation with coefficients ± 1 [7]. But this conjecture has not been proved or disproved.

We now consider the biased Bernoulli convolutions. In the classical Bernoulli convolution if we use the probability weights $(p, 1-p)$, where $p \in (0,1)$, to replace the equal probability weights $(1/2, 1/2)$, then we obtain the biased Bernoulli convolutions $\mu^p = \mu^p_\rho$.

It was shown that [PS1] μ^p is purely singular if $0 < \rho < p^p(1-p)^{(1-p)}$ and it is absolutely continuous for *a.e.-Leb* $\rho \in (p^p(1-p)^{(1-p)}, 0.64)$. Note that the interval $(p^p(1-p)^{(1-p)}, 0.64)$ is nonempty for $p \in (0.165, 0.835)$.

In the unbiased case, at least we know absolute continuity for Garsia numbers as above, but no such example is known in the biased case! Also, the Fourier asymptotics for biased Bernoulli convolutions are unknown either.

(3-8) Find a specific ρ and $p \neq 1/2$ so that μ^p_ρ is absolutely continuous.

(3-9) Fixed a specific ρ, find out the values of $p \in (0,1)$ so that $\hat{\mu}^p(t) \to 0$, as $t \to \infty$, and then study the pointwise asymptotic rate of $\hat{\mu}^p(t)$ as $t \to \infty$ and the asymptotic rate of $\int_{-T}^{T} |\hat{\mu}(t)|^q dt$ as $T \to \infty$ for $q > 0$.

There are some partial results on the problem. Let $\hat{\mu}(t)$ be defined by (2.1). The Erdös-Salem's theorem is generalized to $m \geq 2$: if $b_i = i$ and $p_i = 1/m$ for all $i = 1, ..., m$, then $\limsup_{t\to\infty} |\hat{\mu}(t)| > 0$ if and only if ρ^{-1} is a PV-number and it is not a factor of m [H2]. It is unknown that what happens if $b_i's$ are not consecutive integers.

It was also proved that if ρ^{-1} is an irrational PV-number and $b_i's$ are rational, then $\limsup_{t\to\infty} |\hat{\mu}(t)| > 0$ for any probability weights $p_i's$ [LNR]. But what happens if $\rho^{-1} = 3, 4, ...$? For example, let $\rho^{-1} = 3$, $b_1 = 0$, $b_2 = 1$ and $b_3 = 3$, then we have the (0,1,3)-problem. For what choices of the probability weights we will have $\limsup_{t\to\infty} |\hat{\mu}(t)| > 0$? If $m = 2$, then, up to an affine change of variable, we can assume that $b_1 = -1$ and $b_2 = 1$. It follows that if ρ^{-1} is an irrational PV-number, then for any probability weights we have $\limsup_{t\to\infty} |\hat{\mu}(t)| > 0$ and hence μ is purely singular.

Another interesting question is the convolution of singular measures.

For $i = 1, ..., m$, let ρ_i^{-1} be PV-numbers and let $\hat{\mu}_{\rho_i}$ be the Fourier transform of the Bernoulli measure μ_{ρ_i}. Then $\hat{\mu}_{\rho_i}(c_i t)$, where c_i are constants , does not tend to zero at infinity.

(3-10) When does the product $\prod_{j=1}^{m} \hat{\mu}_{\rho_j}(c_j t)$, where ρ_j^{-1} are PV-numbers, tend to zero at infinity and at what rate? If this product does not tend to zero at infinity, then what are the upper limit and the lower limit?

In 1973, Senge and Strauss [SS] showed that $\lim_{t\to\infty} \hat{\mu}_{\rho_1}(t)\hat{\mu}_{\rho_2}(t) \neq 0$ if and only if $\log \rho_1 / \log \rho_2$ is a rational number. Hu and Lau [HL2] showed that if $q \geq 3$ is an integer and c_j's are constants, then c_i/c_j is an irrational number for some i, j implies that $\prod_{j=1}^{m} \hat{\mu}_{1/q}(c_j t) \to 0$ as $t \to \infty$ and the converse is also true if $q \not\equiv 0 \pmod 4$.

4 Spectral measures, spectral sets and tiling

Let μ be a finite positive Borel measure in $\mathbb{R}^n$. We say that μ is a *spectral measure* if there exists a discrete set $\Lambda \subseteq \mathbb{R}^n$ such that the set of exponentials $E(\Lambda) = \{e_\lambda(x) : \lambda \in \Lambda\}$, where $e_\lambda(x) = e^{i2\pi\lambda\cdot x}$, forms an orthogonal basis for $L^2(\mu)$. In this case we call Λ a *spectrum* of μ, and (μ, Λ) a *spectral pair*. Since any of the $(\mu, \Lambda), (\mu, -\Lambda), (\mu, t+\Lambda)$ with $t \in \mathbb{R}^n$ fixed is a spectral pair implies the other two are also spectral pairs, for simplicity we assume that $0 \in \Lambda$. If μ is a spectral measure and equals the n-dimensional Lebesgue measure restricted on a measurable set $\Omega \subset \mathbb{R}^n$, then we say that Ω is *spectral set*. Thus spectral measures are a natural extension of spectral sets.

The inner product and norm on $L^2(\mu)$ are

$$\langle f, g\rangle = \int f\overline{g}d\mu \text{ and} ||f||^2 = \int |f|^2 d\mu.$$

We have

$$\langle e_\lambda, e_\eta\rangle = \int e^{i2\pi(\lambda-\eta)\cdot x} d\mu = \widehat{\mu}(\lambda - \eta).$$

Hence $E(\Lambda)$ is orthogonal in $L^2(\mu)$ if and only if

$$\widehat{\mu}(\lambda - \eta) = 0 \text{ forall} \lambda, \eta \in \Lambda \text{ with} \lambda \neq \eta. \tag{16}$$

For $E(\Lambda)$ to be complete as well we must in addition have

$$\text{Forany} f \in L^2(\mu) : ||f||^2 = \sum_{\lambda\in\Lambda} | \langle f, e_\lambda\rangle |^2.$$

It suffices to have (4.1) valid for $f(x) = e_t(x)$ for all $t \in \mathbb{R}^n$, since the closed linear span of these functions is all of $L^2(\mu)$, i.e., for all $t \in \mathbb{R}^n$,

$$\sum_{\lambda\in\Lambda} | \widehat{\mu}(t - \lambda)|^2 = 1. \tag{17}$$

Definition 4.1. *Let B and Λ be finite subsets in $\mathbb{R}$ with the same cardinality q. Suppose that the $q \times q$ matrix*

$$\{\frac{1}{\sqrt{q}} e_b(\lambda)\}_{b\in B, \lambda\in\Lambda} \tag{18}$$

is unitary, then $\{B, \Lambda\}$ is called a compatible pair.

Let δ_b denote the probability measure concentrated on the single point b.

Proposition 4.2. *Let B and Λ be finite subsets in $\mathbb{R}$ with the same cardinality q. Let*

$$\mu = \frac{1}{q} \sum_{b\in B} \delta_b \tag{19}$$

be a discrete probability measure supported by the set B. Suppose that $\{B, \Lambda\}$ is a compatible pair, then (μ, Λ) a spectral pair, i.e., Λ is a spectrum for μ.

Proof. From (4.3) we have

$$\widehat{\mu}(t) = \frac{1}{q}\sum_{b\in B} e^{i2\pi bt} = \frac{1}{q}\sum_{b\in B} e_b(t).$$

Then for $\lambda, \eta \in \Lambda$

$$\widehat{\mu}(\lambda-\eta) = \frac{1}{q}\sum_{b\in B} e_b(\lambda-\eta).$$

The condition (4.1) is exactly the orthogonality of the rows of the matrix (4.3), providing the orthogonality of the family $E(\Lambda) = \{e^{i2\pi\lambda x} : \lambda \in \Lambda\}$. To show the completeness, we verify (4.2). Indeed, using the orthogonality of the columns of the matrix (4.3),

$$\begin{aligned}
\sum_{\lambda\in\Lambda} |\ \widehat{\mu}(t-\lambda)|^2 &= \sum_{\lambda\in\Lambda} \widehat{\mu}(t-\lambda)\overline{\widehat{\mu}(t-\lambda)} \\
&= q^{-2}\sum_{\lambda\in\Lambda}\sum_{b\in B}\sum_{b'\in B} e_b(t-\lambda)e_{(-b')}(t-\lambda) \\
&= q^{-2}\sum_{\lambda\in\Lambda}\sum_{b\in B}\sum_{b'\in B} e_{(b-b')}(t-\lambda) \\
&= q^{-2}\sum_{b\in B}\sum_{b'\in B} e_{(b-b')}(t)\sum_{\lambda\in\Lambda}\overline{e_{(b-b')}(\lambda)} \\
&= q^{-1}\sum_{b\in B}\sum_{b'\in B} e_{(b-b')}(t)\delta_{b,b'} \\
&= q^{-1}\sum_{b\in B} e_0(t) = q^{-1}q = 1.
\end{aligned}$$

Example Let $B_n = \{0, 1/n, ..., (n-1)/n\}$ and $\Lambda_n(\mathrm{mod} n) = \{0, 1, ..., n-1\}$, then it is easy to verify that $\{B_n, \Lambda_n\}$ is a compatible pair, thus if we let $\mu_n = \frac{1}{n}\sum_{j=0}^{n-1}\delta_{j/n}$, then (μ_n, Λ_n) is a spectral pair (Hence a spectral measure is not necessary a self-similar measure.) For example, let $n = 3$, $B_3 = \{0, 1/3, 2/3\}$, $\Lambda_3 = \{0, 1, 5\}$, then

$$\{\frac{1}{\sqrt{3}}e_b(\lambda)\}_{b\in B_3,\lambda\in\Lambda_3} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{i2\pi/3} & e^{i4\pi/3} \\ 1 & e^{i4\pi/3} & e^{i2\pi/3}\end{pmatrix} = U$$

so that $\overline{U^T}U$ equals an identity and thus U is unitary. We also have $\mu_3 = \frac{1}{3}(\delta_0 + \delta_{1/3} + \delta_{2/3})$, $\widehat{\mu}_3(t) = \frac{1}{3}(1 + e^{i2\pi t/3} + e^{i4\pi t/3})$ and $\sum_{\lambda\in\Lambda_3} |\ \widehat{\mu}(t-\lambda)|^2 = 1$.

In particular, if we replace n by 2^n, then B_{2^n} is the range of the partial sum $S_n = \sum_{j=1}^{n} 2^{-j}X_j$, where X_j takes two values 0 and 1 with equal probability. Let $n \to \infty$, then S_n converges to the Bernoulli convolution $\sum_{j=1}^{\infty} 2^{-j}X_j$, which has a range $[0, 1]$, and μ_n converges weakly to the Lebesgue measure on $[0, 1]$

and its spectrum $\Lambda = \{0, 1, ..., n, ...\}$. This is a classical result (In the limit case, the proof for (μ, Λ) to be a spectral pair is nontrivial [JP].) In particular, $[0, 1]$ is a spectral set.

Example The first fractal spectral measure (A Cantor measure with scale four) was produced by Jorgensen and Pedersen in 1998 [JP]. Define a scale four Bernoulli convolution by (1.7) and (1.8) with partial sum $S_n = \sum_{j=1}^{n} 4^{-j} X_j$, where X_j take two values 0 and 2 with equal probability. Then the range of S_n is $B_n = \{\sum_{j=1}^{n} 4^{-j}\varepsilon_j : \varepsilon_j = 0, 2\}$. Let $\Lambda_n = \{\sum_{j=0}^{n-1} 4^j \varepsilon_j : \varepsilon_j = 0, 1\}$, then $\{B_n, \Lambda_n\}$ is a compatible pair, and (μ_n, Λ_n) is a spectral pair for $\mu_n = 2^{-n}\sum_{b\in B_n}\delta_b$. Passing to the limit (a nontrivial proof, see [JP],[St5]), we obtain a Cantor set and a Cantor measure μ on [0,1] of scale four, and its spectrum $\Lambda = \{\sum_{j=0}^{n} 4^j\varepsilon_j : \varepsilon_j = 0, 1, n = 0, 1, ...\} = \{0, 1, 4, 5, 16, 17, 20, 21, ...\}$.

It is often nontrivial to prove the exponentials $E(\Lambda) = \{e_\lambda(x) : \lambda \in \Lambda\}$ to be complete in $L^2(\mu)$. See [JP], [St5-6], and [LaW] for various methods employed to achieve this goal.

Example For the Bernoulli convolution μ induced by $S = \sum_{j=0}^{\infty} N^{-j} X_j$, where $N > 1$ is an odd integer and X_j take two values 0 and b with equal probability. Then any set of $\mu-$orthogonal exponentials contains at most two elements [JP]. The proof is simple. Indeed, using $\frac{1}{2}(1 + e^{i2x}) = e^{ix}\cos x$ we obtain the Fourier transform of μ

$$\widehat{\mu}(t) = \prod_{j=0}^{\infty} \frac{1}{2}(1 + e^{\frac{i2\pi bt}{N^j}}) = e^{\frac{i\pi bNt}{N-1}} \prod_{j=0}^{\infty} \cos(\frac{\pi bt}{N^j}).$$

It is easy to see that $\widehat{\mu}(t) = 0$ if and only if t is a root of a factor in the right-hand side product. Thus the zero set of $\widehat{\mu}(t)$ is

$$Z(\widehat{\mu}) = \{\frac{N^k}{2b}(2\mathbb{Z} + 1) : k = 0, 1, ...\}.$$

If $\lambda_j, j = 1, 2, 3$, are such that the $e'_{\lambda_j}s$ are mutually orthogonal in $L^2(\mu)$, by (4.1) $\widehat{\mu}(\lambda_i - \lambda_j) = 0$ for $i \neq j$. Let $\eta_1 = \lambda_1 - \lambda_2, \eta_2 = \lambda_2 - \lambda_3$ and $\eta_3 = \lambda_1 - \lambda_3$ and

$$\eta_j = \frac{N^{k_j}}{2b}(2z_j + 1),\ \ z_j \in \mathbb{Z}.$$

Since $\eta_1 + \eta_2 = \eta_3$, we obtain

$$N^{k_1}(2z_1 + 1) + N^{k_2}(2z_2 + 1) = N^{k_3}(2z_3 + 1)$$

which is impossible since the left-hand side is even while the right-hand side is odd.

By generalizing the original work of Jorgensen and Pedersen [JP], Several authors [St5-6], [LaW] have studied the spectral properties of self-similar measure μ_D induced by the following random variable

$$S = \sum_{j=1}^{\infty} N^{-j} X_j$$

where N is an integer with $|N| > 1$, and X_j take m integral values in $D = \{b_1, ..., b_m\} \subseteq \mathbb{Z}$ with equal probability $\frac{1}{m}$. For any finite set A of $\mathbb{R}$, let

$$\Lambda(A) = \{\sum_{j=0}^{k} N^j a_j : a_j \in A \text{ and} k \geq 1\}.$$

Laba and Wang [LaW] showed that

Theorem 4.1. *Let $S \subseteq Z$ be such that $0 \in S$ and $(\frac{1}{N}D, S)$ is a compatible pair. Then μ_D is a spectral measure. If moreover* $\gcd(D - D) = 1$, *and $S \subseteq [2 - |N|, |N| - 2]$, then $\Lambda(S)$ is a spectrum for μ_D.*

The self-similar spectral measures founded so far are all have equal weights and with contraction ratio $1/N$ with some integer N. It is not clear whether this must be in general true. In [LaW], the following conjecture was raised.

Conjecture 1 Let S_ρ be the random variable defined by (1.7) with each X_n taking values in $B = \{b_1, b_2, \ldots, b_m\}$ with positive probability $p_1, p_2, \ldots, p_m$ respectively. Let μ_ρ be the self-similar measure defined by (1.8). Suppose that μ is a spectral measure. Then

(4-1) $\rho = 1/N$ for some integer N.

(4-2) $p_1 = p_2 = \ldots = p_m = 1/m$.

(4-3) Suppose that $0 \in B$. Then $\alpha B \subseteq \mathbb{Z}$ for some $\alpha \in \mathbb{R}$.

The conjecture has not be settled even in the simple case where μ_ρ is the Bernoulli convolution. Recently, Hu [H3] studied the spectral properties of Bernoulli convolutions (with equal probability weights) and show that any orthogonal set of exponential functions in $L^2(\mu_\rho)$ contains only finite members if and only if ρ is NOT a k^{th} root of a fraction with an even integer as denominator and an odd integer as numerator.

(4-4) Can the result be generalized to the self-similar measure defined by (1.8) with equal probability weights?

The main interest for studying spectral sets comes from its mysterious connection to tiling, first formulated by B. Fuglede [4] in 1974, known today as the *Fuglede Conjecture*:

Conjecture 2 A measurable set $\Omega \subseteq \mathbb{R}^n$ with positive Lebesgue measure is a spectral set if and only if it tiles $\mathbb{R}^n$ by translation.

To understand the conjecture, we first review some concepts in tiling.

A Borel set Ω in $\mathbb{R}^n$ of positive measure is said to *tile* $\mathbb{R}^n$ by translation if there is a discrete set $\mathcal{T}$ in $\mathbb{R}^n$ such that, up to sets of Lebesgue measures 0, the sets $\Omega + t$ are disjoint and

$$\mathbb{R}^n = \bigcup_{t \in \mathcal{T}} (\Omega + t).$$

Then we call Ω a *tile*, $\mathcal{T}$ a *translation set* and $(\Omega, \mathcal{T})$ a *tiling* of $\mathbb{R}^n$. If Ω tiles $\mathbb{R}^n$ by translation, by the Baire Category Theorem, then Ω has nonempty interior. If further, there is $\lambda \neq 0$ such that $\mathcal{T} + \lambda = \mathcal{T}$, then we say that $(\Omega, \mathcal{T})$ is a *periodic* tiling with period λ.

We give some examples in one dimensional case.

For any integer $m \geq 2$, let $\rho = 1/m$ in (1.7) with associated IFS $\{F_1, ..., F_m\}$ consisting $F_j(x) = \frac{1}{m}(x + b_j)$, $j = 1, ..., m$, then the attractor Ω in (1.9) satisfies $\Omega = \bigcup_{j=1}^{m} \frac{1}{m}(\Omega + b_j)$, or

$$m\Omega = \bigcup_{j=1}^{m} (\Omega + b_j). \tag{20}$$

If $Leb(\Omega) > 0$, then we say that Ω is a *self-similar tile*. Since $Leb(m\Omega + b) = mLeb(\Omega)$, hence (4.5) implies $Leb((\Omega + b_i) \cap (\Omega + b_j)) = 0$ for $i \neq j$. If the $b_j's$ are integers, then we say that Ω is an *integral* self-similar tile.

It is known that any self-similar tile Ω tiles $\mathbb{R}$ by translation [GH]. Thus if Ω is a self-similar tile, then $Leb(\Omega) > 0$ if and only if Ω tiles $\mathbb{R}$ by translation. In particular, its interior Ω^o is nonempty, using the similarity, it is easy to show that $\Omega = \overline{\Omega^o}$. Furthermore, since $Leb(F_i(\Omega) \cap F_j(\Omega)) = 0$, hence $F_i(\Omega^o) \cap F_j(\Omega^o) = \phi$, for $i \neq j$. Note that F_j is a bijection on Ω, it sends open set to open set, this along with $F_j(\Omega^o) \subseteq F_j(\Omega) \subseteq \Omega$ implies that $F_j(\Omega^o) \subseteq \Omega^o$. Thus if Ω is a self-similar tile then the IFS $\{F_1, ..., F_m\}$ satisfies the open set condition.

Kenyon, Lagarias and Wang [K], [LW] showed that every 1-dimensional self-similar tile is an integral self-similar tile in essence: there is a real number α such that $\alpha\{b_1, ..., b_m\} \subseteq \mathbb{Z}$. Thus the study of self-similar tiling can be reduced to the study of integral self-similar tiling. On the other hand, a translation of the digit set $\{b_1, ..., b_m\}$ will result in a translation of the tile Ω, hence, without loss of generality, we can always assume that

$$0 \in \{b_1, ..., b_m\} \subseteq \mathbb{Z} \text{ and} g.c.d.(b_1, ..., b_m) = 1.$$

For the structure of the translation set $\mathcal{T}$, it was showed that [K], [LW] self-similar tiling implies that $(\Omega, \mathcal{T})$ is a periodic tiling, Lau and Rao [LR] further showed that $\mathcal{T} = \mathcal{T} + m^k$ for some integer k.

A fundamental problem: Given m, find the digit set $\{b_1, ..., b_m\}$ so that Ω is a self-similar tile.

This question is still largely unsettled. The most basic result is due to Bandt [2]: If $\{b_1, ..., b_m\}$ is a complete residue set modulo m, then Ω is a self-similar tile. This is also a necessary condition for Ω to be a self-similar tile if m is prime [LW]. Lau and Rao [LR] showed that if $b_j \not\equiv 0 \pmod{m}$ for all $b_j \neq 0$, then Ω is a self-similar tile if and only if $\{b_1, ..., b_m\}$ is a complete residue set modulo m.

When m is a power of a prime or is a product of two primes, then sufficient and necessary conditions on the digit set $\{b_1, ..., b_m\}$ for Ω to be a self-similar tile are given [LW], [LR]. But it is still unknown in general.

(4-5) Find a sufficient and necessary condition on the digit set $\{b_1, ..., b_m\}$ for Ω to be a self-similar tile if m is any integer (You may first try the case where $m = pqr$ with p, q and r are primes, for example, let $m = 12$.)

If we assign equal probability weights $p_j = 1/m$ to each map F_j for $j = 1, ..., m$, then we obtain a self-similar measure μ satisfying $\mu = \frac{1}{m}\sum_{j=1}^{m} \mu \circ F_j^{-1}$. It is easy to verify that Ω is a self-similar tile if and only if μ equals the normalized restriction of the Lebesgue measure on Ω, i.e., $\mu = \frac{Leb|_\Omega}{Leb(\Omega)}$. In fact, if Ω is a self-similar tile, then $Leb(F_i(\Omega) \cap F_j(\Omega)) = 0$. This, together with the fact that F_j is bijective from Ω to $F_j(\Omega)$, implies that $Leb|_\Omega$, the restriction of the Lebesgue measure on Ω also satisfies this equation, so does its normalization $\frac{Leb|_\Omega}{Leb(\Omega)}$. By uniqueness, $\mu = \frac{Leb|_\Omega}{Leb(\Omega)}$. Conversely, if $\mu = \frac{Leb|_\Omega}{Leb(\Omega)}$, then $Leb(\Omega) > 0$ and Ω is a self-similar tile.

We now come back to the Fuglede conjecture.

It was disproved very recently that the conjecture is not true in higher dimension. Tao [T] exhibited a spectral set in dimension $n \geq 5$ that is not a tile, and Kolountzakis and Matolsci [KM] exhibited tiles that are not spectral sets in dimension $n \geq 5$. Despite the counterexamples, the connection between spectral sets and tiling is strongly evident, especially in lower dimension. For example, it was proved [IP], [LRW] that if $\Omega = (-1/2, 1/2)^n$ is the unit cube in $\mathbb{R}^n$ and $\Lambda \subseteq \mathbb{R}^n$, then Λ is a spectrum of Ω if and only if $\Omega + \Lambda = \mathbb{R}^n$. The conjecture has also been verified for the convex regions in $\mathbb{R}^2$, namely, the only convex regions in $\mathbb{R}^2$ which are both spectral and tiles are the parallelograms and the symmetric hexagons [IKT]. However, even in the one dimensional case, the conjecture is still unsettled. Let's consider the sets of the type

$$\Omega = A + (0, 1), \text{ where} A \text{ isafinitesubsetofintegers}.$$

Laba [La1], [La2] showed that the conjecture is true for $\#A = 2$ and 3.

(4-6) Is the Fuglede conjecture true for any finite set A?

Another way to look at the problem is to consider the regions of self-similar tile. We know that a self-similar tile may have infinitely many connected components (For example, try the case $m = 3$ with the digit set $\{b_1, b_2, b_3\} = \{0, 1, 5\}$, a complete residue set modulo 3.) If $\{b_1, ..., b_m\}$ is a complete residue set modulo m, then Ω is a self-similar tile. We can verify that in this case Ω is also a spectral set. In fact, let $B_m = \{0, 1/m, ..., (m-1)/m\}$, $\Lambda_m (\mathrm{mod} m) = \{b_1, ..., b_m\} (\mathrm{mod} m) = \{0, 1, ..., m-1\}$ and $0 \in \Lambda_m$, then $\{B_m, \Lambda_m\}$ is a compatible pair. By Theorem 2 (Laba and Wang), the self-similar measure

$$\mu = \frac{1}{m} \sum_{j=1}^{m} \mu \circ F_j^{-1}$$

is a spectral measure, where $F_j(x) = \frac{1}{m}(x + b_j), j = 1, ..., m$. We know that $\mu = \frac{Leb|_\Omega}{Leb(\Omega)}$, thus $Leb|_\Omega$ is a spectral measure, so Ω is a spectral set.

(4-7) Prove or disprove that if Ω is a self-similar tile, then Ω is a spectral set.

If the answer to (4-7) is negative, then the Fuglede conjecture fails even in one dimensional.

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