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ON A SUBCLASS OF 5-DIMENSIONAL SOLVABLE LIE ALGEBRAS WHICH HAVE 3-DIMENSIONAL COMMUTATIVE DERIVED IDEALS

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Abstract

The paper presents a subclass of the class of MD5-algebras and MD5-groups, i.e., five dimensional solvable Lie algebras and Lie groups such that their orbits in the co-adjoint representation (K-orbit) are orbit of zero or maximal dimension. The main results of the paper is the classification up to an isomorphism of all MD5-algebras $\mathcal{G}$ with the derived ideal $\mathcal{G}^1 := [\mathcal{G}, \mathcal{G}]$ is a 3-dimensional commutative Lie algebra.

Introduction

In 1962, studying theory of representations, Kirillov [3] introduced the Orbit Method. This method quickly became the most important method in the theory of representations of Lie groups. The Kirillov's Orbit Method immediately was expanded by Kostant, Auslander, Do Ngoc Diep,etc. Using the Kirillov's Orbit Method, we can obtain all the unitary irreducible representations of solvable and simply connected Lie groups. The importance of Kirillov's Orbits Method is the co-adjoint representation (K-representation). Therefore, it is meaningful to study the K-representation in the theory of representations of Lie groups.

Key words: Lie group, Lie algebra, MD5-group, MD5-algebra, K-orbits.
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The structure of solvable Lie groups and Lie algebras is not too complicated, although the complete classification of them is unsolved up to now. In 1980, studying the Kirillov's Orbit Method, D. N. Diep [2] introduced the class of Lie groups and Lie algebras MD. Let G be an n-dimensional real Lie group. It is called an MDn-group iff its orbits in the K-representation (i.e. K-orbits) are orbits of dimension zero or maximal dimension. The corresponding Lie algebra of G is called an MDn-algebra. Thus, classification and studying of K-representation of the class of MDn-groups and MDn-algebras are problems of great interest. Because all Lie algebras of n dimension (with $n \leq 3$) were listed easily, we will consider MDn-groups and MDn-algebras with $n \geq 4$.

In 1984, Dao Van Tra [5] listed all MD4-algebras. In 1992, all MD4-algebras were classified up to an isomorphism by the author (see [6], [7], [8]). Until now, no complete classifications of MDn-algebras with $n \geq 5$ are known. Three examples of MD5 - algebras and MD5 - groups can be found in [9] and some different MD5 - algebras and MD5 - groups can be found in [10]. In this paper we shall give a classification up to an isomorphism of all MD5-algebras $\mathcal{G}$ with the derived ideal $\mathcal{G}^1 := [\mathcal{G}, \mathcal{G}]$ is a 3-dimensional commutative Lie algebra. The complete classification of all MD5-algebras will be presented.

1 Preliminaries

At first, we recall some preliminary results and notations which will be used later. For more details we refer the readers to references [2], [3], [4].

1.1 Lie Groups and Lie Algebras

Definition 1.1. *A real Lie group of dimension n is a C^∞-manifold G endowed with a group structure such that the map $(g, h) \mapsto g.h^{-1}$ from $G \times G$ into G is C^∞-differentiable.*

Definition 1.2. *A real Lie algebra $\mathcal{G}$ of dimension n is an n-dimensional real vector space together with a skew-symmetric bilinear map $(X, Y) \mapsto [X, Y]$ from $\mathcal{G} \times \mathcal{G}$ into $\mathcal{G}$ (which is called the Lie bracket) such that the following Jacobi identity is satisfied : $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ for every $X, Y, Z \in \mathcal{G}$.*

1.2 The co-adjoint Representation, K-orbits MDn-Groups and MDn-Algebras

Each Lie group G defines a Lie algebra Lie(G) $= \mathcal{G}$ as the tangent space T_eG of G at the unitary element e of G with the Lie bracket is defined by commutators. Conversely, each real Lie algebra $\mathcal{G}$ defines only one connected and simply connected Lie group G such Lie(G) $= \mathcal{G}$. For each g $\in$ G, we denote the internal automorphism associated with g by $A_{(g)}$. So $A_{(g)} : G \longrightarrow G$ is defined as follows

$$A_{(g)}(x) := g.x.g^{-1}, \forall x \in G.$$

This automorphism induces the following map

$$A_{(g)_*} : \mathcal{G} \longrightarrow \mathcal{G}$$

$$X \longmapsto A_{(g)_*}(X) := \frac{d}{dt}[g.exp(tX)g^{-1}] \mid_{t=0}$$

which is called the *tangent map* of $A_{(g)}$.

Definition 1.3. The action

$$Ad : G \longrightarrow Aut(\mathcal{G})$$

$$g \longmapsto Ad(g) := A_{(g)_*}$$

is called *the adjoint representation* of G in $\mathcal{G}$.

Definition 1.4. The action

$$K : G \longrightarrow Aut(\mathcal{G}^*)$$

$$g \longmapsto K_{(g)}$$

such that

$$\langle K_{(g)}F, X\rangle := \langle F, Ad(g^{-1})X\rangle; \quad (F \in \mathcal{G}^*, X \in \mathcal{G})$$

is called the *co-adjoint representation* of G in $\mathcal{G}^*$.

Definition 1.5. Each orbit of the co-adjoint representation of G is called a *K-orbit* of G.

Thus, for every $F \in \mathcal{G}^*$, the K-orbit containing F is defined as follows

$$\Omega_F := \{K_{(g)}F/g \in G\}.$$

The dimension of every K-orbit of G is always even. In order to define the dimension of the K-orbits Ω_F, it is useful to consider the skew-symmetric bilinear form B_F on $\mathcal{G}$ as follows

$$B_F(X, Y) := \langle F, [X, Y]\rangle; \forall X, Y \in \mathcal{G}.$$

Denote the stabilizer of F under the co-adjoint representation of G in $\mathcal{G}^*$ by G_F and $\mathcal{G}_F :=$ Lie(G_F). We shall need in the sequel the following assertion.

Proposition 1.6 (see [3]). *$KerB_F = \mathcal{G}_F$ and $dim\Omega_F = dim\mathcal{G} - dim\mathcal{G}_F$.* $\square$

Definition 1.7. (see [2]) An *MDn-group* is an n-dimensional real solvable Lie group such that its K-orbits are orbits of dimension zero or maximal dimension. The Lie algebra of an MDn-group is called an *MDn-algebra*.

The following proposition give us a necessary condition in order that a Lie algebra belongs to the class of all MD-algebras.

Proposition 1.8 (see [4]). *Let $\mathcal{G}$ be an MD-algebra. Then its second derived ideal $\mathcal{G}^2 := [[\mathcal{G},\mathcal{G}],[\mathcal{G},\mathcal{G}]]$ is commutative.* □

Note, however, that the converse of this statement in general does not hold. In other words, the above necessary condition is not the sufficient one.

2 The Main Result

From now on, $\mathcal{G}$ will denote a Lie algebra of dimension 5. We always choose a suitable basis $(X_1, X_2, X_3, X_4, X_5)$ in $\mathcal{G}$. Then $\mathcal{G}$ isomorphic to $\mathbb{R}^5$ as a real vector space. The notation $\mathcal{G}^*$ will mean the dual space of $\mathcal{G}$. Clearly $\mathcal{G}^*$ can be identified with $\mathbb{R}^5$ by fixing in it the basis $(X_1^*, X_2^*, X_3^*, X_4^*, X_5^*)$ dual to the basis $(X_1, X_2, X_3, X_4, X_5)$.

Theorem 2.1. *Let $\mathcal{G}$ be an MD5-algebra with $\mathcal{G}^1 := [\mathcal{G},\mathcal{G}] \cong \mathbb{R}^3$ (the 3-dimensional commutative Lie algebra).*

I. *Assume that $\mathcal{G}$ is decomposable. Then $\mathcal{G} \cong \mathcal{H} \oplus \mathbb{R}$, where $\mathcal{H}$ is an MD4-algebra.*

II. *Assume that $\mathcal{G}$ is indecomposable. Then we can choose a suitable basis $(X_1, X_2, X_3, X_4, X_5)$ of $\mathcal{G}$ such that $\mathcal{G}^1 = \mathbb{R}.X_3 \oplus \mathbb{R}.X_4 \oplus \mathbb{R}.X_5 \equiv \mathbb{R}^3$, $ad_{X_1} = 0$, $ad_{X_2} \in End(\mathcal{G}^1) \equiv Mat_3(\mathbb{R})$; $[X_1, X_2] = X_3$ and $\mathcal{G}$ is isomorphic to one and only one of the following Lie algebras:*

1. $\mathcal{G}_{5,3,1(\lambda_1,\lambda_2)}$ *:*

$$ad_{X_2} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \lambda_1, \lambda_2 \in \mathbb{R} \setminus \{1\}, \lambda_1 \neq \lambda_2 \neq 0.$$

2. $\mathcal{G}_{5,3,2(\lambda)}$ *:*

$$ad_{X_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}; \quad \lambda \in \mathbb{R} \setminus \{0, 1\}.$$

3. $\mathcal{G}_{5,3,3(\lambda)}$:

$$ad_{X_2} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \lambda \in \mathbb{R} \setminus \{1\}.$$

4. $\mathcal{G}_{5,3,4}$:

$$ad_{X_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

5. $\mathcal{G}_{5,3,5(\lambda)}$:

$$ad_{X_2} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}; \quad \lambda \in \mathbb{R} \setminus \{1\}.$$

6. $\mathcal{G}_{5,3,6(\lambda)}$:

$$ad_{X_2} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}; \quad \lambda \in \mathbb{R} \setminus \{0, 1\}.$$

7. $\mathcal{G}_{5,3,7}$:

$$ad_{X_2} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

8. $\mathcal{G}_{5,3,8(\lambda,\varphi)}$:

$$ad_{X_2} = \begin{pmatrix} cos\varphi & -sin\varphi & 0 \\ sin\varphi & cos\varphi & 0 \\ 0 & 0 & \lambda \end{pmatrix}; \quad \lambda \in \mathbb{R} \setminus \{0\}, \, \varphi \in (0, \pi).$$

In order to prove Theorem 2.1 we need some lemmas.

Lemma 2.2. *Under the above notation. We have* $ad_{X_1} \circ ad_{X_2} = ad_{X_2} \circ ad_{X_1}$.

Proof Using the Jacobi identity for X_1, X_2 and $X_i (i = 3, 4, 5\ )$, we have

$$\begin{aligned} & [[X_1, X_2], X_i] + [[X_2, X_i], X_1] + [[X_i, X_1], X_2] = 0 \\ \Leftrightarrow\ & [X_1, [X_2, X_i]] - [X_2, [X_1, X_i]] = 0 \\ \Leftrightarrow\ & ad_{X_1} \circ ad_{X_2}(X_i) = ad_{X_2} \circ ad_{X_1}(X_i);\ i = 3, 4, 5 \\ \Leftrightarrow\ & ad_{X_1} \circ ad_{X_2} = ad_{X_2} \circ ad_{X_1}. \end{aligned}$$

□

Lemma 2.3 (see[2], [4]). *If $\mathcal{G}$ is an MD-algebra and $F \in \mathcal{G}^*$ is not perfectly vanishing on $\mathcal{G}^1$, i.e. there exists $U \in \mathcal{G}^1$ such that $\langle F, U\rangle \neq 0$, then the K-orbit Ω_F is the one of maximal dimension.*

Proof Assume that Ω_F is not a K-orbit of maximal dimension, i.e. $dim\Omega_F = 0$. This means that

$$dim\mathcal{G}_F = dim\mathcal{G} - dim\Omega_F = dim\mathcal{G}.$$

So $KerB_F = \mathcal{G}_F = \mathcal{G} \supset \mathcal{G}^1$ and F is perfectly vanishing on $\mathcal{G}^1$. This contradicts the supposition of the lemma. Therefore Ω_F is a K-orbit of maximal dimension. □

We are now in a position to prove the main theorem of the paper.

Proof of Theorem 2.1.

Firstly, we can always choose some basis $(X_1, X_2, X_3, X_4, X_5)$ of $\mathcal{G}$ such that $\mathcal{G}^1 = \mathbb{R}.X_3 \oplus \mathbb{R}.X_4 \oplus \mathbb{R}.X_5 \equiv \mathbb{R}^3$; $ad_{X_1}, ad_{X_2} \in End(\mathcal{G}^1) \equiv Mat_3(\mathbb{R})$.

It is obvious that ad_{X_1} and ad_{X_2} cannot be concurrently the trivial operator because $\mathcal{G}^1 \cong \mathbb{R}^3$. There is no loss of generality in assuming $ad_{X_2} \neq 0$. By changing basis, if necessary, we get the similar classification of ad_{X_2} as follows: $\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & 1\end{pmatrix}$, $(\lambda_1, \lambda_2 \in \mathbb{R}\setminus\{1\}, \lambda_1 \neq \lambda_2 \neq 0)$; $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \lambda\end{pmatrix}$, $(\lambda \in \mathbb{R}\setminus\{0,1\})$; $\begin{pmatrix}\lambda & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$, $(\lambda \in \mathbb{R}\setminus\{1\})$; $\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$; $\begin{pmatrix}\lambda & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}$, $(\lambda \in \mathbb{R}\setminus\{1\})$; $\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 0\\ 0 & 0 & \lambda\end{pmatrix}$, $(\lambda \in \mathbb{R}\setminus\{0,1\})$; $\begin{pmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}$; $\begin{pmatrix}cos\varphi & -sin\varphi & 0\\ sin\varphi & cos\varphi & 0\\ 0 & 0 & \lambda\end{pmatrix}$, $(\lambda \in \mathbb{R}\setminus\{0\}, \varphi \in (0,\pi))$.

Assume that $[X_1, X_2] = mX_3 + nX_4 + pX_5; m, n, p \in \mathbb{R}$. We can always change basis to have $[X_1, X_2] = mX_3$. Indeed, if

$$ad_{X_2} = \begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & 1\end{pmatrix}, (\lambda_1, \lambda_2 \in \mathbb{R}\setminus\{1\}, \lambda_1 \neq \lambda_2 \neq 0),$$

then by changing X_1 for $X_1^{'} = X_1 + \frac{n}{\lambda_2}X_4 + pX_5$ we get $[X_1^{'}, X_2] = mX_3$, $m \in \mathbb{R}$. For the other values of ad_{X_2}, we also change basis in the same way. Hence, without restriction of generality, we can assume right from the start that $[X_1, X_2] = mX_3$, $m \in \mathbb{R}$.

There are three cases which contradict each other as follows.

(1) $[X_1, X_2] = 0$ (i.e. $m = 0$) and $ad_{X_1} = 0$. Then $\mathcal{G} = \mathcal{H} \oplus \mathbb{R}.X_1$, where $\mathcal{H}$ is the subalgebra of $\mathcal{G}$ generated by $\{X_2, X_3, X_4, X_5\}$, i.e. $\mathcal{G}$ is decomposable.

(2) $[X_1, X_2] = 0$ and $ad_{X_1} \neq 0$.

(2a) Assume $ad_{X_2} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $(\lambda_1, \lambda_2 \in \mathbb{R}\backslash\{1\},\ \lambda_1 \neq \lambda_2 \neq 0)$.

In view of Lemma 2.2, it follows by a direct computation that

$$ad_{X_1} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & \xi \end{pmatrix};\ \mu, \nu, \xi \in \mathbb{R};\ \mu^2 + \nu^2 + \xi^2 \neq 0.$$

If $\xi \neq 0$, by changing $X_1^{'} = X_1 - \xi X_2$, we get

$$ad_{X_1^{'}} = \begin{pmatrix} \mu^{'} & 0 & 0 \\ 0 & \nu^{'} & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

where $\mu^{'} = \mu - \xi\lambda_1, \nu^{'} = \nu - \xi\lambda_2$. Thus, we can assume from the outset that

$$ad_{X_1} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \nu & 0 \\ 0 & 0 & 0 \end{pmatrix};\ \mu, \nu \in \mathbb{R};\ \mu^2 + \nu^2 \neq 0.$$

Let $F = \alpha X_1{}^* + \beta X_2{}^* + \gamma X_3{}^* + \delta X_4{}^* + \sigma X_5{}^* \in \mathcal{G}^*$ and $U = aX_1 + bX_2 + cX_3 + dX_4 + fX_5 \in \mathcal{G};\ \alpha, \beta, \gamma, \delta, \sigma, a, b, c, d, f \in \mathbb{R}$. So we have

$$\begin{aligned} \mathcal{G}_F &= Ker B_F \\ &= \{U \in \mathcal{G} / \langle F, [U, X_i] \rangle = 0;\ i = 1, 2, , 3, 4, 5\}. \end{aligned}$$

Upon simple computation, we get

$$U \in \mathcal{G}_F \Leftrightarrow M \begin{pmatrix} a \\ b \\ c \\ d \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where

$$M := \begin{pmatrix} 0 & 0 & \mu\gamma & \nu\delta & 0 \\ 0 & 0 & -\lambda_1\gamma & -\lambda_2\delta & -\sigma \\ \mu\gamma & \lambda_1\gamma & 0 & 0 & 0 \\ \nu\delta & \lambda_2\delta & 0 & 0 & 0 \\ 0 & \sigma & 0 & 0 & 0 \end{pmatrix}.$$

Hence, $dim\Omega_F = dim\mathcal{G} - dim\mathcal{G}_F = rank(M)$. According to Lemma 2.3, Ω_F is a K-orbit of maximal dimension if $F|_{\mathcal{G}^1} \neq 0$, i.e. if $\gamma^2 + \delta^2 + \sigma^2 \neq 0$. In particular, $rank(M)$ is a constant if γ, δ, σ are not concurrently zeros. However, it is easily seen that $rank(M) = 2$ when $\gamma = \delta = 0 \neq \sigma$, but $rank(M) = 4$ when all of γ, δ, σ are different zero. This contradiction show that Case (2a) cannot happen.

(2b) By the similar argument and replacing the considered value of ad_{X_2} with the others, we can see that Case (2) cannot happen anyway.

(3) $[X_1, X_2] \neq 0$ (i.e. $m \neq 0$). By changing X_1 by $X_1^{'} = \frac{1}{m}X_1$ one has $[X_1^{'}, X_2] = X_3$. Hence, without loss of generality, we can assume that $[X_1, X_2] = X_3$.

By an argument similar to the one in Case (2a), again we get a contradiction if $ad_{X_1} \neq 0$. Hence, $ad_{X_1} = 0$. Therefore, in the dependence on the value of ad_{X_2}, $\mathcal{G}$ will be isomorphic to one of algebras $\mathcal{G}_{5,3,1(\lambda_1,\lambda_2)}, (\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0,1\}, \lambda_1 \neq \lambda_2 \neq 0)$; $\mathcal{G}_{5,3,2(\lambda)}, (\lambda \in \mathbb{R} \setminus \{0,1\})$; $\mathcal{G}_{5,3,3(\lambda)}, (\lambda \in \mathbb{R} \setminus \{1\})$; $\mathcal{G}_{5,3,4}$; $\mathcal{G}_{5,3,5(\lambda)}$, $(\lambda \in \mathbb{R} \setminus \{1\})$; $\mathcal{G}_{5,3,6(\lambda)}$, $(\lambda \in \mathbb{R} \setminus \{0,1\})$; $\mathcal{G}_{5,3,7}$; $\mathcal{G}_{5,3,8(\lambda,\varphi)}, (\lambda \in \mathbb{R} \setminus \{0\}, \varphi \in (0, \pi)))$. Obviously, these algebras are not isomorphic to each other.

To complete the proof, it remains to show that all of these algebras are MD5-algebras. At first, we shall verify this assertion for $\mathcal{G} = \mathcal{G}_{5,3,1(\lambda_1,\lambda_2)}$, $(\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0,1\}, \lambda_1 \neq \lambda_2 \neq 0)$. Consider an arbitrary linear form $F = \alpha X_1{}^* + \beta X_2{}^* + \gamma X_3{}^* + \delta X_4{}^* + \sigma X_5{}^* \in \mathcal{G}^*$; ($\alpha, \beta, \gamma, \delta, \sigma \in \mathbb{R}$). We need prove that $dim\Omega_F = dim\mathcal{G} - dim\mathcal{G}_F$ is zero or maximal.

Let $U = aX_1 + bX_2 + cX_3 + dX_4 + fX_5 \in \mathcal{G}$; ($a, b, c, d, f \in \mathbb{R}$). Upon simple computation which is similar to one in Case (2a), we get

$$U \in \mathcal{G}_F \Leftrightarrow N \begin{pmatrix} a \\ b \\ c \\ d \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where

$$N := \begin{pmatrix} 0 & -\gamma & 0 & 0 & 0 \\ \gamma & 0 & -\lambda_1\gamma & -\lambda_2\delta & -\sigma \\ 0 & \lambda_1\gamma & 0 & 0 & 0 \\ 0 & \lambda_2\delta & 0 & 0 & 0 \\ 0 & \sigma & 0 & 0 & 0 \end{pmatrix}.$$

Hence, $dim\Omega_F = dim\mathcal{G} - dim\mathcal{G}_F = rank(N)$. It is plain that

$$rank(N) = \begin{cases} 0 & if \quad \gamma = \delta = \sigma = 0; \\ 2 & if \quad \gamma^2 + \delta^2 + \sigma^2 \neq 0. \end{cases}$$

Therefore, Ω_F is the orbit of dimension zero or two (maximal dimension) for any $F \in \mathcal{G}^*$, i.e. $\mathcal{G} = \mathcal{G}_{5,3,1(\lambda_1,\lambda_2)}$ is an MD5-algebra, $(\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0,1\}, \lambda_1 \neq \lambda_2 \neq 0)$. By the same way, we can be also seen that the remaining algebras are MD5-algebras. The proof is complete. $\square$

Remark

Let us recall that each real Lie algebra $\mathcal{G}$ defines only one connected and simply connected Lie group G such Lie(G) $= \mathcal{G}$. Therefore we obtain a collection of eight families of connected and simply connected MD5-groups corresponding to given MD5-algebras in Theorem 2.1. For convenience, each MD5-group from this collection is also denoted by the same indices as corresponding MD5-algebra. For example, $G_{5,3,1(\lambda_1,\lambda_2)}$ is the connected and simply connected MD5-group corresponding to $\mathcal{G}_{5,3,1(\lambda_1,\lambda_2)}$. Especially, we have eight families of MD5-groups as follows: $G_{5,3,1(\lambda_1,\lambda_2)}, (\lambda_1, \lambda_2 \in \mathbb{R}\setminus\{0,1\}, \lambda_1 \neq \lambda_2 \neq 0)$; $G_{5,3,2(\lambda)}, (\lambda \in \mathbb{R}\setminus\{0,1\})$; $G_{5,3,3(\lambda)}, (\lambda \in \mathbb{R}\setminus\{1\})$; $G_{5,3,4}$; $G_{5,3,5(\lambda)}, (\lambda \in \mathbb{R}\setminus\{1\})$; $G_{5,3,6(\lambda)}, (\lambda \in \mathbb{R} \setminus \{0,1\})$; $G_{5,3,7}$; $G_{5,3,8(\lambda,\varphi)}, (\lambda \in \mathbb{R} \setminus \{0\}, \varphi \in (0,\pi)))$. All of them are indecomposable MD5-groups.

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