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ON $\overline{Z}_M$-SEMIPERFECT MODULES

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Abstract

Let τ_M be any preradical for $\sigma[M]$ and N any module in $\sigma[M]$. N is called a τ_M-*semiperfect module* if for every submodule K of N, there is a decomposition $K = A \oplus B$ such that A is a projective direct summand of N in $\sigma[M]$ and $B \subseteq \tau_M(N)$. In this paper we prove that any finite direct sum of τ_M-semiperfect modules is τ_M-semiperfect. It is also shown that if M is a local projective module in $\sigma[M]$, then for every index set Λ, the sum $M^{(\Lambda)}$ is $\overline{Z}_M$-semiperfect in $\sigma[M]$ if and only if every factor module of $M^{(\Lambda)}$ has a projective $\overline{Z}_M$-cover.

1 Introduction

Throughout this paper all rings are associative with identity and modules are unitary right modules. For any module M, τ_M will denote a preradical in $\sigma[M]$. Like in [2], a module $N \in \sigma[M]$ is called τ_M-lifting if for every submodule K of N, there exists a decomposition $K = A \oplus B$ such that A is a direct summand of N and $B \subseteq \tau_M(N)$. According to [9], any module N in $\sigma[M]$ is called *semiperfect* in $\sigma[M]$ if every factor module of N has a projective cover. By [9, 41.14 and 42.1], if $P \in \sigma[M]$ is projective in $\sigma[M]$, then P is semiperfect if and only if for every submodule K of P, there exists a decomposition $K = A \oplus B$ such that A is a direct summand of P and $B \ll P$. Recently, Özcan and Alkan [7] have defined the τ_M-semiperfect modules in $\sigma[M]$ for any preradical τ_M on $\sigma[M]$. Inspired by this work, we mainly study $\overline{Z}_M$-semiperfect modules in

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$\sigma[M]$ in this paper. A module $N \in \sigma[M]$ is called $\overline{Z}_M$-*semiperfect* in $\sigma[M]$ if for every submodule K of N, there is a decomposition $K = A \oplus B$ such that A is a projective direct summand of N in $\sigma[M]$ and $B \subseteq \overline{Z}_M(N)$.

In Section 2, we will be concerned with the structure of $\overline{Z}_R(R_R)$. It is shown that if R is a right good ring, then $\overline{Z}_R(R_R) = ann_r(Rad(R))$. Section 3 is devoted to the study of τ_M-lifting modules. Some results deal with the case $\tau_M = \overline{Z}_M$. In Section 4, we prove that any finite direct sum of τ_M-semiperfect modules is τ_M-semiperfect. In [2], the authors called a module $L \in \sigma[M]$ τ_M-semiperfect, if every factor module of N has a projective τ_M-cover in $\sigma[M]$. Section 5 establishes the relation between this definition of τ_M-semiperfect modules and the one given in [7] in some special cases. In particular, we prove the following Proposition:

Let M be a local projective module in $\sigma[M]$. The following are equivalent for a module N in $\sigma[M]$ which is isomorphic to $M^{(\Lambda)}$ for some index set Λ:

(1) Every factor module of N has a projective $\overline{Z}_M$-cover.

(2) N is $\overline{Z}_M$-semiperfect.

2 Some Properties of $\overline{Z}_R(R_R)$

Let M be an R-module. A module $N \in \sigma[M]$ is said to be M-*small* if there exists a module $L \in \sigma[M]$ such that $N \ll L$.

Let $N \in \sigma[M]$. In [8], Talebi and Vanaja define $\overline{Z}_M(N)$ as follows: $\overline{Z}_M(N) = \cap\{Ker(g) \mid g \in Hom(N, L), L \text{ is } M\text{-small}\}$. The module N is called M-*cosingular* (*non-M-cosingular*) if $\overline{Z}_M(N) = 0$ ($\overline{Z}_M(N) = N$).

The following proposition maybe well-known. We give here its proof for the sake of completeness.

Proposition 2.1. *Let M be any R-module. Then we have:*

(1) $Rad(M)\overline{Z}_R(R_R) = 0$.

(2) $M\overline{Z}_R(R_R) \leq \overline{Z}_R(M)$.

Proof (1) Let $x \in Rad(M)$ and let $a \in \overline{Z}_R(R_R)$. Consider the homomorphism $f : R \longrightarrow xR$ defined by $f(r) = xr$. Since $xR \ll M$, $\overline{Z}_R(R_R) \leq Ker(f)$. Thus $f(a) = 0$. That is $xa = 0$.

(2) Let $x \in M$ and consider the homomorphism $f : R \longrightarrow M$ defined by $f(r) = xr$. Since $f(\overline{Z}_R(R_R)) \leq \overline{Z}_R(M)$ (see [8, Proposition 2.1(2)]), it follows that $x\overline{Z}_R(R_R) \leq \overline{Z}_R(M)$. Therefore $M\overline{Z}_R(R_R) \leq \overline{Z}_R(M)$. □

Corollary 2.2. *If R is a ring having a radical R-module M with $ann_r(M) = 0$, then $\overline{Z}_R(R_R) = 0$.*

Proof Since $Rad(M)\overline{Z}_R(R_R) = 0$ and $Rad(M) = M$, we have $M\overline{Z}_R(R_R) = 0$. But $ann_r(M) = 0$. So $\overline{Z}_R(R_R) = 0$. □

Example 2.3. *We consider the ring* $\mathbb{Z}$. *We know that* $\mathbb{Q}$ *is a radical faithful* $\mathbb{Z}$*-module. Then* $\overline{Z}_{\mathbb{Z}}(\mathbb{Z}) = 0$ *by Corollary 2.2.*

The proof of the next Proposition is similar to the proof of [1, Proposition 17.10].

Proposition 2.4. *Let* R *be any ring with* $I = \overline{Z}_R(R_R)$. *If* P *is a projective right* R*-module, then* $\overline{Z}_R(P) = PI$.

Proof Since P is projective, P is a direct summand of a free module $R^{(A)}$. So there exists a submodule Q of $R^{(A)}$ such that $R^{(A)} = P \oplus Q$. By [8, Proposition 2.1(4)], $\overline{Z}_R(P) \oplus \overline{Z}_R(Q) = (\overline{Z}_R(R_R))^{(A)} = I^{(A)} = R^{(A)}.I = PI \oplus QI$. But $PI \leq \overline{Z}_R(P)$ by Proposition 2.1. Then $\overline{Z}_R(P) = PI$. □

Definition 2.5. *Following [3, p. 236], a ring* R *is called a right good ring if for every right* R*-module* M *we have* $MRad(R) = Rad(M)$.

Clearly, any semilocal ring is a good ring (see [3, Theorem 9.7.1]). Especially, every artinian ring is a good ring.

Proposition 2.6. *Let* R *be a right good ring. We have* $\overline{Z}_R(R_R) = ann_r(Rad(R))$.

Proof By Proposition 2.1(1), $\overline{Z}_R(R_R) \subseteq ann_r(Rad(R))$.

Now let $r \in ann_r(Rad(R))$ i.e. $Rad(R)r = 0$ and let $f : R \longrightarrow L$ be a homomorphism where L is a small submodule of an R-module X. Since $Rad(X) = XRad(R)$, we have $Rad(X)r = 0$. Thus $Lr = 0$ and hence $f(1)r = 0$. That is $r \in Ker(f)$. Therefore $r \in \overline{Z}_R(R_R)$ and so $ann_r(Rad(R)) \subseteq \overline{Z}_R(R_R)$. Consequently, $\overline{Z}_R(R_R) = ann_r(Rad(R))$. □

Corollary 2.7. *Let* R *be a semilocal ring. Then:*

(1) $\overline{Z}_R(R_R) = Soc(_RR)$ *is an ideal of* R.

(2) If P *is a projective* R*-module, then* $\overline{Z}_R(P) = PSoc(_RR)$.

(3) If $Soc(_RR) = Soc(R_R)$, *then for every projective right* R*-module* P *we have* $\overline{Z}_R(P) = Soc(P)$.

Proof (1) By Proposition 2.6, $\overline{Z}_R(R_R) = ann_r(Rad(R))$. By [1, Proposition 15.17], $Soc(_RR) = ann_r(Rad(R))$. The result follows.

(2) By (1) and Proposition 2.4.

(3) It is known that $Soc(P) = PSoc(R_R)$ (See [1, Exercise 17.12]). But by hypothesis, we have $Soc(_RR) = Soc(R_R)$. So $Soc(P) = PSoc(_RR)$. By (2), we get $\overline{Z}_R(P) = Soc(P)$.

□

Corollary 2.8. *Let* R *be a QF ring. Then for every projective right* R*-module* P *we have* $\overline{Z}_R(P) = Soc(P)$.

Proof By Corollary 2.7 and [5, Corollary 15.7]. □

Examples 2.9. *(1) We consider the ring* $\mathbb{Z}$. *We have* $Rad(\mathbb{Z}) = 0$. *Then* $ann_r(Rad(\mathbb{Z})) = \mathbb{Z}$. *But* $\overline{Z}_{\mathbb{Z}}(\mathbb{Z}) = 0$. *Therefore* $\overline{Z}_{\mathbb{Z}}(\mathbb{Z}) \neq ann_r(Rad(\mathbb{Z}))$.

(2) Let R *be a Dedekind domain and let* M *be any radical* R*-module. Then* $\overline{Z}_R(M) = M$. *In fact, if* $f : M \longrightarrow L$ *is a homomorphism where* L *is a small* R*-module, then* $\frac{M}{Ker(f)} \cong Im(f)$. *Hence* $Im(f)$ *is radical. Since* R *is a Dedekind domain,* $Im(f)$ *is injective. But* $Im(f)$ *is a small* R*-module. Thus* $Im(f) = 0$ *and* $Ker(f) = M$. *In particular, we have* $\overline{Z}_{\mathbb{Z}}(\mathbb{Q}) = \mathbb{Q}$.

(3) If R *is semisimple, then* $\overline{Z}_R(R_R) = R$ *(because the only* R*-small module is* 0*).*

Recall that an R-module M is called *V-module* if every simple module in $\sigma[M]$ is M-injective. By [8, Proposition 2.5], if M is an R-module, then M is a V-module if and only if every module in $\sigma[M]$ is non-M-cosingular. Moreover, by [8, Corollary 2.6], R is a right V-ring if and only if the module R is non-R-cosingular. We can also give the following Proposition:

Proposition 2.10. *A ring* R *is a right* V*-ring if and only if for every right* R*-modules* M *and* N *with* $N \in \sigma[M]$*, we have* $\overline{Z}_M(N) = N$.

Proof Assume that R is a right V-ring. Then for every right R-module L we have $Rad(L) = 0$. So 0 is the only M-small R-module. Then $\overline{Z}_M(N) = N$. The converse is clear by [8, Corollary 2.6]. □

3 τ_M-lifting modules

Lemma 3.1. *Let* $N \in \sigma[M]$*. The following are equivalent:*

(i) For every submodule K *of* N*, there is a decomposition* $K = A \oplus B$ *such that* A *is a direct summand of* N *and* $B \subseteq \tau_M(N)$.

(ii) For every submodule K *of* N*, there is a direct summand* A *of* N *such that* $A \subseteq K$ *and* $K/A \subseteq \tau_M(N/A)$.

(iii) For every submodule K *of* N*, there is a decomposition* $N = A \oplus B$ *such that* $A \subseteq K$ *and* $B \cap K \subseteq \tau_M(N)$.

Proof This is clear. □

A module $N \in \sigma[M]$ is called τ_M*−lifting* if it satisfies one of the equivalent conditions of Lemma 3.1. τ_M-lifting modules are studied in [2] for any radical τ on $\sigma[M]$. Note that by [2, Proof of 2.10(2)], any direct summand of a τ_M*−lifting* module is again τ_M*−lifting* for every preradical τ_M.

Theorem 3.2. *Let $N \in \sigma[M]$ with $N = N_1 \oplus N_2$ be a direct sum of relatively projective modules N_1 and N_2 such that N_1 is semisimple and N_2 is τ_M-lifting. Then N is τ_M-lifting.*

Proof We follow the proof in [4, Theorem 6]. Let L be any submodule of N. Note that $K = N_1 \cap (L + N_2)$ is a submodule of N_1. Then $N_1 = K \oplus K^{'}$ for some submodule $K^{'}$ of N_1 and hence $N = K \oplus K^{'} \oplus N_2 = L + (N_2 \oplus K^{'})$. It is easy to see that K is $(N_2 \oplus K^{'})$-projective. Then there exists a submodule $L^{'}$ of L such that $N = L^{'} \oplus (N_2 \oplus K^{'})$ by [9, 41.14]. Since $K^{'} \cap (L + N_2) = K^{'} \cap K = 0$, it is easy to check that $L \cap (X + K^{'}) = X \cap (L + K^{'})$ for every $X \leq N_2$. Note that N_2 is τ_M-lifting and $N_2 \cap (L + K^{'})$ is a submodule of N_2. Therefore there exists a decomposition $N_2 = A_1 \oplus A_2$ such that $A_1 \subseteq N_2 \cap (L + K^{'})$ and $A_2 \cap (L + K^{'}) \subseteq \tau_M(N_2)$. Thus $N = (L^{'} \oplus A_1) \oplus (A_2 \oplus K^{'})$, $L^{'} \oplus A_1 \subseteq L$ and $L \cap (A_2 \oplus K^{'}) = A_2 \cap (L + K^{'}) \subseteq \tau_M(N)$. Consequently, N is τ_M-lifting. □

The following result can be found also in [2] or in [7].

Lemma 3.3. *Let N be τ_M-lifting in $\sigma[M]$. Then $Rad(N) \subseteq \tau_M(N)$.*

Proof Since N is τ_M-lifting in $\sigma[M]$, $N/\tau_M(N)$ is semisimple. The result follows. □

Proposition 3.4. *Let $N \in \sigma[M]$ be an indecomposable module such that N is τ_M-lifting in $\sigma[M]$. Then either $\tau_M(N) = N$ or N is a local module with $\tau_M(N) = Rad(N)$.*

Proof Suppose that $\tau_M(N) \neq N$. Let X be any submodule of N. Then $X = A \oplus B$ with A is a direct summand of N and $B \leq \tau_M(N)$. But N is indecomposable. Thus $A = 0$ or $A = N$. So $X \leq \tau_M(N)$ or $X = N$. Therefore N is a local module and $\tau_M(N)$ is the maximal submodule of N. □

Corollary 3.5. *Let $N \in \sigma[M]$ be a local module. The following are equivalent:*
(1) N is τ_M-lifting in $\sigma[M]$.
(2) either $\tau_M(N) = N$ or $\tau_M(N) = Rad(N)$.

Proof (1) ⇒ (2) By Proposition 3.4.
(2) ⇒ (1) Immediate. □

The following result gives some examples of modules M such that M is $\overline{Z}_M$-lifting and shows that, in general, a local module over a ring R need not be τ_M-lifting.

Proposition 3.6. *Let R be a local ring with maximal ideal m. Then the following are equivalent:*
(i) R_R is $\overline{Z}_R$-lifting.
(ii) $m^2 = 0$.

Proof From Proposition 2.6 we have $\overline{Z}_R(R_R) = ann_r(Rad(R)) = ann_r(m)$. The result is a consequence of Corollary 3.5. □

Proposition 3.7. *Let N be a τ_M-lifting module in $\sigma[M]$ such that $N = \oplus_{j\in J} N_j$ is a finite direct sum of indecomposable submodules N_j $(j \in J)$. Then $N = K \oplus L$ such that $\tau_M(K) = K$ and L is a direct sum of local submodules with $\tau_M(L) = Rad(L)$.*

Proof By [2, Proof of 2.10(2)], every N_j $(j \in J)$ is τ_M-lifting. It follows that for every $j \in J$, we have $\tau_M(N_j) = N_j$ or $\tau_M(N_j) = Rad(N_j)$ by Proposition 3.4. The result follows. □

Corollary 3.8. *Let $N \in \sigma[M]$ be a module with finite hollow dimension such that N is projective in $\sigma[M]$. If N is τ_M-lifting, then $N = K \oplus L$ with K is semiperfect in $\sigma[M]$ and $L = \tau_M(L)$.*

Proof It is well-known that N is a finite direct sum of indecomposable submodules. The result follows from Proposition 3.7 and [9, 42.3]. □

Corollary 3.9. *Let N be a $\overline{Z}_M$-lifting module in $\sigma[M]$ such that $N = \oplus_{i\in I} N_i$ is a direct sum of indecomposable submodules N_i $(i \in I)$. Then $N = K \oplus L$ such that $\overline{Z}_M(K) = K$, L is a direct sum of local submodules and $\overline{Z}_M(L) = Rad(L)$.*

Proof By Proposition 3.4 and [8, Proposition 2.1(4)]. □

Proposition 3.10. *Let R be a commutative ring and M an R-module. Let L be a local $\overline{Z}_M$-lifting module in $\sigma[M]$. Then L is simple or $\overline{Z}_M(L) = Rad(L)$.*

Proof Suppose that L is not simple. Let $x \in L$ such that $L = xR$ and let N be the maximal submodule of L. By Lemma 3.3, we have $N \subseteq \overline{Z}_M(L)$. Let $a \in N$ such that $a \neq 0$. It is clear that aR is an M-small module. Consider the homomorphism $f : xR \longrightarrow aR$ defined by $f(xr) = ar$ for all $r \in R$. It is well defined, because if $xr = 0$ for some $r \in R$, then $r \in ann_r(L)$ (R is commutative), and hence $ar = 0$. This gives $\overline{Z}_M(L) \subseteq Ker(f)$. Since $Ker(f) \neq L$ $(f \neq 0)$, $Ker(f) \subseteq N$. Thus $\overline{Z}_M(L) \subseteq N$. Therefore $\overline{Z}_M(L) = Rad(L)$. □

Corollary 3.11. *Suppose that R is commutative. Let P be a projective semiperfect module in $\sigma[M]$. Then the following are equivalent:*

(1) P is $\overline{Z}_M$-lifting.

(2) $P = P_1 \oplus P_2$ with $Rad(P_1) = \overline{Z}_M(P_1)$ and P_2 is semisimple.

Proof (1) $\Rightarrow$ (2) Since P is projective semiperfect in $\sigma[M]$, P is a direct sum of local submodules and $Rad(P) \ll P$ by [9, 42.4]. By Proposition 3.10 and [8, Proposition 2.1], $P = P_1 \oplus P_2$ with $Rad(P_1) = \overline{Z}_M(P_1)$ and P_2 is semisimple.

(2) $\Rightarrow$ (1) Since P_1 is projective semiperfect in $\sigma[M]$, as a direct summand of P, and $Rad(P_1) = \overline{Z}_M(P_1)$, P_1 is $\overline{Z}_M$-lifting. From Theorem 3.2, we conclude that P is $\overline{Z}_M$-lifting. □

4 τ_M-Semiperfect Modules

Let τ_M be any preradical for $\sigma[M]$ and let N be a module in $\sigma[M]$. The module N is called a *τ_M-semiperfect module* in $\sigma[M]$ if it satisfies one of the following equivalent two conditions (See [7, Proposition 2.1]):

(1) For every submodule K of N, there is a decomposition $K = A \oplus B$ such that A is a projective direct summand of N in $\sigma[M]$ and $B \subseteq \tau_M(N)$;

(2) For every submodule K of N, there is a decomposition $N = A \oplus B$ such that A is projective in $\sigma[M]$, $A \leq K$ and $K \cap B \leq \tau_M(N)$.

It is easy to see that every τ_M-semiperfect module is τ_M-lifting and any projective module in $\sigma[M]$ is τ_M-lifting if and only if it is τ_M-semiperfect in $\sigma[M]$.

Examples 4.1. *(1) Let K be a field and let $R = \prod_{n\geq 1} K_n$ with $K_n = K$ for all $n \geq 1$. Then the ring R is a von Neumann regular ring which is not semisimple (See [3, p. 264]). Hence the R-module R is not semiperfect* $(Rad(R) = 0)$. *On the other hand, since $\overline{Z}_R(R) = R$, the module R is $\overline{Z}_R$-semiperfect.*

(2) If R is a DVR, then the R-module R_R is semiperfect but by Proposition 3.6, the module R_R is not $\overline{Z}_R$-semiperfect.

Theorem 4.2. *Any finite direct sum of τ_M-semiperfect modules is τ_M-semiperfect.*

Proof To prove this result it is sufficient by induction to prove that a direct sum of two τ_M-semiperfect modules in $\sigma[M]$ is again τ_M-semiperfect in $\sigma[M]$. Let $N = N_1 \oplus N_2$ such that N_1 and N_2 are τ_M-semiperfect modules in $\sigma[M]$. Let L be any submodule of N. Note that $N_1 \cap (L + N_2)$ is a submodule of N_1 and N_1 is τ_M-semiperfect in $\sigma[M]$. Thus there exists a decomposition $N_1 = A_1 \oplus B_1$ such that A_1 is projective in $\sigma[M]$ with $A_1 \subseteq N_1 \cap (L+N_2)$ and $B_1 \cap (L + N_2) \subseteq \tau_M(N_1)$. Then $N = L + (B_1 \oplus N_2)$. Since $N_2 \cap (L + B_1)$ is a submodule of N_2, there exists a decomposition $N_2 = A_2 \oplus B_2$ such that A_2 is projective in $\sigma[M]$ with $A_2 \subseteq N_2 \cap (L + B_1)$ and $B_2 \cap (L + B_1) \subseteq \tau_M(N_2)$. Then $N = L + (B_1 \oplus B_2) = (A_1 \oplus A_2) \oplus (B_1 \oplus B_2)$. Since A_1 and A_2 are projective in $\sigma[M]$, $A_1 \oplus A_2$ is $(B_1 \oplus B_2)$-projective. Hence by [9, 41.14], $N = L^{'} \oplus (B_1 \oplus B_2)$ for some submodule $L^{'}$ of N with $L^{'} \subseteq L$. Clearly, $L^{'}$ is projective in $\sigma[M]$. Now, $L \cap (B_1 \oplus B_2) \subseteq B_1 \cap (L + B_2) + B_2 \cap (L + B_1)$

implies that $L \cap (B_1 \oplus B_2) \subseteq \tau_M(N_1) \oplus \tau_M(N_2) = \tau_M(N)$. Therefore N is τ_M-semiperfect in $\sigma[M]$. □

Note that Theorem 4.2 generalizes [7, Theorem 2.10].

Corollary 4.3. *Suppose that R is commutative. Let P be a projective module in $\sigma[M]$ with $Rad(P) \ll P$ and P has finite hollow dimension. Then the following are equivalent:*

(1) P is $\overline{Z}_M$-semiperfect.

(2) $P = P_1 \oplus P_2 \oplus P_3$ with P_1 is semiperfect and $Rad(P_1) = \overline{Z}_M(P_1)$, P_2 is semisimple and $\overline{Z}_M(P_3) = P_3$.

Proof (1) $\Rightarrow$ (2) By Corollary 3.8, $P = K \oplus L$ with K is semiperfect and $L = \overline{Z}_M(L)$. Now following Corollary 3.11, $K = K_1 \oplus K_2$ with $Rad(K_1) = \overline{Z}_M(K_1)$ and K_2 is semisimple. It is clear that K_1 is semiperfect in $\sigma[M]$.

(2) $\Rightarrow$ (1) It is clear that P_1, P_2 and P_3 are all $\overline{Z}_M$-semiperfect in $\sigma[M]$. So P is $\overline{Z}_M$-semiperfect by Theorem 4.2. □

5 $\overline{Z}_M$-semiperfect Modules and $\overline{Z}_M$-cover

Let $N \in \sigma[M]$. We call an epimorphism $f : P \longrightarrow N$ a projective τ_M-cover of N in $\sigma[M]$ if P is projective in $\sigma[M]$ and $Ker(f) \subseteq \tau_M(P)$.

In [2], the authors called a module $L \in \sigma[M]$ τ_M-semiperfect, if every factor module of N has a projective τ_M-cover. In [7, Theorem 2.23], the authors showed that this definition agrees with the one given in this paper for a projective module in $\sigma[M]$ and for the preradical *Soc*. It is of interest to know whether these two definitions coincide in the case $\tau_M = \overline{Z}_M$ for projective modules. It is not our purpose to answer the question. We will touch on only some special cases.

Lemma 5.1. *Let $N \in \sigma[M]$. If every factor module of N has a projective $\overline{Z}_M$-cover, then $Rad(N) \leq \overline{Z}_M(N)$.*

Proof By [2, 2.17], $\frac{N}{\overline{Z}_M(N)}$ is semisimple. Therefore $Rad(N) \leq \overline{Z}_M(N)$. □

Proposition 5.2. *Let $N \in \sigma[M]$ be a projective module in $\sigma[M]$. If N is τ_M-semiperfect in $\sigma[M]$, then every factor module of N has a projective τ_M-cover.*

Proof Let A be a submodule of N. By hypothesis, $N = N_1 \oplus N_2$ such that $N_1 \subseteq A$ and $A \cap N_2 \subseteq \tau_M(N_2)$. Now consider the canonical epimorphism $f : N_2 \longrightarrow N/A$ with $Ker(f) = A \cap N_2$. Clearly, f is a projective τ_M-cover of N/A. □

Proposition 5.3. *Let M be a module such that every simple module in $\sigma[M]$ is M-small. Then for every module N in $\sigma[M]$, we have $\overline{Z}_M(N) \leq Rad(N)$.*

Proof By the definition of the radical (see [1, p. 109 and 120]), we have $Rad(N) = \cap\{Kerh \mid h \in Hom_R(N,S)$ and S is a simple R-module$\}$. Thus $Rad(N) = \cap\{Kerh \mid h \in Hom_R(N,S)$ and S is a simple module with $S \in \sigma[M]$ $\}$. By hypothesis and the definition of $\overline{Z}_M(N)$, we have $\overline{Z}_M(N) \leq Rad(N)$. □

Corollary 5.4. *Let M be an R-module such that every simple module in $\sigma[M]$ is M-small and let N be any module in $\sigma[M]$. Consider the following conditions:*

(1) Every factor module of N has a projective $\overline{Z}_M$-cover.

(2) Every factor module of N has a projective Rad-cover.

Then (1) implies (2).

Proof By Proposition 5.3. □

The following example shows that, in general, the converse of Corollary 5.4 is false.

Example 5.5. *Let R be a local ring with maximal ideal m such that $m \neq 0$. Suppose that every factor of R has a projective $\overline{Z}_R$-cover. By Lemma 5.1, $m \subseteq \overline{Z}_R(R_R)$. But $\overline{Z}_R(R_R) = ann_r(m)$. Then $\overline{Z}_R(R_R) = m$. By Proposition 3.6, R_R is $\overline{Z}_R$-semiperfect. This proves that R_R is $\overline{Z}_R$-semiperfect if and only if every factor of R has a projective $\overline{Z}_R$-cover.*

Let R be a DVR with maximal ideal m. Consider the ring $S = \frac{R}{m^3}$. It is clear that S is not a von Neuman regular ring ($m \neq m^3$). Thus the simple S-module $\frac{S}{m/m^3}$ is not an injective S-module. Hence every simple S-module is S-small. Since $(\frac{m}{m^3})^2 \neq 0$, the module S_S is not $\overline{Z}_S$-semiperfect by Proposition 3.6. Thus the S-module S_S does not satisfy the condition (1) of Corollary 5.4. But it is easily seen that the module S_S satisfies the condition (2).

Lemma 5.6. *If M is a local R-module with maximal submodule K, then $\frac{M}{K}$ is M-small or $\overline{Z}_M(N) = N$ for every module $N \in \sigma[M]$.*

Proof Let I be a right ideal of R such that $M \cong \frac{R}{I}$ and let J be a right maximal ideal of R such that $\frac{M}{K} \cong \frac{R}{J}$. Since M is local, J is the only right maximal ideal over I. Let S be a simple module in $\sigma[M]$. Then S is isomorphic to a submodule of a factor module of a direct sum $M^{(\Lambda)}$ for some index set Λ. Since $MI = 0$, we have $SI = 0$. Hence $I \subseteq ann_r(S)$. But $ann_r(S)$ is a right maximal ideal of R. Therefore $ann_r(S) = J$. So $S \cong \frac{M}{K}$. Suppose that $\frac{M}{K}$ is not M-small. By [6, 5.1.4], every simple module in $\sigma[M]$ is M-injective. Thus the module M is a V-module. Hence $\overline{Z}_M(N) = N$ for every module $N \in \sigma[M]$ by [8, Proposition 2.5]. □

Proposition 5.7. *Let N be a nonzero module in $\sigma[M]$ which has a projective Rad-cover in $\sigma[M]$. Then N has a maximal submodule.*

Proof Let $f : P \longrightarrow N$ be a projective *Rad*-cover of N in $\sigma[M]$. Then $Ker(f) \subseteq Rad(P)$. By [9, 22.3], P contains a maximal submodule K. Therefore $Ker(f) \subseteq K$. It is easy to check that $f(K)$ is a maximal submodule of N. □

Proposition 5.8. *Let M be a local projective module in $\sigma[M]$ with maximal submodule K. The following are equivalent for a module N in $\sigma[M]$ which is isomorphic to $M^{(\Lambda)}$ for some index set Λ:*

(1) Every factor module of N has a projective $\overline{Z}_M$-cover.

(2) N is $\overline{Z}_M$-semiperfect.

Proof (1)⇒(2) Since M is local, $\overline{Z}_M(L) = L$ for every module $L \in \sigma[M]$ or $\frac{M}{K}$ is M-small by Lemma 5.6. If $\overline{Z}_M(L) = L$ for every module $L \in \sigma[M]$, then $\overline{Z}_M(N) = N$. So N is $\overline{Z}_M$-semiperfect. By the proof of Lemma 5.6, every simple module in $\sigma[M]$ is isomorphic to $\frac{M}{K}$. So if $\frac{M}{K}$ is M-small, then Corollary 5.4 shows that every factor module of N has a projective *Rad*-cover. By Proposition 5.7, every proper submodule of N is contained in a maximal submodule of N. Hence $Rad(N) \ll N$. But N is projective in $\sigma[M]$ and it is a direct sum of local modules. Thus N is semiperfect in $\sigma[M]$ by [9, 42.3(1)] and [10, Satz 1.4(A)]. Therefore N is $\overline{Z}_M$-semiperfect in $\sigma[M]$ since $Rad(N) \leq \overline{Z}_M(N)$ (See Lemma 5.1).

(2)⇒(1) By Proposition 5.2. □

Proposition 5.9. *Let R be a semilocal ring such that $Soc({}_RR) = Soc(R_R)$ and let N be a projective R-module. The following are equivalent:*

(1) Every factor module of N has a projective $\overline{Z}_R$-cover.

(2) N is $\overline{Z}_R$-semiperfect.

(3) N is Soc-semiperfect.

Proof Let P be any projective R-module. By Corollary 2.7, we have $\overline{Z}_R(P) = Soc(P)$. The result follows from [7, Theorem 2.23]. □

Corollary 5.10. *Let R be a QF ring. Let N be a projective R-module. The following are equivalent:*

(1) Every factor module of N has a projective $\overline{Z}_R$-cover.

(2) N is $\overline{Z}_R$-semiperfect.

(3) N is Soc-semiperfect.

Proof By Proposition 5.9 and [5, Corollary 15.7]. □

Proposition 5.11. *Suppose that τ_M is a radical for $\sigma[M]$. Let N be a projective module in $\sigma[M]$ such that N is Soc-lifting. The following are equivalent:*
(1) Every factor module of N has a projective τ_M-cover.
(2) N is τ_M-semiperfect in $\sigma[M]$.

Proof (1)⇒(2) By [2, 2.17], $\frac{N}{\tau_M(N)}$ is semisimple. Hence $Rad(N) \leq \tau_M(N)$. On the other hand, $N = N_1 \oplus N_2$ where N_1 is semisimple and $\tau_M(N_2)$ is essential in N_2 ([2, 2.2 and 2.16]). Therefore $Soc(N_2) \leq \tau_M(N_2)$. Since N is *Soc*-lifting, N_2 is *Soc*-lifting and hence it is τ_M-lifting. By Theorem 3.2, N is τ_M-lifting. Therefore N is τ_M-semiperfect since it is projective in $\sigma[M]$.

(2)⇒(1) Clear by Proposition 5.2. □

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