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WEAK GENERATORS FOR CLASSES OF R-MODULES

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Abstract

Let R be a ring. An R-module M is called a weak generator for a class $\mathcal{C}$ of R-modules if $\mathrm{Hom}_R(M, V)$ is non-zero for every non-zero module V in $\mathcal{C}$. A projective module M is a weak generator for $\mathcal{C}$ if and only if $M \neq MA$ for every annihilator A of a non-zero module V in $\mathcal{C}$. Given any class $\mathcal{C}$ of R-modules, a finitely annihilated R-module M is a weak generator for the class of injective hulls of modules in $\mathcal{C}$ if and only if the R-module R/A is a weak generator for $\mathcal{C}$, where A is the annihilator of M. Moreover a finitely annihilated R-module M is a weak generator for the class of all injective R-modules if and only if the annihilator of M is a left T-nilpotent ideal. In case the ring R is commutative, a finitely generated R-module M is a weak generator for the class of all R-modules if and only if M is a weak generator for the class of injective R-modules. In addition, if the ring R is Morita equivalent to a commutative semiprime Noetherian ring, then M is a weak generator for the class of all R-modules if and only if the trace of M in R is an essential right ideal of R.

1. Introduction

All rings are associative with unit elements and all modules are unitary right modules. Let R be any ring. By a *class* $\mathcal{C}$ of R-modules we mean a collection

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$\mathcal{C}$ of R-modules which contains a non-zero module and which is closed under taking isomorphisms. An R-module M is called a *weak generator* for a class $\mathcal{C}$ of R-modules if $\mathrm{Hom}_R(M, V) \neq 0$ for every non-zero module V in $\mathcal{C}$. We call an R-module M a *weak generator* if M is a weak generator for Mod-R, the class of all R-modules. Weak generators are discussed in [12]. Note that if M is a weak generator for a class $\mathcal{C}$ of R-modules then $M \neq 0$.

Clearly any generator is a weak generator (for Mod-R) and every weak generator is a weak generator for $\mathcal{C}$ for every class $\mathcal{C}$ of R-modules. More generally, if $\mathcal{C}' \subseteq \mathcal{C}$ are classes of R-modules then every weak generator for $\mathcal{C}$ is also a weak generator for $\mathcal{C}'$. Moreover, the converse holds in case every non-zero module in $\mathcal{C}$ contains a non-zero submodule in $\mathcal{C}'$. Let M be an R-module. It is clear that if there exists a submodule K of M with M/K a weak generator for $\mathcal{C}$, then M is a weak generator for $\mathcal{C}$. In particular, if M/K is a weak generator, for some submodule K of M, then M is a weak generator. On the other hand, if $\mathbb{Z}$ denotes the ring of integers and $\mathbb{Q}$ the field of rational numbers then the $\mathbb{Z}$-module $\mathbb{Q}$ contains the submodule $\mathbb{Z}$ which is a generator but the $\mathbb{Z}$-module $\mathbb{Q}$ is not a weak generator because $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Z}) = 0$. Let S and T be rings and let R denote the ring $S \oplus T$, let $A = S \oplus 0$ and $B = 0 \oplus T$. Then A and B are non-zero idempotent ideals of R such that $R = A \oplus B$. The R-module R is a generator but the R-modules A and B are not weak generators, by [12, Lemma 3.12].

For any non-empty subset Y of an R-module M, we denote the *annihilator* of Y (in R) by $\mathrm{ann}_R(Y)$, i.e. $\mathrm{ann}_R(Y) = \{r \in R : xr = 0 \text{ for all } x \in Y\}$. In particular, $\mathrm{ann}_R(M)$ denotes the annihilator of M. Any terminology not defined here may be found in [1],[7], [10].

We begin with the following elementary observation.

Lemma 1.1 *Let M and V be R-modules such that $Hom_R(M, R/A) \neq 0$, where $A = ann_R(V)$. Then $Hom_R(M, V) \neq 0$.*

Proof Since the R-module R/A embeds in a product $\prod V$ of copies of V, it follows that $\mathrm{Hom}_R(M, R/A)$ embeds in $\mathrm{Hom}_R(M, \prod V) \simeq \prod \mathrm{Hom}_R(M, V)$ by [1, 20.2]. Thus $\mathrm{Hom}_R(M, V) \neq 0$.

Let $\mathcal{C}$ be any class of R-modules. We denote the collection of annihilators (in R) of non-zero modules in $\mathcal{C}$ by $\mathcal{A}(\mathcal{C})$. For any I in $\mathcal{A}(\mathcal{C})$, the set $\{V \in \mathcal{C} \mid VI = 0\}$ will be denoted by $\mathcal{Z}_{\mathcal{C}}(I)$. Note that if $\mathcal{C} = $ Mod-R then $\mathcal{Z}_{\mathcal{C}}(I) = $ Mod-(R/I).

Proposition 1.2 *Let $\mathcal{C}$ be any non-empty class of R-modules. Consider the following statements for an R-module M.*
(i) $Hom_R(M, R/A) \neq 0$ for every ideal A in $\mathcal{A}(\mathcal{C})$.

(ii) $Hom_R(M/MA, R/A) \neq 0$ *for every ideal* A *in* $\mathcal{A}(\mathcal{C})$.
(iii) M *is a weak generator for* $\mathcal{C}$.
(iv) The (R/A)*-module* M/MA *is a weak generator for* $\mathcal{Z}_{\mathcal{C}}(A)$ *for every ideal* A *in* $\mathcal{A}(\mathcal{C})$.
Then (i) $\Leftrightarrow$*(ii)*$\Rightarrow$*(iii)*$\Rightarrow$*(iv). Furthermore, statements (i)-(iv) are equivalent if* (R/A) *belongs to* $\mathcal{C}$ *for every ideal* A *in* $\mathcal{A}(\mathcal{C})$.

Proof (i) $\Leftrightarrow$(ii) By [12, Lemma 2.1].
(ii)$\Rightarrow$(iii) By Lemma 1.1.
(iii)$\Rightarrow$(iv) Let A be any ideal in $\mathcal{A}(\mathcal{C})$. There exists a non-zero module in $\mathcal{Z}_{\mathcal{C}}(A)$. Now let V be any non-zero module in $\mathcal{C}$ such that $VA = 0$. By our assumption, there exists a non-zero R-homomorphism $\varphi : M \to V$. Note that $\varphi(MA) = \varphi(M)A \subseteq VA = 0$. Thus φ induces a non-zero (R/A)-homomorphism $\theta : M/MA \to V$. It follows that the (R/A)-module M/MA is a weak generator for $\mathcal{Z}_{\mathcal{C}}(A)$.
The last statement is now clear because, by our assumption, the R-module (R/A) belongs to $\mathcal{Z}_{\mathcal{C}}(A)$ for every ideal A in $\mathcal{A}(\mathcal{C})$ so that (iv)$\Rightarrow$(i).

Corollary 1.3 *The* R*-module* M *is a weak generator if and only if* $Hom_R(M, R/A) \neq 0$ *for every proper ideal* A *in* R.

Proof Apply Proposition 1.2 in case $\mathcal{C} =$ Mod-R.

Corollary 1.4 Let $\mathcal{C}$ be any class of R-modules and let M be a weak generator for $\mathcal{C}$. Then $M \neq MA$ for every ideal A in $\mathcal{A}(\mathcal{C})$.

Proof By Proposition 1.2.

Corollary 1.5 *Let* $B \subseteq A$ *be proper ideals of a ring* R*. Then the* R*-module* R/B *is a weak generator for Mod-*(R/A).

Proof Apply Proposition 1.2 in case $M = R/B$ and $\mathcal{C} =$ Mod-R/A. Note that if D is an ideal in $\mathcal{A}(\mathcal{C})$ then $A \subseteq D$ and so $M/MD \simeq R/D$.

For projective modules the converse of Corollary 1.4 holds.

Theorem 1.6 *Let* $\mathcal{C}$ *be any class of* R*-modules. Then a projective* R*-module* M *is a weak generator for* $\mathcal{C}$ *if and only if* $M \neq MA$ *for every ideal* A *in* $\mathcal{A}(\mathcal{C})$.

Proof The necessity follows from Corollary 1.4. Conversely, suppose that V is a non-zero module in $\mathcal{C}$, $A = \mathrm{ann}_R(V)$ and $M \neq MA$. Then M/MA is a non-zero projective (R/A)-module so that $\mathrm{Hom}_R(M/MA, R/A) \neq 0$. It follows that $\mathrm{Hom}_R(M, R/A) \neq 0$. By Lemma 1.1, $\mathrm{Hom}_R(M, V) \neq 0$. The result follows.

Corollary 1.7 *The following statements are equivalent for a projective R-module M.*
(i) M is a generator.
(ii) M is a weak generator.
(iii) $Hom_R(M, X) \neq 0$ for every simple R-module X.
(iv) $M \neq MP$ for right primitive ideal P of R.

Proof (i)⇒(ii)⇒(iii). Clear.
(iii)⇔(iv) By Theorem 1.6.
(iii)⇒(i) By [1, Proposition 17.9].

It is very easy to give an example of a ring R and a (non-projective) R-module M such that $\mathrm{Hom}_R(M, X) \neq 0$ for every simple R-module X, but M is not a weak generator. For example, let R be any ring with zero right socle and let $\{U_\lambda : \lambda \in \Lambda\}$ be a set of representatives of the isomorphism classes of simple R-modules. Let $M = \oplus_{\lambda \in \Lambda} U_\lambda$. Clearly $\mathrm{Hom}_R(M, X) \neq 0$ for every simple R-module X, but M is not a weak generator because $\mathrm{Hom}_R(M, R) = 0$.

Let M be any R-module. The *singular submodule* $\mathrm{Z}(M)$ of M is defined to be the set of all elements m in M such that $mE = 0$ for some essential right ideal E of R. The *socle* $\mathrm{Soc}(M)$ of M is defined to be the sum of all simple submodules of M or 0 if M has no simple submodule.

Proposition 1.8 *Let R be any ring with $Z = Z(R_R)$ and $S = Soc(R_R)$ and let A be any right ideal of R. Then the R-modules $A \oplus (R/A)$ and $(R/Z) \oplus (R/S)$ are both weak generators.*

Proof Suppose that $M = A \oplus (R/A)$ and I is a proper ideal of R. If $A \subseteq I$ then there exists a non-zero homomorphism $\varphi : R/A \to R/I$ defined by $\varphi(r + A) = r + I$ for all $r \in R$. On the other hand, if $A \nsubseteq I$ then there exists a non-zero homomorphism $\theta : A \to R/I$ defined by $\theta(a) = a + I$ for all $a \in A$. In any case, $\mathrm{Hom}_R(M, R/I) \neq 0$. By Corollary 1.3, M is a weak generator.
Next, suppose that $L = (R/Z) \oplus (R/S)$ and V is a non-zero R-module. Suppose that $\mathrm{Z}(V) \neq 0$. Let $0 \neq z \in \mathrm{Z}(V)$. By [1, 9.7], $zS = 0$. The mapping $\varphi : R/S \to V$ defined by $\varphi(r + S) = zr$ for all $r \in R$ is a non-zero homomorphism. Now suppose that $\mathrm{Z}(V) = 0$. Because $VZ \subseteq \mathrm{Z}(V)$, we have $VZ = 0$. Let $0 \neq v \in V$. Then the mapping $\theta : R/Z \to V$ defined by $\theta(r + Z) = vr$ for all $r \in R$ is a non-zero homomorphism. In any case, $\mathrm{Hom}_R(L, V) \neq 0$. It follows that L is a weak generator.

Let M be an R-module. For any R-module N, the submodule

$$\mathrm{Tr}(M, N) = \Sigma\{\mathrm{Im}\, f \mid f : M \to N\} \subseteq N$$

is called the *trace of M in N*.

Lemma 1.9 Let $\mathcal{C}$ be any class of R-modules closed under taking submodules and let M be a weak generator for $\mathcal{C}$. Then $\mathrm{Tr}(M, N)$ is an essential submodule of N for each non-zero module N in $\mathcal{C}$.

Proof Clear.

Proposition 1.10 *A ring R is semiprime if and only if every weak generator R-module is faithful.*

Proof The sufficiency follows by Proposition 1.8. Conversely, let R be a semiprime ring and M be a weak generator. Then $I = \mathrm{Tr}(M, R)$ is an essential right ideal in the semiprime ring R by Lemma 1.9. Thus $\mathrm{r.ann}_R(I) = 0$. It follows that M is a faithful R-module.

For certain rings R, Corollary 1.3 can be improved. First we prove:

Lemma 1.11 *Let $\mathcal{P}$ be a non-empty collection of proper ideals of a ring R such that for every proper ideal A of R there exists a right ideal B properly containing A such that $ann_R(B/A)$ belongs to $\mathcal{P}$. Then an R-module M is a weak generator if and only if $Hom_R(M, R/P) \neq 0$ for every P in $\mathcal{P}$.*

Proof The necessity is clear. Conversely, suppose that $\mathrm{Hom}_R(M, R/P) \neq 0$ for all $P \in \mathcal{P}$. Let A be any proper ideal of R. By hypothesis there exists a right ideal B properly containing A such that if $P = \mathrm{ann}_R(B/A)$ then $P \in \mathcal{P}$. By Lemma 1.1 and Corollary 1.3, M is a weak generator.

We shall be interested in the following two properties of a ring R:

(P_1) For every proper ideal A of R there exists a positive integer n and prime ideals $P_i (1 \leq i \leq n)$, each containing A, such that $P_1 \cdots P_n \subseteq A$.

(P_2) R satisfies the ascending chain condition on prime ideals.

Clearly simple rings satisfy (P_1) and (P_2). More generally, any ring which satisfies the ascending chain condition on (two-sided) ideals satisfies (P_1) and (P_2) (see [11, Lemma 1]). Next suppose that R is a ring with right or left Krull dimension. Then R satisfies (P_1) by [8, Theorem 7.4] and R satisfies (P_2) by [8, Theorem 7.1]. On the other hand, in [13] an example is given of a ring R which satisfies (P_1) and (P_2) but does not have ascending chain condition on ideals, nor does it have right or left Krull dimension.

Lemma 1.12 (See [11, Lemma 2].) *Let R be any ring satisfying (P_1) and (P_2) and let M be a non-zero R-module. Then there exists a submodule K of M and a prime ideal P of R such that $P = ann_R(K)$.*

Proposition 1.13 *Let R be a ring which satisfies (P_1) and (P_2). Then an R-module M is a weak generator if and only if $Hom_R(M, R/P) \neq 0$ for every prime ideal P of R.*

Proof By Lemmas 1.11 and 1.12.

Note that Proposition 1.13 generalizes [12, Theorem 3.7]. Now we show that for the rings of Proposition 1.13, to investigate weak generators we can suppose that R is a prime ring. Let R be any ring which satisfies (P_1). Then R contains only a finite number of minimal prime ideals.

Lemma 1.14 *Let R be any ring, let n be a positive integer and let $I_j (1 \leq j \leq n)$ be ideals of R such that $I_1 I_2 \ldots I_n = 0$. Then the R-module M is a weak generator if and only if the (R/I_j)-module M/MI_j is a weak generator for all $1 \leq j \leq n$.*

Proof The necessity follows by Proposition 1.2. Conversely, let X be any non-zero R-module. Then $X \neq 0$ but $XI_1 \ldots I_n = 0$. There exists $1 \leq j \leq n$ such that $XI_1 \ldots I_{j-1} \neq 0$ but $XI_1 \ldots I_j = 0$. Let $Y = XI_1 \ldots I_{j-1}$. Then Y is a non-zero (R/I_j)-module. By hypothesis, there exists a non-zero (R/I_j)-homomorphism $\phi : M/MI_j \to Y$. Clearly ϕ is an R-homomorphism. If $\pi : M \to M/MI_j$ is the canonical projection and $i : Y \to X$ is inclusion then $i\phi\pi : M \to X$ is a non-zero R-homomorphism. Thus $\mathrm{Hom}_R(M, X) \neq 0$ for every non-zero R-module X. Hence M is a weak generator.

Theorem 1.15 *Let R be any ring which satisfies (P_1) and let $P_1, \cdots, P_n$ be the minimal prime ideals of R for some positive integer n. Then an R-module M is a weak generator if and only if the (R/P_i)-module M/MP_i is a weak generator for all $1 \leq i \leq n$.*

Proof The necessity follows by Proposition 1.2. Conversely, suppose that the (R/P_i)-module M/MP_i is a weak generator for all $1 \leq i \leq n$. Because R satisfies (P_1), there exists a positive integer k and prime ideals $Q_i \in \{P_1, \cdots, P_n\}$ $(1 \leq i \leq k)$ such that $Q_1 \cdots Q_k = 0$. By Lemma 1.14, M is a weak generator.

We end this section with the following result that will be useful later.

Lemma 1.16 *Let R and S be rings such that there exists a ring homomor-*

phism from R to S. If the R-module M is a weak generator then the S-module $M \otimes_R S$ is a weak generator. The converse is true if ${}_RS$ is free.

Proof Let M_R be a weak generator . For any S-module V, we have V is an R-module and by [1, Proposition 20.6], $\mathrm{Hom}_S(M \otimes_R S, V) \simeq \mathrm{Hom}_R(M, \mathrm{Hom}_S(S, V))$ $\simeq \mathrm{Hom}_R(M, V)$ as $\mathbb{Z}$-modules. It follows that the S-module $M \otimes_R S$ is a weak generator. Conversely, suppose that ${}_RS$ is free and $M \otimes_R S$ is a weak generator and $L \in$ Mod-R. Let $V = L \otimes_R S$, in the above relations, then $\mathrm{Hom}_R(M, V)$ $\simeq \bigoplus \mathrm{Hom}_R(M, L)$. It follows that M_R is a weak generator.

2. Weak generators for module classes

Let R be any ring. For any R-module V, $\mathrm{E}(V)$ will denote the injective envelope of V. Let $\mathcal{C}$ be any class of R modules. Then $\mathcal{X}(\mathcal{C})$ (resp. $\mathcal{I}(\mathcal{C})$) will denote the collection of extensions (resp. injective envelopes) of modules in $\mathcal{C}$, i.e. a non-zero module V belongs to $\mathcal{X}(\mathcal{C})$ (resp. $\mathcal{I}(\mathcal{C})$) if and only if V contains a non-zero submodule U such that $U \in \mathcal{C}$ (resp. $V = \mathrm{E}(U)$ for some non-zero $U \in \mathcal{C}$). Note that $\mathcal{X}(\mathcal{C})$ and $\mathcal{I}(\mathcal{C})$ are also classes of R-modules. Note further that $\mathcal{C} \bigcup \mathcal{I}(\mathcal{C}) \subseteq \mathcal{X}(\mathcal{C})$. Thus an R-module M is a weak generator for a class $\mathcal{C}$ of R-modules if and only if M is a weak generator for $\mathcal{X}(\mathcal{C})$. In this section we shall investigate when an R-module M is a weak generator for a class $\mathcal{C}$ of R-modules. We begin with the following elementary lemma.

Lemma 2.1 *Let $\mathcal{C}$ be any class of R modules and let an R-module M be a weak generator for $\mathcal{C}$. Then M is a weak generator for $\mathcal{C}'$ for any class $\mathcal{C}' \subseteq \mathcal{X}(\mathcal{C})$.*

Proof By the above remarks.

Lemma 2.2 *Let an R-module M be a weak generator for a class $\mathcal{C}$ of R-modules. Then the R-module R/A is also a weak generator for $\mathcal{C}$, where $A = ann_R(M)$.*

Proof Suppose that $\mathcal{T}$ is the collection of R-modules N such that $N = \phi(M)$ where ϕ is a homomorphism from M to V for some V in $\mathcal{C}$. Note that by hypothesis $\mathcal{T}$ is a class of R-modules. Clearly, $\mathcal{C} \subseteq \mathcal{X}(\mathcal{T})$. Thus, by Lemma 2.1, it is enough to shown that R/A is a weak generator for $\mathcal{T}$. If $D \in \mathcal{A}(\mathcal{T})$, then $D = \mathrm{ann}_R W$ for some non-zero $W \in \mathcal{T}$. Consequently, $A \subseteq D$. Thus $\mathrm{Hom}_R(R/A, R/D) \neq 0$. Now apply Proposition 1.2 in case $M = R/A$ and $\mathcal{C} = \mathcal{T}$.

The converse of Lemma 2.2 is false in general and it is easy to give a counter example. Let R be the ring $\mathbb{Z}$ of integers and M be the R-module $\mathbb{Q}$. If $A = \mathrm{ann}_R(M)$ then $A = 0$ and hence the R-module R/A is a generator. However, $\mathrm{Hom}_R(M, R) = 0$ so that M is not a weak generator (for Mod-R). In view of Lemma 2.2 we shall consider, for any ring R, when an R-module of the form R/A, where A is a proper ideal of R, is a weak generator for a class $\mathcal{C}$ of R-modules.

Lemma 2.3 *Let A be a proper (right) ideal of a ring R. Then the R-module R/A is a weak generator for a class $\mathcal{C}$ of R-modules if and only if each non-zero module V in $\mathcal{C}$ contains a non-zero element v such that $vA = 0$.*

Proof This is clear because, for any R-module V, $\mathrm{Hom}_R(R/A, V) \neq 0$ if and only if there exists a non-zero $v \in V$ such that $vA = 0$.

Corollary 2.4 *Let A be an ideal of a ring R and let M be any non-zero R-module. Then the R-module M/MA is a weak generator for the class $\mathcal{S}$ of simple R-modules if and only if A is contained in the Jacobson radical J of R and M is a weak generator for $\mathcal{S}$.*

Proof ($\Rightarrow$) Clearly, M is a weak generator for $\mathcal{S}$. Let $B = \mathrm{ann}_R(M/MA)$. Then $A \subseteq B$ and by Lemma 2.2, R/B is a weak generator for $\mathcal{S}$. Now, by Lemma 2.3, $B \subseteq J$. Thus $A \subseteq J$.
($\Leftarrow$) Apply Proposition 1.2.

Corollary 2.5 *Let A be an ideal of a ring R. Then the R-module R/A is a weak generator for the class of simple R-modules if and only if A is contained in the Jacobson radical of R.*

Proof By Corollary 2.4.

Theorem 2.6 *Let $\mathcal{C}$ be a class of R-modules and let A be a proper ideal of R. Then the following statements are equivalent.*
(i) The R-module R/A is a weak generator for $\mathcal{C}$.
(ii) The R-module R/A is a weak generator for $\mathcal{I}(\mathcal{C})$.
(iii) The R-module R/A is a weak generator for $\mathcal{X}(\mathcal{C})$.

Proof (i)$\Leftrightarrow$(iii) By the remarks preceding Lemma 2.1.
(i)$\Rightarrow$(ii) By Lemma 2.1 because $\mathcal{I}(\mathcal{C}) \subseteq \mathcal{X}(\mathcal{C})$.
(ii)$\Rightarrow$(i) Let V be any non-zero R-module in $\mathcal{C}$. By Lemma 2.3, there exists $0 \neq e \in \mathrm{E}(V)$ such that $eA = 0$. Now $eR \cap V \neq 0$ and $(eR \cap V)A = 0$. Applying Lemma 2.3 again we conclude that R/A is a weak generator for $\mathcal{I}(\mathcal{C})$.

Let R be any ring and let $\mathcal{C}$ be a class of R-modules such that $\mathcal{C}$ is closed under taking submodules, i.e. if V is any module in $\mathcal{C}$ then every submodule of V also belongs to $\mathcal{C}$. In particular, $\mathcal{C}$ contains a non-zero cyclic submodule X. If $X = xR$ for some $0 \neq x \in X$ and $F = \{r \in R\ :\ xr = 0\}$, then F is a proper right ideal of R such that R/F belongs to $\mathcal{C}$. In this case, let $\mathcal{F}(\mathcal{C})$ denote the collection of proper right ideals E of R such that the R-module R/E belongs to $\mathcal{C}$.

Theorem 2.7 *Let A be a proper ideal of a ring R and let $\mathcal{C}$ be a class of R-modules which is closed under taking submodules and homomorphisms. Then the R-module R/A is a weak generator for $\mathcal{C}$ if and only if for each F in $\mathcal{F}(\mathcal{C})$ and each sequence $a_1, a_2, a_3, \cdots$ of elements of A there exists a positive integer n such that $a_1 a_2 \cdots a_n \in F$.*

Proof Suppose first that R/A is a weak generator for $\mathcal{C}$. Let $F \in \mathcal{F}(\mathcal{C})$ and let $a_1, a_2, a_3, \cdots$ be any sequence of elements of A. Suppose that $a_1 a_2 \cdots a_n \notin F$ for all $n \geq 1$. By Zorn's Lemma there exists a right ideal E of R such that $F \subseteq E$ and E is maximal in the collection of right ideals G of R such that $F \subseteq G$ and $a_1 a_2 \cdots a_k \notin G$ for all $k \geq 1$. Clearly $E \neq R$. Because R/E is a homomorphic image of R/F, the module R/E belongs to $\mathcal{C}$. By Lemma 2.3, there exists an element $r \in R \setminus E$ such that $rA \subseteq E$. By the choice of E, $a_1 \cdots a_t \in E + rR$ for some positive integer t. But this implies that $a_1 \cdots a_t a_{t+1} \in (E + rR)A \subseteq E + rA \subseteq E$, a contradiction. Thus there exists a positive integer n such that $a_1 \cdots a_n \in F$.
Conversely, suppose that for each F in $\mathcal{F}(\mathcal{C})$ and each sequence $a_1, a_2, a_3, \cdots$ of elements of A there exists a positive integer n such that $a_1 a_2 \cdots a_n \in F$. Let V be any non-zero module in $\mathcal{C}$. We claim that there exists $0 \neq v \in V$ such that $vA = 0$. Suppose not. Let $0 \neq u \in V$. There exists $b_1 \in A$ such that $ub_1 \neq 0$. Next $ub_1 A \neq 0$ so that $ub_1 b_2 \neq 0$ for some $b_2 \in A$. Repeat this process. We obtain a sequence $b_1, b_2, \cdots$ of elements of A such that $ub_1 \cdots b_n \neq 0$ for every positive integer n. Let $E = \{r \in R \mid ur = 0\}$. Then E is a right ideal of R such that $E \in \mathcal{F}(\mathcal{C})$ because $R/E \simeq uR$. We have proved that $b_1 \cdots b_n \notin E$ for all $n \geq 1$, a contradiction. Thus $vA = 0$ for some non-zero element v of V. By Lemma 2.3, the R-module R/A is a weak generator for $\mathcal{C}$.

Theorem 2.7 can be applied to many classes of R-modules, We mention only one application. Let $\mathcal{G}$ denote the class of singular R-modules V, i.e. modules V such that $V = \mathrm{Z}(V)$. Note that a module $V \in \mathcal{X}(\mathcal{G})$ if and only if $V = 0$ or V contains a non-zero singular submodule and this occurs if and only if $V = 0$ or $V \neq 0$ and V is not nonsingular. Note that an R-module M is a weak generator for $\mathcal{G}$ if and only if it is a weak generator for $\mathcal{X}(\mathcal{G})$. Moreover we have the following result.

Corollary 2.8 *Let $\mathcal{G}$ denote the class of singular R-modules and let A be a* *proper ideal of the ring R. Then the R-module R/A is a weak generator for $\mathcal{G}$ if and only if for each essential right ideal E of R and sequence $a_1, a_2, a_3, \cdots$ of elements of A there exists a positive integer n such that $a_1 \cdots a_n \in E$.*

Proof By Theorem 2.7.

In contrast to Theorem 2.6, if an R-module M is a weak generator for $\mathcal{I}(\mathcal{C})$, for some class $\mathcal{C}$ of R-modules, then it does not follow that M is a weak generator for $\mathcal{C}$. For example, if R is the ring $\mathbb{Z}$ of integers and M is the R-module $\mathbb{Q}$ then M is a (weak) generator for the class of all injective R-modules but $\mathrm{Hom}_R(M, R) = 0$. However there is a fact we can mention at this point.

Theorem 2.9 *Let $\mathcal{C}$ be any class of R-modules. Then an R-module M is a weak generator for $\mathcal{I}(\mathcal{C})$ if and only if there exist an index set Λ and an (essential) submodule L of the module $M^{(\Lambda)}$ such that L is a weak generator for $\mathcal{C}$.*

Proof Suppose first that M is a weak generator for $\mathcal{I}(\mathcal{C})$. Let $\mathcal{U} = \{V_\lambda : \lambda \in \Lambda\}$ be a set of representatives of the isomorphism classes of R-modules in $\mathcal{C}$. Let V be any non-zero module in $\mathcal{U}$. Then there exists a non-zero homomorphism $\varphi : M \to \mathrm{E}(V)$. Let $L_V = \varphi^{-1}(V)$ and note that L_V is an essential submodule of M such that $\mathrm{Hom}_R(L_V, V) \neq 0$. Let $M' = \bigoplus_{V \in \mathcal{U}} M_V$, where $M_V = M$ for all V in $\mathcal{U}$, and let $L = \bigoplus_{V \in \mathcal{U}} L_V$. Then L is an essential submodule of M' such that $\mathrm{Hom}_R(L, V) \neq 0$ for all V in $\mathcal{C}$, i.e. L is a weak generator for $\mathcal{C}$.
Conversely, suppose that there exist an index set Λ and a submodule L of $M^{(\Lambda)}$ such that L is a weak generator for $\mathcal{C}$. Let U be any non-zero module in $\mathcal{C}$. Then there exists a non-zero homomorphism $\theta : L \to U$. If $i : U \to \mathrm{E}(U)$ denotes the inclusion mapping then $i\theta : L \to \mathrm{E}(U)$ is a non-zero homomorphism. Because $\mathrm{E}(U)$ is injective, the homomorphism $i\theta$ can be lifted to a (non-zero) homomorphism $\theta' : M^{(\Lambda)} \to \mathrm{E}(U)$. Hence $\mathrm{Hom}_R(M^{(\Lambda)}, \mathrm{E}(U)) \neq 0$ so that $\mathrm{Hom}_R(M, \mathrm{E}(U)) \neq 0$. It follows that M is a weak generator for $\mathcal{I}(\mathcal{C})$.

Let $\mathcal{C}$ be a class of R-modules. Theorem 2.9 gives a characterization of when an R-module M is a weak generator for $\mathcal{I}(\mathcal{C})$ in terms of a related module being a weak generator for $\mathcal{C}$. A simpler characterization can be given in case M is finitely annihilated. An R-module M is called *finitely annihilated* provided there exist a positive integer n and elements $m_i \in M$ $(1 \leq i \leq n)$ such that $A := \mathrm{ann}_R(M) = \{r \in R \mid m_i r = 0 \text{ for all } 1 \leq i \leq n\}$, equivalently there exists an embedding $\theta : R/A \to M^{(n)}$. Finitely annihilated modules are considered by various authors (see, for example, [3], [15]). It is proved independently by Faith [5, Theorem 17A], Ghorbani [6], Hajarnavis (unpublished), Lenagan (unpublished) and essentially also by Beachy [2] that a ring R is right Artinian if and only if every right R-module is finitely annihilated. For an improvement of this theorem see [14].

Theorem 2.10 *Let $\mathcal{C}$ be a class of R-modules and let M be a finitely annihilated R-module with $A = ann_R(M)$. Then the following statements are equivalent.*
(i) M is a weak generator for $\mathcal{I}(\mathcal{C})$.
(ii) R/A is a weak generator for $\mathcal{C}$.
(iii) R/A is a weak generator for $\mathcal{I}(\mathcal{C})$.

Proof (i)$\Rightarrow$(iii). By Lemma 2.2.
(ii)$\Leftrightarrow$(iii). By Theorem 2.6.
(iii)$\Rightarrow$(i). Let $0 \neq V \in \mathcal{I}(\mathcal{C})$. Then there exists a non-zero homomorphism $\varphi : R/A \to V$. But M finitely annihilated implies that R/A embeds in $M^{(n)}$ for some positive integer n. Because V is injective, φ lifts to a non-zero homomorphism $\theta : M^{(n)} \to V$. Hence $\mathrm{Hom}_R(M, V) \neq 0$. It follows that M is a weak generator for $\mathcal{I}(\mathcal{C})$.

Note that Theorem 2.10 is false in general if M is not finitely generated. Let R be the ring $\mathbb{Z}$ of integers and M be the Prufer p-group $\mathbb{Z}_{p^\infty}$ for some prime number p. Then $A = \mathrm{ann}_R(M) = 0$ so that the R-module R/A is a generator. However, $\mathrm{Hom}_R(M, \mathbb{Q}) = 0$, so that M is not a weak generator for injective R-modules. Note that any finitely generated module over a commutative ring is finitely annihilated and for such modules we have the following result.

Theorem 2.11 *Let R be a commutative ring and let $\mathcal{C}$ be a class of R-modules. Then a finitely generated R-module M is a weak generator for $\mathcal{I}(\mathcal{C})$ if and only if M is a weak generator for $\mathcal{C}$.*

Proof The sufficiency follows by Lemma 2.1. Conversely, suppose that M is a weak generator for $\mathcal{I}(\mathcal{C})$. Let V be any non-zero module in $\mathcal{C}$. There exists a non-zero homomorphism $\varphi : M \to \mathrm{E}(V)$. Note that $M = m_1R + \cdots + m_nR$ for some positive integer n. Because $\varphi(M) = \varphi(m_1)R + \cdots + \varphi(m_n)R$ is a non-zero finitely generated submodule of $\mathrm{E}(V)$, there exists an element $r \in R$ such that $0 \neq \varphi(M)r \subseteq V$. Define a mapping $\theta : M \to V$ by $\theta(m) = \varphi(m)r$, for all $m \in M$. It is clear that θ is a non-zero homomorphism. It follows that M is a weak generator for $\mathcal{C}$.

3. T-nilpotence

Let R be an arbitrary ring. It will be convenient to denote the class Mod-R of all R-modules by $\mathcal{M}$. In this section we apply the results of §2 to the case when $\mathcal{C}$ is $\mathcal{M}$. Note that $\mathcal{I}(\mathcal{M})$ is the class of all injective R-modules. Following [1], a non-empty subset Y of R is called *left T-nilpotent* provided for each sequence $y_1, y_2, y_3, \cdots$ of elements of Y there exists a positive integer n such that $y_1 y_2 \ldots y_n = 0$.

Theorem 3.1 *The following statements are equivalent for an ideal A of a ring R.*
(i) A is left T-nilpotent.
(ii) For every non-zero R-module V there exists a non-zero element $v \in V$ such that $vA = 0$.
(iii) The R-module R/A is a weak generator (for $\mathcal{M}$).
(iv) The R-module R/A is a weak generator for the class $\mathcal{I}(\mathcal{M})$ of injective R-modules.

Proof (i)$\Leftrightarrow$(iii). By Theorem 2.7.
(ii)$\Leftrightarrow$(iii). By Lemma 2.3.
(iii)$\Leftrightarrow$(iv). By Theorem 2.6.

We saw in Proposition 1.2 that an R-module M is a weak generator (for $\mathcal{M}$) if and only if the (R/A)-module M/MA is a weak generator (for Mod-R/A) for every proper ideal A of R. Compare the following result.

Corollary 3.2 *Let A be any left T-nilpotent ideal of R. Then an R-module M is a weak generator if and only if the (R/A)-module M/MA is weak generator.*

Proof Note first that A is a proper ideal of R. The necessity follows by applying Proposition 1.2 to $\mathcal{C} = \mathcal{M}$. Conversely, suppose that the (R/A)-module M/MA is a weak generator. Let U be any non-zero R-module. By Theorem 3.1 there exists a non-zero element $u \in U$ such that $uA = 0$. Thus U contains a submodule uR which is annihilated by A. By hypothesis, $\mathrm{Hom}_{R/A}(M/MA, uR) \neq 0$ so that $\mathrm{Hom}_R(M, U) \neq 0$. It follows that the R-module M is a weak generator.

Corollary 3.3 *Let R be any ring and I be a proper ideal of R. Then the following statements are equivalent.*
(i) I is a left T-nilpotent ideal.
(ii) $I[x]$ is a left T-nilpotent ideal of the polynomial ring $R[x]$.
(iii) $I[[x]]$ is a left T-nilpotent ideal of the power series ring $R[[x]]$.

Proof (iii)$\Rightarrow$(ii) and (ii)$\Rightarrow$(i) are clear.
(i)$\Rightarrow$(iii). Note that $R[[x]]/I[[x]] \simeq R/I \otimes_R R[[x]]$ as right $R[[x]]$-modules. Apply Theorem 3.1 and Lemma 1.16.

Theorem 3.4 *Let M be any non-zero R-module. If M is a weak generator* *for $\mathcal{I}(\mathcal{M})$, then $ann_R(M)$ is left T-nilpotent. Moreover, the converse holds in case M is a finitely annihilated module.*

Proof The first statement is proved by applying Lemma 2.2 to $\mathcal{C} = \mathcal{I}(\mathcal{M})$ and then using Theorem 3.1. The converse is proved by applying Theorem 2.10 to $\mathcal{C} = \mathcal{M}$ and then again using Theorem 3.1.

In [12, Theorem 3.5], rings are characterized over which every non-zero module is a weak generator. As a corollary of Theorem 3.4, we can characterize rings over which all non-zero finitely annihilated modules are weak generators for $\mathcal{I}(\mathcal{M})$.

Corollary 3.5 *Let R be any ring and let J be the Jacobson radical of R. Then the following statements are equivalent.*
(i) There exists a left T-nilpotent ideal A of R such that R/A is a simple ring.
(ii) Every proper ideal of R is left T-nilpotent.
(iii) R/J is a simple ring and J is a left T-nilpotent ideal.
(iv) Every finitely annihilated R-module is a weak generator for $\mathcal{I}(\mathcal{M})$.

Proof (i)⇒(ii). Let I be a proper ideal of R. Since A is a left T-nilpotent ideal of R, it must be small so that $I + A \neq R$. Thus $(I + A)/A$ is a proper ideal of R/A. Now by hypothesis, I lies in A. It follows that I is a left T-nilpotent ideal.
(ii)⇒(iii). This is true because any left T-nilpotent ideal of R lies in J.
(iii)⇒(i). Clear.
(ii)⇒(iv). By Theorem 3.4.
(iv)⇒(ii). Note that if I is a proper ideal of R then $M = R/I$ is a non-zero finitely annihilated R-module with $\mathrm{ann}_R(M) = I$. Thus the result follows from Theorem 3.4.

Corollary 3.6 *Let R be any ring and let n be any positive integer. Then an ideal I of R is left T-nilpotent if and only if the ideal $Mat_{n\times n}(I)$ of $Mat_{n\times n}(R)$ is left T-nilpotent.*

Proof The sufficiency is clear. Conversely, suppose that $S = \mathrm{Mat}_{n\times n}(R)$ and let $F : \text{Mod-}R \to \text{Mod-}S$ be the standard Morita equivalence of R with S. Then $\mathrm{Mat}_{n\times n}(I) = \mathrm{ann}_S F(R/I)$. Clearly, being a weak generator is a Morita invariant property. Now apply Theorems 3.1 and 3.4.

Next we consider rings R over which an R-module M is a weak generator for $\mathcal{I}(\mathcal{M})$ if and only if $\mathrm{Hom}_R(M,\mathrm{E}(R_R)) \neq 0$. However, we first prove the following lemma.

Lemma 3.7 *Let M and N be non-zero R-modules and $\mathcal{C}$ be any class of R-modules. If $Tr(M,N)$ is a weak generator for $\mathcal{C}$ then so is M.*

Proof Let $L = \mathrm{Tr}(M,N)$ be a weak generator for $\mathcal{C}$. Note that if there exists a non-zero homomorphism $\varphi : L \to V$ for some $V \in \mathcal{C}$ then there exists a non-zero homomorphism $\phi : M \to N$ such that $\varphi\phi \neq 0$ (otherwise, φ must be zero). Thus $\mathrm{Hom}_R(M,V) \neq 0$ for any non-zero V in $\mathcal{C}$. It follows that M is a weak generator for $\mathcal{C}$.

Theorem 3.8 *Let R be any ring. Then the following statements are equivalent.*
(i) For any R-module M, if $Hom_R(M,E(R_R)) \neq 0$ then M is a weak generator for $\mathcal{I}(\mathcal{M})$.
(ii) Every non-zero ideal of R is a weak generator for $\mathcal{I}(\mathcal{M})$.

Proof (i)⇒(ii). This is clear.
(ii)⇒(i). Let M be an R-module such that $\mathrm{Hom}_R(M,\mathrm{E}(R_R)) \neq 0$. Let $L = \mathrm{Tr}(M,\mathrm{E}(R_R))$ and $I = L \cap R$. Then I is a non-zero ideal of R. By hypothesis, I is a weak generator for $\mathcal{I}(\mathcal{M})$. Thus L is a weak generator for $\mathcal{I}(\mathcal{M})$ by Theorem 2.9. Consequently, M is a weak generator for $\mathcal{I}(\mathcal{M})$ by applying Lemma 3.7 in case $N = \mathrm{E}(R_R)$ and $\mathcal{C} = \mathcal{I}(\mathcal{M})$.

Corollary 3.9 *Let R be a left Noetherian ring with nilpotent Jacobson radical J such that R/J is a simple ring. Then an R-module M is a weak generator for $\mathcal{I}(\mathcal{M})$ if and only if $Hom_R(M,E(R_R)) \neq 0$.*

Proof The necessity is clear. Conversely, by hypothesis and Corollary 3.5, every finitely annihilated R-module is a weak generator for $\mathcal{I}(\mathcal{M})$. On the other hand, because R is left Noetherian, it is easy to verify that every ideal of R is finitely annihilated as a right ideal. Therefore every non-zero ideal of R is a weak generator for $\mathcal{I}(\mathcal{M})$. Now apply Theorem 3.8.

Recall from [9] that a ring R is said to be *right strongly prime* if for any $r \neq 0$ in R, there exists a set $\{r_1, \cdots, r_k\} \subseteq R$ such that $rr_ia = 0$ for each i, $a \in R$, implies $a = 0$. Equivalently, R is right strongly prime if the injective hull $\mathrm{E}(R_R)$ has no non-trivial fully invariant R-submodules; see, for example [4, Proposition 1.2]. Clearly, any domain is right strongly prime and so too is any prime right Goldie ring. More generally, as we will observe in the next lemma, any prime ring which satisfies the descending chain condition on right annihilators is right strongly prime. We shall show that, for a right strongly prime ring R, any R-module M with the property $\mathrm{Hom}_R(M,\mathrm{E}(R_R)) \neq 0$ is a weak generator for the class of injective R-modules. However, we first prove the following lemma. This may be found in the literature, but we state a proof for completeness.

Lemma 3.10 *The following statement are equivalent for a ring R.*
(i) R is a right strongly prime ring.
(ii) Every non-zero (right) ideal of R is faithful and finitely annihilated as a right ideal.

Proof (i)$\Rightarrow$(ii). Let I be any non-zero right ideal of R. Then $\text{r.ann}_R(I) = 0$. Let $0 \neq r \in I$. By hypothesis, there exist a positive integer n and elements $a_i \in R$ such that $\cap_{i=1}^n \text{r.ann}_R(ra_i) = 0$. Thus I is finitely annihilated.
(ii)$\Rightarrow$(i). Suppose that every ideal of R is a faithful finitely annihilated right R-module. Let $0 \neq r \in R$ and let I denote the non-zero ideal RrR. By (ii) there exist a positive integer k and elements $b_i,\ c_i \in R$ $(1 \leq i \leq k)$ such that $\cap_{i=1}^k \text{r.ann}_R(b_i rc_i) = 0$ and hence $\cap_{i=1}^k \text{r.ann}_R(rc_i) = 0$. Now (i) follows.

Theorem 3.11 *Let R be a right strongly prime ring. Then an R-module M is a weak generator for $\mathcal{I}(\mathcal{M})$ if and only if $Hom_R(M,E(R_R)) \neq 0$.*

Proof The necessity is clear. Conversely, by Lemma 3.10, every non-zero ideal of R is faithful and finitely annihilated as a right ideal. Thus if I is a non-zero ideal of R then I_R is a weak generator for $\mathcal{I}(\mathcal{M})$ by Theorem 3.4. The result now follows from Theorem 3.8.

Remarks 3.12 (1) In Theorem 3.11, it is crucial that R be a prime ring. For let S and T be any two non-zero rings and let $R = S \oplus T$. Let A denote the ideal $S \oplus 0$ of R. Then $\text{Hom}_R(A,\text{E}(R_R)) \neq 0$ but $\text{Hom}_R(A,\text{E}(B_R)) = 0$ where B is the ideal $0 \oplus T$ of R.
(2) In a semiprime right Goldie ring R, an ideal I is a weak generator for $\mathcal{I}(\mathcal{M})$ if and only if I is an essential (right) ideal. For every essential right ideal I of R contains a regular element x so that I contains the weak generator $xR \simeq R$ and hence I_R is a weak generator for $\mathcal{I}(\mathcal{M})$. Conversely, if I_R is a weak generator for $\mathcal{I}(\mathcal{M})$, then $\text{r.ann}_R(I) = 0$ or equivalently I is an essential right ideal (see Proposition 1.10 and Theorem 2.9).

Compare the next result with Theorem 3.4.

Theorem 3.13 *Let $M = m_1R + \ldots + m_nR$ be a finitely generated module over an arbitrary ring R and let $A = \bigcap_i^n ann_R(m_i)$. Then the following statements are equivalent.*
(i) M is a weak generator for $\mathcal{I}(\mathcal{M})$.
(ii) The R-module R/A is a weak generator for $\mathcal{I}(\mathcal{M})$.
(iii) For every non-zero injective R-module E there exists a non-zero element e of E such that $eA = 0$.

Proof (ii)⇔(iii). By Lemma 2.3.
(ii)⇒(i). By our assumption, the R-module R/A embeds in M^n and so M is a weak generator for $\mathcal{I}(\mathcal{M})$ by applying Theorem 2.9 in case $L = R/A$ and $\mathcal{C} = \mathcal{I}(\mathcal{M})$.
(i)⇒(iii). Let E be any non-zero injective R-module. There exists a non-zero homomorphism $\varphi : M \to E$. Therefore, $\varphi(m_i) \neq 0$ for some $1 \leq i \leq n$. It follows that $\varphi(m_i)A = \varphi(m_i A) = \varphi(0) = 0$.

We complete this section by proving the following result for rings R which are Morita equivalent to a commutative ring.

Theorem 3.14 *Let R be a ring which is Morita equivalent to a commutative ring S. Then the following statements are equivalent for a finitely generated R-module M.*
(i) The R-module M is a weak generator.
(ii) The R-module M is a weak generator for the class of injective R-modules.
(iii) The ideal $ann_R(M)$ is left T-nilpotent.

Proof Let $F : \text{Mod-}R \to \text{Mod-}S$ be an equivalence. By [1, Proposition 21.8], $F(M)$ is a finitely generated S-module. Applying [1, Proposition 21.2] we see that (i) holds if and only if the S-module $F(M)$ is a weak generator. Next by [1, Propositions 21.2 and 21.6], (ii) holds if and only if $\text{Hom}_S(F(M), E') \neq 0$ for every non-zero injective S-module E'. Let B denote the annihilator in S of the S-module $F(M)$ and let I denote the annihilator in R of the R-module M. By [1, Proposition 21.11], R/I is Morita equivalent to S/A where $A = \text{ann}_S(F(R/I))$. Since M is faithful as an R/I-module, $B = A$ by [1, Proposition 21.6]. Now, by Theorem 3.1, (iii) holds if and only if the ideal B is T-nilpotent.
Thus we can suppose that R is commutative, then:
(i)⇔(ii). By Theorem 2.12.
(ii)⇔(iii). By Theorem 3.4.

Finally, we give a number of applications of Theorem 3.14.

Corollary 3.15 *Let R be a ring which is Morita equivalent to a commutative domain. Then a finitely generated R-module M is a weak generator if and only if $Hom_R(M, R) \neq 0$.*

Proof The necessity is clear and the sufficiency follows by Theorems 3.11 and 3.14.

Note that in Corollary 3.15, the condition that M be finitely generated is necessary. For example, let R be any commutative domain which contains a proper non-zero idempotent ideal. It is well known that in this case I cannot be finitely generated. Also I is not a weak generator by [12, Lemma 3.12].

Corollary 3.16 Let R be a commutative ring such that $P_1 \dots P_n = 0$ for some positive integer n and (not necessarily distinct) prime ideals $P_i (1 \leq i \leq n)$. Then a finitely generated R-module M is a weak generator if and only if $\mathrm{Hom}_R(M, R/P_i) \neq 0$ for all $1 \leq i \leq n$.

Proof The necessity is clear. Conversely, suppose that $\mathrm{Hom}_R(M, R/P_i) \neq 0$ for all $1 \leq i \leq n$. Let $1 \leq i \leq n$ and let $\phi : M \to R/P_i$ be any non-zero R-homomorphism. Note that $\phi(MP_i) = \phi(M)P_i \subseteq (R/P_i)P_i = 0$, so ϕ induces a non-zero R/P_i-homomorphism $\theta : M/MP_i \to R/P_i$. Thus by Corollary 3.15, the R/P_i-module M/MP_i is a weak generator for all $1 \leq i \leq n$. By Lemma 1.14, M is a weak generator.

In [12, Theorem 3.9], it is proved that over a commutative Noetherian domain R, an R-module M is a weak generator if and only if $\mathrm{Hom}_R(M, R) \neq 0$ (or equivalently $\mathrm{Tr}(M, R) \neq 0$). We generalize this result as follows:

Theorem 3.18 *Let R be a ring which is Morita equivalent to a commutative semiprime Noetherian ring. Then M is a weak generator if and only if the ideal $Tr(M, R)$ is an essential (right) ideal of R.*

Proof The necessity follows by Lemma 1.9. Conversely, let M be an R-module such that $I := \mathrm{Tr}(M, R)$ is an essential right ideal of R. By Remark 3.12(ii), I is a weak generator for the class of injective R-modules. Now by Theorem 3.14, the finitely generated R-module I is a weak generator. Thus M is a weak generator by Lemma 3.7.

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