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WEAK HOPF-ALGEBRAS AND SMASH PRODUCTS

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Abstract

The definitions of a u-weak Hopf algebra and the quantum dimension $\underline{det}_u M$ of a representation M by u are given. It is shown that a u-weak Hopf algebra H is semisimple if and only if there is a finite-dimensional projective H-module P such that $\underline{det}_u P$ is invertible. Let X be an associative algebra and A is a weak Hopf algebra. We investigate the global dimension and the weak dimension of the smash product $H \bowtie_R A$ and show that $lD(H) \leq rD(A) + lD(X)$ and $wD(H) \leq wD(A) + wD(X)$.

1 Introduction

Weak Hopf algebras have been proposed([3], [9], [14]) as a new generalization of ordinary Hopf algebras that replaces Ocneanu's paragroup ([11]), in the depth 2 case, with a concrete "Hopf algebra" object. A weak Hopf algebra is a vector space that has both algebra and coalgebra structures related to each other in a certain self-dual way and that possesses an antipode. The main difference between ordinary and weak Hopf algebras comes from the fact that the comultiplication of the latter is no longer required to preserve the unit and results in the existence of two canonical subalgebras playing the role of "non-commutative basis" in a "quantum groupoid".

So far weak Hopf algebras have been considered only under the additional assumption of finite dimensionality. Although a good deal of the results can be generalized to the infinite-dimensional case, finite dimension is particularly

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attractive because it implies self-duality. Just like finite Abelian groups or finite-dimensional Hopf algebras, the finite-dimensional weak Hopf algebras are self-dual in the following sense. If A is a weak Hopf algebra then its dual space A^* is canonically equipped with a weak Hopf algebra structure. Furthermore this duality is reflexive, $(A^*)^* \cong A$. This is a feature which makes weak Hopf algebras more natural objects of study than either finite (non-Abelian) groups or finite-dimensional (weak) quasi-Hopf algebras.

A weak Hopf algebra satisfying $S^2(h) = uhu^{-1}$ for some invertible element $u \in H$ and all $h \in H$ is called a u-weak Hopf algebra. For example, quasi-triangular weak Hopf algebras are u-weak Hopf algebras. In this paper, we will characterize the semisimplicity of u-weak Hopf algebras by using the quantum dimension.

In [6] and [4], the global dimensions and the weak dimensions of the crossproduct and R-smash product of an associative algebra with a Hopf algebra have been invested. In this paper, we will prove a similar result. Let $H = X \bowtie_R A$ be an R-smash product of an associative algebra X and a weak Hopf algebra A. We get $lD(H) \leq lD(X) + rD(A)$ and $wD(H) \leq wD(X) + wD(A)$.

2 Preliminaries

Throughout this paper k denotes a field and all vector spaces are defined over k. We use Sweedler's notation for a comultiplication:$\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$.

Definition of a weak Hopf algebra. Below we collect the definition and basic properties of weak Hopf algebras.

Definition 2.1. ([1], [3]) *A weak bialgebra is a vector space H with the structures of an associative algebra $(H, m, 1)$ with a multiplication $m : H \otimes_k H \longrightarrow H$ and unit $1 \in H$ and a coassociative coalgebra (H, Δ, ε) with a comultiplication $\Delta : H \longrightarrow H \otimes_k H$ and counit $\varepsilon : H \longrightarrow k$ such that:*

(i) *The comultiplication Δ is a (not necessarily unit-preserving) homomorphism of algebras:*

$$\Delta(gh) = \Delta(g)\Delta(h), \qquad h, g \in H.$$

(ii) *The unit and counit satisfy the following identities:*

$$(\Delta \otimes id)\Delta(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1), \tag{1}$$

$$\varepsilon(fgh) = \sum \varepsilon(fg_{(1)})\varepsilon(g_{(2)}h) = \sum \varepsilon(fg_{(2)})\varepsilon(g_{(1)}h), \tag{2}$$

for all $f, g, h \in H$.

A weak bialgebra is called a weak Hopf algebra if there is a linear map $S : H \longrightarrow H$, *called an antipode, such that*

(iii)

$$m(id \otimes S)\Delta(h) = (\varepsilon \otimes id)(\Delta(1)(h \otimes 1)), \tag{3}$$

$$m(S \otimes id)\Delta(h) = (id \otimes \varepsilon)(1 \otimes h)\Delta(1), \tag{4}$$

$$S(h) = \sum S(h_{(1)})h_{(2)}S(h_{(3)}), \tag{5}$$

for all $h \in H$.

Remark 1. *A weak Hopf algebra is a Hopf algebra if and only if the comultiplication is unit-preserving and if and only if* ε *is a homomorphism of algebras.*

Counital maps and subalgebras. The linear maps defined in (3) and (4) are called target and source counital maps and are denoted ε_t and ε_s respectively:

$$\varepsilon_t(h) = \sum \varepsilon(1_{(1)}h)1_{(2)}, \qquad \varepsilon_s(h) = \sum 1_{(1)}\varepsilon(h1_{(2)}), \tag{6}$$

for all $h \in H$. In the next proposition we collect several useful properties of the counital maps.

Proposition 2.2. ([1], [10]) *For all* $h, g \in H$ *we have*

(i) *Counital maps are idempotents in* $End_k(H)$:

$$\varepsilon_t(\varepsilon_t(h)) = \varepsilon_t(h), \qquad \varepsilon_s(\varepsilon_s(h)) = \varepsilon_s(h). \tag{7}$$

(ii) *The relations between* $\varepsilon_t, \varepsilon_s$, *and comultiplication are as follows*

$$(id \otimes \varepsilon_t)\Delta(h) = \sum 1_{(1)}h \otimes 1_{(2)}, \quad (\varepsilon_s \otimes id)\Delta(h) = \sum 1_{(1)} \otimes h1_{(2)}, \tag{8}$$

$$\sum \varepsilon_s(1_{(1)}h) \otimes 1_{(2)} = \sum \varepsilon_s(h_{(1)}) \otimes \varepsilon_t(h_{(2)}) = \sum 1_{(1)} \otimes \varepsilon_t(h1_{(2)}). \tag{9}$$

(iii) *The images of counital maps are characterized by*

$$h = \varepsilon_t(h) \quad \Leftrightarrow \quad \Delta(h) = \sum 1_{(1)}h \otimes 1_{(2)}, \tag{10}$$

$$h = \varepsilon_s(h) \quad \Leftrightarrow \quad \Delta(h) = \sum 1_{(1)} \otimes h1_{(2)}. \tag{11}$$

(iv) $\varepsilon_t(H)$ *and* $\varepsilon_s(H)$ *commute.*
(v) *One also has identities dual to* (8)

$$h\varepsilon_t(g) = \sum \varepsilon(h_{(1)}g)h_{(2)}, \quad \varepsilon_s(h)g = \sum h_{(1)}\varepsilon(gh_{(2)}), \tag{12}$$

(vi) *Eqs.*(8) *imply the relations*

$$\sum 1_{(1)}1_{(1')} \otimes 1_{(2)} \otimes 1_{(2')} = \sum 1_{(1)} \otimes \varepsilon_t(1_{(2)}) \otimes 1_{(3)}, \tag{13}$$

$$\sum 1_{(1)} \otimes 1_{(1')} \otimes 1_{(2)}1_{(2')} = \sum 1_{(1)} \otimes \varepsilon_s(1_{(2)}) \otimes 1_{(3)}. \tag{14}$$

(vii) *The antipode* S *satisfies the following relations*

$$\sum h_{(1)} \otimes S(h_{(2)})h_{(3)} = \sum h1_{(1)} \otimes S(1_{(2)}), \tag{15}$$

$$\sum h_{(1)}S(h_{(2)}) \otimes h_{(3)} = \sum S(1_{(1)}) \otimes 1_{(2)}h. \tag{16}$$

The images of the counital maps

$$H_t = \varepsilon_t(H) = \{h \in H|\Delta(h) = \sum 1_{(1)}h \otimes 1_{(2)}\}, \tag{17}$$

$$H_s = \varepsilon_s(H) = \{h \in H|\Delta(h) = \sum 1_{(1)} \otimes h1_{(2)}\}, \tag{18}$$

play the role of basis of H*. The next proposition summarizes their properties.*

Proposition 2.3. ([1], [10]) $H_t(resp.H_s)$ *is a left (resp. right) coideal subalgebra of* H*. These subalgebras commute with each other, moreover*

$$H_t = \{(\phi \otimes id)\Delta(1)|\phi \in \hat{H}\}, \qquad H_s = \{(id \otimes \phi)\Delta(1)|\phi \in \hat{H}\},$$

i.e., $H_t(resp.H_s)$ *is generated by the right (resp. left) tensorands of* $\Delta(1)$.

We call $H_t(resp.H_s)$ a target (resp. source) counital subalgebra.

The properties of the antipode of a weak Hopf algebra are similar to those of a finite-dimensional Hopf algebra.

Proposition 2.4. ([10]) (i) *The antipode* S *is unique and bijective. Also, it is both algebra and coalgebra anti-homomorphism.*

(ii) *We have* $S \circ \varepsilon_s = \varepsilon_t \circ S$ *and* $\varepsilon_s \circ S = S \circ \varepsilon_t$*. The restriction of* S *defines an algebra anti-isomorphism between counital subalgebras* H_t *and* H_s.

Remark 2. (1) *The set of axioms of Definition* 2.1 *is selfdual. This allows to define a natural weak Hopf algebra structure on the dual vector space* $\hat{H} = Hom_k(H, k)$ *by revering the arrows:*

$$< \phi\psi, h >=< \phi \otimes \psi, \Delta(h) >,$$

$$< \hat{\Delta}(\phi), h \otimes g >=< \phi, hg >,$$

$$< \hat{S}(\phi), h >=< \phi, S(h) >,$$

for all $\phi, \psi \in \hat{H}, h, g \in H$. *The unit* $\hat{1}$ *of* $\hat{H}$ *is* ε *and counit* $\hat{\varepsilon}$ *is* $\phi \longrightarrow < \phi, 1 >$.

(2) *The opposite algebra* H^{op} *is also a weak Hopf algebra with the same coalgebra structure and the antipode* S^{-1}. *Similarly, the co-opposite coalgebra* H^{cop} *(with the same algebra structure as* H *and the antipode* S^{-1}*) and* $(H^{op/cop}, S)$ *are weak Hopf algebras.*

Definition of Rep(H) For a weak Hopf algebra H, let $Rep(H)$ be the category of representations of H, whose objects are H-modules of finite rank and whose morphism are H-linear homomorphism.

For objects V, W of $Rep(H)$ set

$$V \diamond W = \{x \in V \otimes_k W | x = \Delta(1)x\}$$

with the obvious action of H via the comultiplication Δ (here $\otimes_k$ denotes the usual tensor product of vector spaces).

Since $\Delta(1)$ is an idempotent, $V \diamond W = \Delta(1)(V \otimes_k W)$. The tensor product of morphisms is the restriction of usual tensor product of homomorphisms. The standard associativity isomorphisms $\Phi_{U,V,W} : (U \diamond V) \diamond W \longrightarrow U \diamond (V \diamond W)$ are functorial and satisfy the pentagon condition, since Δ is coassociative. We will suppress these isomorphisms and write simply $U \diamond V \diamond W$.

The target counital subalgebra $H_t \subset H$ has an H-module structure given by $h \cdot z = \varepsilon_t(hz)$, where $h \in H, z \in H_t$.

Lemma 2.5. ([10]) H_t *is the unit object of* $Rep(H)$.

Using the antipode S of H, we can provide $Rep(H)$ with a duality . For any object V of $Rep(H)$, we define the action of H on $V^* = Hom_k(V, k)$ by $(h.\phi)(v) = \phi(S(h).v)$, where $h \in H, v \in V, \phi \in V^*$. For any morphism $f : V \to W$, let $f^* : W^* \to V^*$ be the morphism dual to f.

For any V in $Rep(H)$, we define the duality homomorphisms

$$d_V : V^* \diamond V \longrightarrow H_t, \qquad b_V : H_t \longrightarrow V \diamond V^*,$$

as follows. For $\sum_j \phi^j \otimes v_j \in V^* \otimes V$, set

$$d_V(\Delta(1) \cdot \sum_j \phi^j \otimes v_j) = \sum_j (\sum_{(1)} \phi^j (1_{(1)} \cdot v_j) 1_{(2)}).$$

Let $\{g_i\}_i$ and $\{\gamma^i\}_i$ be basis of V and V^* respectively, dual to each other. The element $\sum_i g_i \otimes \gamma^i$ does not depend on choice of these basis; moreover, for all $v \in V, \phi \in V^*$, one has $\phi = \sum_i \phi(g_i)\gamma^i$ and $v = \sum_i g_i \gamma^i(v)$. Set

$$b_V(z) = \Delta(1) \cdot \sum_i z \cdot g_i \otimes \gamma^i.$$

Proposition 2.6. ([2]) *The category Rep(H) is a monoidal category with duality.*

Quasitriangular weak Hopf algebra. A quasitriangular weak Hopf algebra is a pair (H, R) where H is a weak Hopf algebra and $R \in \Delta^{op}(1)(H \otimes_k H)\Delta(1)$ satisfying the following conditions:

$$\Delta^{op}(h)R = R\Delta(h),$$

for all $h \in H$, where Δ^{op} denotes the comultiplication opposite to Δ,

$$(id \otimes \Delta)R = R_{13}R_{12},$$

$$(\Delta \otimes id)R = R_{13}R_{23},$$

where $R_{12} = R \otimes 1, R_{23} = 1 \otimes R$, etc., as usual, and such that there exists $\bar{R} \in \Delta(1)(H \otimes_k H)\Delta^{op}(1)$ with

$$R\bar{R} = \Delta^{op}(1), \qquad \bar{R}R = \Delta(1).$$

Proposition 2.7. ([10]) *Let (H, R) be a quasitriangular weak Hopf algebra. Then*

$$S^2(h) = uhu^{-1}$$

for all $h \in H$, where $u = \sum S(R^{(2)})R^{(1)}$ is an invertible element of H such that

$$u^{-1} = \sum R^{(2)}S^2(R^{(1)}), \qquad \Delta(u) = \bar{R}\bar{R}_{21}(u \otimes u),$$

likewise, $v = S(u) = \sum R^{(1)}S(R^{(2)})$ obeys $S^{-2}(h) = vhv^{-1}$ and

$$v^{-1} = \sum S^2(R^{(1)})R^{(2)}, \qquad \Delta(v) = \bar{R}\bar{R}_{21}(v \otimes v),$$

where $R = \sum R^{(1)} \otimes R^{(2)}$. The element u is called the Drinfeld element of H.

Quantum Dimension. For a u-weak Hopf algebra H and $M \in Rep(H)$, we define a quantum dimension of M. Write

$$\Delta(1) = \sum_{i=1}^{n} x_i \otimes y_i$$

with $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ linearly independent. For $M \in Rep(H)$, we define a k-map $u_{ij} : M \longrightarrow M$ given by $u_{ij}(m) = S(x_i)uy_j \cdot m$ for all $m \in M$. Set $A = (tr(u_{ij}))_{n\times n}$ and we call $\underline{det}_u M = det(A)$ the quantum dimension of M by u. In this paper, we will show that a u-weak Hopf algebra H is semisimple if and only if there is a finite-dimensional projective H-module P such that $\underline{det}_u P$ is invertible in k.

R-smash Product. Let k be a field. For two vector spaces V and W and a k-linear map $R : V \otimes W \longrightarrow W \otimes V$, we write

$$R(v \otimes w) = \sum w_R \otimes v_R$$

for all $v \in V, w \in W$.

Let A and B be associative k-algebras with units, and consider a k-linear map $R : B \otimes A \longrightarrow A \otimes B$. By definition $A \bowtie_R B$ is equal to $A \otimes B$ as a k-vector space with multiplication given by the formula

$$m_{A\bowtie_R B} = (m_A \otimes m_B)(I_A \otimes R \otimes I_B)$$

or

$$(a \bowtie_R b)(c \bowtie_R d) = \sum ac_R \bowtie_R b_R d$$

for all $a, c \in A, b, d \in B$.

Definition 2.8. *Let A and B be k-algebras with units, and $R : B\otimes A \longrightarrow A\otimes B$ a k-linear map. If $A \bowtie_R B$ is an associative k-algebra with unit $1_A \bowtie 1_B$, we call $A \bowtie_R B$ an R-smash product.*

Lemma 2.9. ([4]) *Let A,B be two algebras and let $R : B \otimes A \longrightarrow A \otimes B$ be a k-linear map. Then $A \bowtie_R B$ is an R-smash product if and only if*

$$(AR1) \qquad R(b \otimes 1_A) = 1_A \otimes b$$

$$(AR2) \qquad R(1_B \otimes a) = a \otimes 1_B$$

$$(AR3) \qquad R(bd \otimes a) = \sum a_{Rr} \otimes b_r d_R$$

$$(AR4) \qquad R(b \otimes ac) = \sum a_R c_r \otimes b_{Rr}$$

for all $a, c \in A, b, d \in B$.

Proposition 2.10. ([4]) *Let $X \bowtie_R A$ be an R-smash product, then $i_X : X \longrightarrow X \bowtie_R A, x \longrightarrow x \bowtie 1_A$ and $i_A : A \longrightarrow X \bowtie_R A, a \longrightarrow 1_X \bowtie a$ are injective algebra morphisms and*

$$m_{X\bowtie_R A}(i_A \otimes i_X) = m_{X\bowtie_R A}(i_X \otimes i_A)R$$

or

$$i_A(a)i_X(x) = \sum x_R \otimes a_R$$

for all $x \in X$ and $a \in A$.

For simplicity, we write a for $i_A(a)$ (resp. x for $i_X(x)$). Then $X \bowtie_R A$ is generated by elements xa for $x \in X$, $a \in A$, and $ax = \sum x_R \otimes a_R$.

3 Semisimplicity of u-weak Hopf Algebra

Many results for Hopf algebras can be generalized directly to weak Hopf algebras. Suppose M and N are H-modules. Then $M \diamond N = \Delta(1)(M \otimes_k N)$ is an H-module given by

$$h \cdot (\Delta(1)(m \otimes n)) = \sum \Delta(1)(h_{(1)} \cdot m \otimes h_{(2)} \cdot n) \tag{19}$$

for all $h \in H, m \in M$ and $n \in N$. Similarly, $Hom(M,N) = 1 \cdot Hom_k(M,N)$ (with the obvious action of H via the comultiplication Δ and antipode S) is an H-module given by

$$(h \cdot f)(m) = \sum h_{(1)} \cdot f(S(h_{(2)}) \cdot m) \tag{20}$$

for all $h \in H, m \in M$ and $f \in Hom_k(M,N)$.

The following lemma for weak Hopf algebras corresponds to the relevant case for Hopf algebras, whose proofs are omitted.

Lemma 3.1. *Let H be a weak Hopf algebra and M a finite-dimensional H-module. If $\{m_i\}_{i=1}^n$ is a k-basis for M and $\{m_i^*\}_{i=1}^n$ is the dual basis; then the morphism $\rho = b_M : H_t \longrightarrow M \diamond M^*$ given by $\rho(z) = \Delta(1)(\sum_i (z \cdot m_i \otimes m_i^*))$ is an H-module homomorphism.*

Lemma 3.2. *Let M, N and K be H-modules. Then the morphism*

$$\phi : Hom_H(M \diamond N, K) \longrightarrow Hom_H(M, Hom(N,K))$$

given by $\phi(f)(m)(n) = f(\Delta(1)(m \otimes n))$ for all $f \in Hom_H(M \diamond N, K), m \in M$ and $n \in N$, is an isomorphism of H-modules, which is functorial in M and K.

Proof (i) For all $f \in Hom_H(M \diamond N, K)$ and $m \in M$, $\phi(f)(m) \in Hom(N,K)$. In fact, for all $n \in N$, we have

$$\begin{aligned}
[1.(\phi(f)(m))](n) &= \sum 1_{(1)}(\phi(f)(m))(S(1_{(2)}).n) \\
&= \sum 1_{(1)} f(\Delta(1)(m \otimes S(1_{(2)}).n)) \\
&= \sum f(\Delta(1_{(1)})\Delta(1)(m \otimes S(1_{(2)}).n)) \\
&= \sum f(\Delta(1)(1_{(1)}.m \otimes 1_{(2)} S(1_{(3)}).n)) \\
&= \sum f(\Delta(1)(1_{(1)}.m \otimes \varepsilon_t(1_{(2)}).n)) \\
&= \sum f(\Delta(1)(1_{(1)}.m \otimes 1_{(2)}.n)) \\
&= f(\Delta(1).(m \otimes n)) \\
&= \phi(f)(m)(n).
\end{aligned}$$

Hence $\phi(f)(m) = 1.\phi(f)(m) \in Hom(N, K)$.

(ii) For all $f \in Hom_H(M \diamond N, K)$, $\phi(f) \in Hom_H(M, Hom(N, K))$. For all $m \in M, n \in N$, and $h \in H$, we have

$$\phi(f)(h.m)(n) = f(\Delta(1)(h.m \otimes n))$$

and

$$\begin{aligned}(h.\phi(f)(m))(n) &= \sum h_{(1)}\phi(f)(m)(S(h_{(2)}).n) \\ &= \sum h_{(1)}f(\Delta(1)(m \otimes S(h_{(2)}).n)) \\ &= \sum f(\Delta(h_{(1)})(m \otimes S(h_{(2)}).n)) \\ &= \sum f(h_{(1)}.m \otimes h_{(2)}S(h_{(3)}).n) \\ &= \sum f(h_{(1)}.m \otimes \varepsilon_t(h_{(2)}).n) \\ &= \sum f(1_{(1)}h.m \otimes 1_{(2)}.n) \\ &= f(\Delta(1)(h.m \otimes n)).\end{aligned}$$

Hence $\phi(f)(h.m) = h.\phi(f)(m)$.

(iii) ϕ is an H-module map.

For all $f \in Hom_H(M \diamond N, K), m \in M, n \in N$, and $h \in H$, we have

$$\begin{aligned}\phi(h.f)(m)(n) &= (h.f)(\Delta(1)(m \otimes n)) \\ &= \sum h_{(1)}f(\Delta(S(h_{(2)}))(m \otimes n)) \\ &= \sum f(\Delta(h_{(1)}S(h_{(2)}))(m \otimes n)) \\ &= f(\Delta(\varepsilon_t(h))(m \otimes n))\end{aligned}$$

and

$$\begin{aligned}(h.\phi(f))(m)(n) &= \sum [h_{(1)}\phi(f)(S(h_{(2)}).m)](n) \\ &= \sum h_{(1)}\phi(f)(S(h_{(3)}).m)(S(h_{(2)}).n) \\ &= \sum h_{(1)}f(\Delta(1)(S(h_{(3)}).m \otimes S(h_{(2)}).n)) \\ &= \sum f(h_{(1)}S(h_{(4)}).m \otimes h_{(2)}S(h_{(3)}).n) \\ &= \sum f(h_{(1)}S(h_{(3)}).m \otimes \varepsilon_t(h_{(2)}).n) \\ &= \sum f(1_{(1)}h_{(1)}S(h_{(2)}).m \otimes 1_{(2)}.n) \\ &= \sum f(1_{(1)}\varepsilon_t(h).m \otimes 1_{(2)}.n) \\ &= f(\Delta(\varepsilon_t(h))(m \otimes n).\end{aligned}$$

Hence $\phi(h.f) = h.\phi(f)$.

(iv) ϕ is an invertible map.

We define $\psi : Hom_H(M, Hom(N, K)) \longrightarrow Hom_H(M \diamond N, K)$ by

$$\psi(g)(\Delta(1)(m \otimes n)) = g(m)(n).$$

First, for all $g \in Hom_H(M, Hom(N, K))$, $\psi(g)$ is well defined, i.e., $\psi(g)$ is a $k-$map. In fact, for all $m \in M, n \in N$, the map $\psi(g) : M \times N \longrightarrow K$ given by $\psi(g)(m, n) = g(m)(n)$ is bilinear, since maps g and $g(m)$ are homomorphisms. Hence $\psi(g)$ induces a k-map $\psi(g) : M \otimes N \longrightarrow K$, and it also induces a map $\psi(g) : M \diamond N = \Delta(1)(M \otimes N) \longrightarrow K$.

Second, for all $g \in Hom_H(M, Hom(N, K))$, $\psi(g) \in Hom_H(M \diamond N, K)$. In fact, for all $m \in M, n \in N$ and $h \in H$, we have

$$\begin{aligned}
\psi(g)(h.(\Delta(1)(m \otimes n))) &= \sum \psi(g)(\Delta(1).(h_{(1)}.m \otimes h_{(2)}.n)) \\
&= \sum g(h_{(1)}.m)(h_{(2)}.n) \\
&= \sum (h_{(1)}.g(m))(h_{(2)}.n) \\
&= \sum h_{(1)}g(m)(S(h_{(2)})h_{(3)}.n) \\
&= \sum h1_{(1)}g(m)(S(1_{(2)}).n) \\
&= \sum h[1_{(1)}g(m)(S(1_{(2)}).n)] \\
&= h[(1.g(m))(n)] \\
&= h[g(m)(n)] \\
&= h\psi(g)(\Delta(1)(m \otimes n)).
\end{aligned}$$

Hence $\psi(g)$ is an H-module map.

It is clear that $\psi\phi = id$ and $\phi\psi = id$. Therefor ϕ is an invertible map, ϕ is an isomorphism. □

Suppose H is a weak Hopf algebra and u is an invertible element in H. Let M be an H-module. Then M^{**} is an H-module by (20). Let $\psi_u : M \longrightarrow M^{**}$ be given by

$$\psi_u(m)(f) = f(u.m)$$

for all $m \in M$ and $f \in M^*$. In general, ψ_u is not an H-module homomorphism.

Proposition 3.3. *ψ_u is an H-module homomorphism for all H-modules M if and only if H is a u-weak Hopf algebra.*

Proof For all $h \in H$, if $S^2(h) = uhu^{-1}$, then

$$\begin{aligned}
(h.\psi_u(m))(f) &= \psi_u(m)(S(h).f) \\
&= (S(h).f)(u.m) \\
&= f(S^2(h)u.m) \\
&= f(uh.m) \\
&= \psi_u(h.m)(f),
\end{aligned}$$

for all $m \in M$ and $f \in M^*$. Hence , ψ_u is an H-module homomorphism.

Conversely, if ψ_u is an H-module homomorphism for all H-module M, in particular, for the regular H-module H, then $\psi_u(h)(f) = f(u.h)$ for all $h \in H$ and $f \in H^*$. On the other hand,

$$\begin{aligned}
\psi_u(h)(f) &= (h.\psi_u(1))(f) \\
&= \psi_u(1)(S(h).f) \\
&= (S(h).f)(u.1) \\
&= f(S^2(h)u).
\end{aligned}$$

Thus , $uh = S^2(h)u$. Hence, H is a u-weak Hopf algebra. □

Let $\mu : M \diamond M^* \longrightarrow H_t$ be the k-map given by

$$\mu(\Delta(1)(m \otimes \alpha)) = \sum \alpha(S(1_{(1)})u.m)1_{(2)},$$

for all $m \in M$ and $\alpha \in M^*$.

Proposition 3.4. *μ is an H-module homomorphism for all H-module M if and only if H is a u-weak Hopf algebra.*

Proof For all $h \in H$, if $S^2(h) = uhu^{-1}$, then

$$\begin{aligned}
\mu(h.\Delta(1)(m \otimes \alpha)) &= \sum \mu(h_{(1)}.m \otimes h_{(2)}.\alpha) \\
&= \sum (h_{(2)}.\alpha)(S(1_{(1)})uh_{(1)}.m)1_{(2)} \\
&= \sum \alpha(S(h_{(2)})(1_{(1)})uh_{(1)}.m)1_{(2)} \\
&= \sum \alpha(S(h_{(2)})S(1_{(1)})S^2(h_{(1)})u.m)1_{(2)} \\
&= \sum \alpha(S(S(h_{(1)})1_{(1)}h_{(2)})u.m)1_{(2)} \\
&= \sum \alpha(S(S(h_{(1)})h_{(2)})u.m)\varepsilon_t(h_{(3)}) \\
&= \sum \alpha(S(\varepsilon_s(h_{(1)})u.m)\varepsilon_t(h_{(2)})
\end{aligned}$$

and

$$\begin{aligned}
h.\mu(\Delta(1)(m \otimes \alpha)) &= \sum h.\alpha(S(1_{(1)})u.m)1_{(2)} \\
&= \sum \alpha(S(1_{(1)})u.m)\varepsilon_t(h1_{(2)}) \\
&= \sum \alpha(S(\varepsilon_s(h_{(1)})u.m)\varepsilon_t(h_{(2)}).
\end{aligned}$$

Hence μ is an H-module homomorphism for all H-modules.

Conversely, if μ is an H-module homomorphism for all H-module, then

$$\sum S(h_{(2)})S(1_{(1)})uh_{(1)} \otimes 1_{(2)} = \sum S(\varepsilon_s(1_{(1)}h)) \otimes 1_{(2)}.$$

Hence

$$\sum S(h_{(2)})uh_{(1)} = S(\varepsilon_s(h))u$$

and

$$\begin{aligned}
S^2(h)u &= S(S(h))u \\
&= \sum S(\varepsilon_s(h_{(1)})S(h_{(2)}))u \\
&= \sum S^2(h_{(2)})S(\varepsilon_s(h_{(1)}))u \\
&= \sum S^2(h_{(3)})S(h_{(2)})uh_{(1)} \\
&= \sum S(h_{(2)}S(h_{(3)}))uh_{(1)} \\
&= \sum S(\varepsilon_t(h_{(2)}))uh_{(1)} \\
&= \sum S(1_{(2)})u1_{(1)}h \\
&= S(\varepsilon_s(1))uh \\
&= uh.
\end{aligned}$$

□

Applying the quantum dimension of a representation, we can characterize the semisimplicity of a u-weak Hopf algebra. First, we prove the following proposition.

Proposition 3.5. *Let H be a u-weak Hopf algebra and P a projective H-module. Then $P \diamond M$ is a projective H-module for any H-module M.*

Proof Suppose
$$0 \longrightarrow C' \longrightarrow C \longrightarrow C'' \longrightarrow 0$$
is an exact sequence of H-modules. Since $F(-) = Hom(M, -)$ is an exact functor, by lemma 3.2, we have the following commutative diagram:

$$\begin{array}{ccccc} Hom_H(P, F(C')) & \longrightarrow & Hom_H(P, F(C)) & \longrightarrow & Hom_H(P, F(C'')) \longrightarrow 0 \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ Hom_H(P \diamond M, C') & \longrightarrow & Hom_H(P \diamond M, C) & \longrightarrow & Hom_H(P \diamond M, C'') \end{array}$$

This shows that the sequence
$$Hom_H(P \diamond M, C') \longrightarrow Hom_H(P \diamond M, C) \longrightarrow Hom_H(P \diamond M, C'') \longrightarrow 0$$
is exact. Hence $P \diamond M$ is projective. □

As an immediate consequence of proposition 3.5 and lemma 2.5, we see that H is semisimple if and only if the trivial module module H_t is projective.

Theorem 3.6. *Let H be a u-weak Hopf algebra over a field k. Then H is semisimple if and only if there is finite-dimensional projective H-module P such that $\underline{det}_u P$ is invertible in k.*

Proof If there exits a finite dimensional projective H-module P such that $\underline{det}_u P$ is invertible in k, by proposition 3.5, $P \diamond P^*$ is projective. The map $\rho : H_t \longrightarrow P \diamond P^*$ given in lemma 3.1 is an H-module homomorphism. Since $S^2(h) = uhu^{-1}$, the map $\mu : P \diamond P^* \longrightarrow H_t$ in proposition 3.4 is also an H-module homomorphism. Now
$$\mu \circ \rho(y_j) = \sum m_t^*(S(x_i)uy_j.m_t)y_i = \sum_{i=1}^{n} tr(\mu_{ij})y_i,$$
for $j = 1, ..., n$. Therefore μ is a splitting H-module homomorphism and H_t is a projective H-module. It follows that H is semisimple by proposition 3.5.

Conversely, if H is semisimple, then the trivial module H_t is projective and $\mu \circ \rho$ is an isomorphism. Hence $\underline{det}_u H_t$ is invertible in k. □

As a consequence, some other interesting results can be deduced.

Corollary 3.7. *Let H be a quasi-triangular weak Hopf algebra. Then H is semisimple if and only if there exits a finite-dimensional projective H-module such that its quantum dimension is invertible in k.*

Proof Suppose (H, R) is a quasi-triangular weak Hopf algebra. Then H is a u-weak Hopf algebra by proposition 2.7, hence corollary 3.7 is obvious by theorem 3.6. □

4 Spectral Sequence and Homological Dimension of Smash Product $X \bowtie_R A$

Let X be an associative algebra and A a weak Hopf algebra with invertible antipode S, and let $R : A \otimes X \longrightarrow X \otimes A$ be a k-linear map such that $H = X \bowtie_R A$ is an R-smash product. In this section, we assume that A_s is semisimple and

$$\Delta(1) = \sum 1_{(1)} \otimes 1_{(2)} = \sum 1_{(2)} \otimes 1_{(1)}. \tag{21}$$

We also assume that the map R satisfies the following condition:

$$\sum xS(a_{(1)}) \otimes a_{(2)} = \sum S(a_{(1)})x_R \otimes (a_{(2)})_R, \tag{22}$$

in $(X \bowtie_R A) \otimes_k (X \bowtie_R A)$, for all $x \in X$ and $a \in A$.

Proposition 4.1. *The assumption* (21) *implies* $A_s = A_t$ *and* A_s *is a commutative subalgebra of* A.

Proof It is clear by the assumption (21) and proposition 2.3. □

Remark 3. *By* (17), (18) *and proposition* 4.1, *for any* $a \in A$, *we have*

$$\Delta(\varepsilon_s(a)) = \sum 1_{(1)} \otimes \varepsilon_s(a)1_{(2)} = \sum 1_{(1)}\varepsilon_s(a) \otimes 1_{(2)}. \tag{23}$$

Let V, W be left H-modules. For each $\phi \in Hom_X(V, W)$ and $a \in A$, define $\phi.a : V \longrightarrow W$ by

$$(\phi.a)(v) = \sum S(a_{(1)})\phi(a_{(2)}v) \tag{24}$$

for all $v \in V$. Then $\phi.a \in Hom_X(V, W)$. In fact, for any $x \in X$ and $v \in V$, we have

$$\begin{aligned} (\phi.a)(xv) &= \sum S(a_{(1)})\phi(a_{(2)}xv) \\ &= \sum S(a_{(1)})\phi(x_R(a_{(2)})_R v) \\ &= \sum S(a_{(1)})x_R\phi((a_{(2)})_R v) \\ &= \sum xS(a_{(1)})\phi(a_{(2)}v) \\ &= x(\phi.a)(v). \end{aligned}$$

Let $\mathbb{H}om_X(V, W) = Hom_X(V, W).1_A$.

Lemma 4.2. (i) *The above definition makes* $\mathbb{H}om_X(V, W)$ *a right* A*-module.*

(ii) $\mathbb{H}om_H(V, W)$ *is a right* A_s*-submodule of* $\mathbb{H}om_X(V, W)$ *and there is a canonical right* A_s*-linear isomorphism*

$$Hom_A(A_S, \mathbb{H}om_X(V, W)) \cong \mathbb{H}om_H(V, W).$$

(iii) *W is a right A_s-module by the action*

$$w.\varepsilon_s(a) = \varepsilon_t(S(a))w, \quad \text{for all } w \in W, \quad a \in A,$$

and

$$\mathbb{H}om_X(H, W) \cong Hom_{A_s}(A, W) \tag{25}$$

as right A-modules (where A acts on the right hand side by $(\phi.a)(b) = \phi(ab)$, for $\phi \in Hom_{A_s}(A, W)$ and $a, b \in A$.)

(iv) *If $f : V \longrightarrow V'$ and $g : W \longrightarrow W'$ are H-module maps, then $g_* f^* : \mathbb{H}_X(V', W) \longrightarrow \mathbb{H}om_X(V, W')$ is an A-module map.*

Proof (i) Suppose $a, b \in A,\ v \in V$ and $\phi \in \mathbb{H}om_X(V, W)$, we have

$$\begin{aligned} (\phi.(ab))(v) &= \sum S(a_{(1)}b_{(1)})\phi((a_{(2)}b_{(2)})v) \\ &= \sum S(b_{(1)})S(a_{(1)})\phi(a_{(2)}(b_{(2)}v)) \\ &= \sum S(b_{(1)})(\phi.a)(b_{(2)}v) \\ &= ((\phi.a).b)(v). \end{aligned}$$

Hence $\phi.(ab) = (\phi.a).b$ and so $\mathbb{H}om_X(V, W)$ is a right A-module.

(ii) Note that there is a canonical isomorphism of $Hom_A(A_S, \mathbb{H}om_X(V, W))$ with the A_s-submodule of A-invariants in $\mathbb{H}om_X(V, W)$, that is with

$$\mathbb{H}om_X(V, W)^A = \{\phi \in \mathbb{H}om_X(V, W) | \phi.a = \varepsilon_s(a)\phi, \quad \text{for all } a \in A\}.$$

Thus it suffices to show that $\mathbb{H}om_X(V, W)^A = \mathbb{H}om_H(V, W)$.

Let $\phi \in \mathbb{H}om_X(V, W),\ a \in A,\ v \in V$. Then

$$\begin{aligned} &(\phi.a)(v) = \varepsilon_s(a)\phi(v) \\ &\Leftrightarrow \sum S(a_{(1)})\phi(a_{(2)}v) = \sum S(a_{(1)})a_{(2)}\phi(v) \\ &\Leftrightarrow \sum S(1_{(1)})\phi(1_{(2)}av) = \sum a_{(1)}S(a_{(2)})a_{(3)}\phi(v) \\ &\Leftrightarrow \phi(av) = \sum a1_{(1)}S(1_{(2)})\phi(v) = a\phi(v) \end{aligned}$$

Since $H = XA$, the last condition is equivalent with $\phi \in \mathbb{H}om_H(V, W)$. This proves the desired isomorphism and we also get $\mathbb{H}om_H(V, W)$ is a right A_s-submodule of $\mathbb{H}om_X(V, W)$.

(iii) By proposition 2.4, we have

$$\begin{aligned} \varepsilon_s(c) = \varepsilon_s(a)\varepsilon_s(b) &\Leftrightarrow S(\varepsilon_s(c)) = S(\varepsilon_s(b))S(\varepsilon_s(a)) \\ &\Leftrightarrow \varepsilon_t(S(c)) = \varepsilon_t(S(b))\varepsilon_t(S(a)). \end{aligned}$$

Hence

$$\begin{aligned} w.(\varepsilon_s(a)\varepsilon_s(b)) &= w.\varepsilon_s(c) \\ &= \varepsilon_t(S(c))w \\ &= (\varepsilon_t(S(b))\varepsilon_t(S(a)))w \\ &= (\varepsilon_t(S(b))(\varepsilon_t(S(a))w)) \\ &= (w.(\varepsilon_s(a))).\varepsilon_s(b). \end{aligned}$$

Thus W is a right A_s-module.

Now consider the map $f:\ \mathbb{H}om_X(H,W) \longrightarrow Hom_{A_s}(A,W)$ that is defined by

$$f(\phi)(a) = (\phi.a)(1) = \sum S(a_{(1)})\phi(a_{(2)})$$

for $\phi \in \mathbb{H}om_X(H,W)$ and $a \in A$. Then f is well defined. In fact, for $\phi \in \mathbb{H}om_X(H,W)$ and $a,b \in A$, set $\psi = \phi.a$ and $x = \varepsilon_s(b) \in A_t = A_s$. By (23), we have

$$\begin{aligned}
(\psi.\varepsilon_s(b))(1) &= (\psi.x)(1) \\
&= \sum S(1_{(1)}x)\psi(1_{(2)}) \\
&= \sum S(x)S(1_{(1)})\psi(1_{(2)}) \\
&= S(x)\psi(1) \\
&= S(\varepsilon_s(b))\psi(1) \\
&= \varepsilon_t(S(b))\psi(1).
\end{aligned}$$

Hence we have

$$\begin{aligned}
f(\phi)(a\varepsilon_s(b)) &= (\phi.a\varepsilon_s(b))(1) \\
&= (\psi.\varepsilon_s(b))(1) \\
&= \varepsilon_t(S(b))\psi(1) \\
&= \varepsilon_t(S(b))(\phi.a)(1) \\
&= (\phi.a)(1).\varepsilon_s(b) \\
&= f(\phi)(a).\varepsilon_s(b).
\end{aligned}$$

Thus $f(\phi) \in Hom_{A_s}(A,W)$.

The map f is A-linear. In fact, for all $\phi \in \mathbb{H}om_X(H,W)$ and $a,b \in A$, we have

$$f(\phi.a)(b) = ((\phi.a).b)(1) = (\phi.ab)(1)$$

and

$$(f(\phi).a)(b) = f(\phi)(ab) = (\phi.ab)(1).$$

Hence $f(\phi.a) = f(\phi).a$.

Define a map $g:\ Hom_{A_s}(A,W) \longrightarrow \mathbb{H}om_X(H,W)$ by

$$g(\psi)(a) = \sum a_{(1)}\psi(a_{(2)})$$

for $\psi \in Hom_{A_s}(A,W)$ and $a \in A$. Note that $g(\psi)$ is well defined because $H = XA$. One readily checks that f and g are inverse to each other, whence the isomorphism (25) follows.

Finally, the last assertion (iv) is trivial and so the lemma is proved. □

Let V_H and ${}_HW$ be H-modules. For $v \otimes w \in V \otimes_X W$ and $a \in A$, define

$$a.(v \otimes w) \in V \otimes_X W, \quad \text{by} \quad a.(v \otimes w) = \sum vS(a_{(1)}) \otimes_X a_{(2)}w.$$

This definition is well defined, in fact, let $x \in X$, we have

$$\begin{aligned} a.(vx \otimes w) &= \sum vxS(a_{(1)}) \otimes a_{(2)}w \\ &= \sum vS(a_{(1)})x_R \otimes (a_{(2)})_R w \\ &= \sum vS(a_{(1)}) \otimes a_{(2)}xw \\ &= a.(v \otimes xw). \end{aligned}$$

Let $V \bar{\otimes}_X W = 1_A.(V \otimes W)$.

Lemma 4.3. (i) *The above definition makes $V\bar{\otimes}_X W$ a left A-module.*

(ii) *$V \otimes_H W$ is a left A_s-module and there is a canonical A_s-linear isomorphism*

$$A_s \otimes_A (V\bar{\otimes}_X W) \cong V\bar{\otimes}_H W.$$

(iii) *V is a left A_s-module by the action*

$$\varepsilon_s(a).v = v\varepsilon_t(S(a)), \quad \text{for all } a \in A, \ v \in V,$$

and

$$V\bar{\otimes}_X H \cong A \otimes_{A_s} V$$

as left A-modules, where the A-action on the right hand side is via the action on the factor A.

(iv) *If $f : V \longrightarrow V'$ and $g : W \longrightarrow W'$ are H-module maps, then $g \otimes f : V\bar{\otimes}_X W \longrightarrow V'\bar{\otimes} W'$ is an A-module map.*

Proof (i) Let $a, b \in A$, we have

$$\begin{aligned} (ab).(v \otimes w) &= \sum vS(a_{(1)}b_{(1)}) \otimes a_{(2)}b_{(2)}w \\ &= \sum vS(b_{(1)})S(a_{(1)}) \otimes a_{(2)}b_{(2)}w \\ &= a.\sum vS(b_{(1)}) \otimes b_{(2)}w \\ &= a.(b.(v \otimes w)). \end{aligned}$$

Hence $V\bar{\otimes}_X W$ is a left A-module.

(ii) Note that $A_s \otimes (V\bar{\otimes}_X W) \cong V\bar{\otimes}_X W/\mathrm{Ker}\varepsilon_s(V\bar{\otimes}_X W)$ and $\mathrm{Ker}\varepsilon_s(V\bar{\otimes}_X W)$ is the A_s-submodule of $V\bar{\otimes}_X W$ that is generated by the elements of the form $a.(v \otimes w) - \varepsilon_s(a).(v \otimes w)$, for $a \in A, \ v \in V, \ w \in W$. But

$$\begin{aligned} a.(v \otimes w) - \varepsilon_s(a).(v \otimes w) &= \sum vS(a_{(1)}) \otimes a_{(2)}w - \sum vS(1_{(1)}) \otimes \varepsilon_s(a)1_{(2)}w \\ &= \sum vS(a_{(1)}) \otimes a_{(2)}w - \sum vS(1_{(1)}) \otimes 1_{(2)}\varepsilon_s(a)w \\ &= \sum vS(a_{(1)}) \otimes a_{(2)}w - \sum v \otimes \varepsilon_s(a)w \\ &= \sum vS(a_{(1)}) \otimes a_{(2)}w - \sum v \otimes S(a_{(1)})a_{(2)}w, \end{aligned}$$

and hence $\mathrm{Ker}\varepsilon_s(V\bar{\otimes}_X W)$ equals the A_s-submodule of $V\bar{\otimes}_X W$ that is generated by the elements of the form $va \otimes w - v \otimes aw$. Since $H = XA$, this proves the isomorphism and $V\bar{\otimes}_H W$ is a left A_s-module.

(iii) The proof that V is a left A_s-module is similarly as the proof of lemma 4.2 (iii). Now set

$$g : V\bar{\otimes}_X H \longrightarrow A \otimes_{A_s} V, \ \ g(v \otimes a) = \sum a_{(2)} \otimes va_{(1)}$$

and

$$f : A \otimes_{A_s} V \longrightarrow V\bar{\otimes}_X H, \ \ f(a \otimes v) = a.(v \otimes 1) = \sum vS(a_{(1)}) \otimes a_{(2)}.$$

Then

(a) It is clear that g is well defined. Now let $a, b \in A$ and $v \in V$. Note that $\varepsilon_s(b) \in A_s = A_t$, we have

$$\begin{aligned} f(a\varepsilon_s(b) \otimes v) &= \sum vS(\varepsilon_s(b)_{(1)})S(a_{(1)}) \otimes a_{(2)}\varepsilon_s(b)_{(2)} \\ &= \sum vS(\varepsilon_s(b))S(1_{(1)})S(a_{(1)}) \otimes a_{(2)}1_{(2)} \\ &= \sum v\varepsilon_t(S(b))S(a_{(1)}) \otimes a_{(2)} \\ &= f(a \otimes v\varepsilon_t(S(b))) \\ &= f(a \otimes \varepsilon_s(b).v) \end{aligned}$$

Hence $f(a\varepsilon_s(b) \otimes v) = f(a \otimes \varepsilon_s(b).v)$, i.e., f is well defined.

(b) f and g are A-linear. For any $a, b \in A$ and $v \in V$, we have

$$f(b.(a\otimes v)) = f(ba\otimes v) = \sum vS(b_{(1)}a_{(1)})\otimes b_{(2)}a_{(2)} = b.(\sum vS(a_{(1)})\otimes a_{(2)}) = b.f(a\otimes v),$$

and

$$\begin{aligned} g(b.(v \otimes a)) &= g(\sum vS(b_{(1)}) \otimes b_{(2)}a) \\ &= \sum b_{(3)}a_{(2)} \otimes vS(b_{(1)})b_{(2)}a_{(1)} \\ &= \sum b_{(2)}a_{(2)} \otimes v\varepsilon_s(b_{(1)})a_{(1)} \\ &= \sum b1_{(2)}a_{(2)} \otimes v1_{(1)}a_{(1)} \\ &= \sum ba_{(2)} \otimes va_{(1)} \\ &= b.(\sum a_{(2)} \otimes va_{(1)}) \\ &= b.g(v \otimes a). \end{aligned}$$

(c) $fg = 1_{V\bar{\otimes}_X H}$ and $gf = 1_{A\otimes_{A_s} V}$. In fact, we have

$$\begin{aligned} fg(v \otimes a) &= \sum f(a_{(2)} \otimes va_{(1)}) \\ &= \sum va_{(1)}S(a_{(2)}) \otimes a_{(3)} \\ &= \sum v\varepsilon_t(a_{(1)}) \otimes a_{(2)} \\ &= \sum vS(1_{(1)}) \otimes 1_{(2)}a \\ &= v \otimes a, \end{aligned}$$

and

$$\begin{aligned} gf(a \otimes v) &= g(\sum vS(a_{(1)}) \otimes a_{(2)}) \\ &= \sum a_{(3)} \otimes vS(a_{(1)})a_{(2)} \\ &= \sum a_{(2)} \otimes v\varepsilon_s(a_{(1)}) \\ &= \sum a1_{(2)} \otimes v1_{(1)} \\ &= \sum a \otimes v1_{(1)}S(1_{(2)}) \\ &= a \otimes v. \end{aligned}$$

Thus we get the isomorphism $V\bar{\otimes}_X H \cong A \otimes_{A_s} V$ by (b) and (c).

(iv) The last assertion is again clear and so the lemma is proved. □

The above A-actions can extend to A-actions on Ext and Tor. We expand this for Ext. The case of Tor can be treated analogously. So let V and W be left H-modules and let

$$\mathbb{P}: \quad \cdots \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow 0$$

be a projective resolution of V. So $H_n(\mathbb{P}) = 0$ for all $n \neq 0$ and $H_0(\mathbb{P}) \cong V$. Since H is free as left X-module, the restriction of $\mathbb{P}$ to X is a projective resolution of V as a left X-module. So we define

$$\mathbb{E}xt^*_X(V, W) = H^*(\mathbb{H}om_X(\mathbb{P}, W)).$$

By lemma 4.2, the components of the complex $\mathbb{H}om_X(\mathbb{P}, W)$ are right A-modules and the differential $(f^*_n)_n$ is A-linear. Thus the cohomology $H^*(\mathbb{H}om_X(\mathbb{P}, W))$ is a right A-module and hence so is $\mathbb{E}xt^*_X(V, W)$.

Proposition 4.4. (i) *Let V and W be left H-modules. Then there is a third quadrant spectral sequence*

$$E_2^{p,q} = Ext^p_A(A_s, \mathbb{E}xt^q_X(V, W)) \Rightarrow_p \mathbb{E}xt^n_H(V, W).$$

(ii) *Let V be a right H-module and W a left H-module. Then there is a first quadrant spectral sequence*

$$E^2_{p,q} = Tor^A_p(A_s, \mathbb{T}or^X_q(V, W)) \Rightarrow_p \mathbb{T}or^H_n(V, W).$$

Proof Both spectral sequences can be obtained as applications of the Grothendieck spectral sequence (c.f. Rotman, 1979, chap. 11). We let ${}_H\mathfrak{M}$, $\mathfrak{M}_A$ and $\mathfrak{M}_{A_s}$ denote the category of left H-modules, the category of right A-modules and the category of right A_s-modules respectively.

We construct two functors F and G. The The rest of the proof is analogous to proposition 3.1 in Lorenz and Lorenz (1995).

(i) Let ${}_HW$ be a given left H-module. Define functors

$$G: {}_H\mathfrak{M} \longrightarrow \mathfrak{M}_A, \quad G(V) = \mathbb{H}om_X(V, W)$$

and

$$F: \mathfrak{M}_A \longrightarrow \mathfrak{M}_{A_s}, \quad F(N) = Hom_A(A_s, N).$$

By lemma 4.2, FG is equivalent with the functor $\mathbb{H}om_H(-, W)$ and so the right derived functor $R^n(FG)$ are equivalent with $\mathbb{E}xt^n_H(-, W)$. It is easy to prove that F and G satisfy the conditions of Theorem 11.8 in Rotman(1979) under the assumption that A_s is semisimple, hence the required spectral sequence exists.

(ii) Let V_H be a given H-module. Define functors

$$G :_H \mathfrak{M} \longrightarrow_A \mathfrak{M}, \quad G(W) = V\bar{\otimes}_X W$$

and

$$F :_A \mathfrak{M} \longrightarrow_{A_s} \mathfrak{M}, \quad F(N) = A_s \otimes N.$$

By lemma 4.3 FG is equivalent with the functor $V\bar{\otimes}_H -$, and so the left derived functor $L_n(FG)$ are equivalent with $\mathbb{T}or_n^H(V, -)$. F and G also satisfy the conditions of Theorem 11.39 in Rotman(1979), thus the required spectral sequence exists. □

Note that

$$\mathbb{E}xt_X^n(V, W) = H^n(\mathbb{H}om_X(\mathbb{P}, W)) = H^n(Hom_X(\mathbb{P}, W)) = Ext_X^n(V, W), \quad n \geq 1.$$

Then the above proposition implies immediately the following estimates for the projective dimension and the flat dimension of modules.

Corollary 4.5. (i) *Let V be a left H-module. Then $pd({}_H V) \leq pd({A_s}_A) + pd({}_X V)$. Consequently, $lD(H) \leq rD(A) + lD(X)$. In particular, if X and A are semisimple, then so is H.*

(ii) *Let V be a right H-module. Then $fd(V_H) \leq fd({A_s}_A) + fd(V_X)$. Therefore $wD(H) \leq wD(A) + wD(X)$. In particular, if X and A are both von Neumann regular then so is H.*

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