

UNIQUE RANGE SETS FOR p -ADIC MEROMORPHIC FUNCTIONS IN SEVERAL VARIABLES

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Abstract

In this paper we give some unique range sets for p -adic meromorphic functions sharing four values in several variables.

1. Introduction.

In 1926, Nevanlinna proved that two nonconstant meromorphic functions of one complex variable which attain same five distinct values at the same points, must be identical.

It is observed that p -adic entire functions of one variable behave in many ways more like polynomials than like entire functions of one complex variable. In 1971, Adams and Straus [1] proved the following theorem.

Theorem A. *Let f, g be two nonconstant p -adic entire functions such that for two distinct (finite) values a, b we have $f(x) = a \Leftrightarrow g(x) = a$ and $f(x) = b \Leftrightarrow g(x) = b$. Then $f \equiv g$.*

For p -adic meromorphic functions, Adams and Straus [1] obtained the following result similar to Nevanlinna's.

Theorem B. *Let f, g be two nonconstant p -adic meromorphic functions such that for four distinct values a_1, a_2, a_3, a_4 we have $f(x) = a_i \Leftrightarrow g(x) = a_i, i = 1, 2, 3, 4$. Then $f \equiv g$.*

Key words: Unique range set, p -adic meromorphic functions, several variables.
(2000) AMS Mathematics Subject Classification: 11G, 30D35.

Ru [9] and Hu and Yang [4] extended Theorem B to p -adic holomorphic curves.

The main tool in the cited above papers is the Nevanlinna theory in one variable for the non-Archimedean case. The aim of this paper is to extend Theorem B to the case of p -adic meromorphic functions in several variables .

In this paper by using the p -adic Nevanlinna theory in high dimension, developed in [2], [3], [5], [7],[8] , we givesome range sets for p -adic meromorphic functions in several variables .

2. Height of p -adic holomorphic functions of several variables

Let p be a prime number, \mathbb{Q}_p the field of p -adic numbers and \mathbb{C}_p the p -adic completion of the algebraic closure of \mathbb{Q}_p . The absolute value in \mathbb{Q}_p is normalized so that $|p| = p^{-1}$. We further use the notion $v(z)$ for the additive valuation on \mathbb{C}_p which extends ord_p . We use the notations $b_{(m)} = (b_1, \dots, b_m)$, $b_i(b) = (b_1, \dots, b_{i-1}, b, b_{i+1}, \dots, b_m)$, $b_{(m, i_s)} = b_i(b_{i_s})$, $(\widehat{b_i}) = (b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_m)$, $D_r = \{z \in \mathbb{C}_p : |z| \leq r, r > 0\}$, $D_{<r>} = \{z \in \mathbb{C}_p : |z| = r, r > 0\}$, $D_{r_{(m)}} = D_{r_1} \times \dots \times D_{r_m}$, where $r_{(m)} = (r_1, \dots, r_m)$ for $r_i \in \mathbb{R}_+^*$, $D_{<r_{(m)}>} = D_{<r_1>} \times \dots \times D_{<r_m>}$, $|\gamma| = \gamma_1 + \dots + \gamma_m$, $z^\gamma = z_1^{\gamma_1} \dots z_m^{\gamma_m}$, $r^\gamma = r_1^{\gamma_1} \dots r_m^{\gamma_m}$, $\gamma = (\gamma_1, \dots, \gamma_m)$, where $\gamma_i \in \mathbb{N}$, $|\cdot| = |\cdot|_p$, $\log = \log_p$. Notice that the set of $(r_1, \dots, r_m) \in \mathbb{R}_+^{*m}$ such that there exist $x_1, \dots, x_m \in \mathbb{C}_p$ with $|x_i| = r_i, i = 1, \dots, m$, is dense in \mathbb{R}_+^{*m} . Therefore, without loss of generality one may assume that $D_{<r_{(m)}>} \neq \emptyset$.

Let f be a non-zero holomorphic function in $D_{r_{(m)}}$ and

$$f = \sum_{|\gamma| \geq 0} a_\gamma z^\gamma, \quad |z_i| \leq r_i \text{ for } i = 1, \dots, m.$$

Then we have

$$\lim_{|\gamma| \rightarrow \infty} |a_\gamma| r^\gamma = 0.$$

Hence, there exists an $(\gamma_1, \dots, \gamma_m) \in \mathbb{N}^m$ such that $|a_\gamma| r^\gamma$ is maximum.

Define

$$|f|_{r_{(m)}} = \max_{0 \leq |\gamma| < \infty} |a_\gamma| r^\gamma.$$

Lemma 2.1 ([7]) *For each $i = 1, \dots, m$, let r_{i_1}, \dots, r_{i_q} be positive real numbers such that $r_{i_1} \geq \dots \geq r_{i_q}$. Let $f_s(z_{(m)}), s = 1, 2, \dots, q$, be q non-zero holomorphic functions on $D_{r_{(m, i_s)}}$. Then there exist $u_{(m, i_s)} \in D_{r_{(m, i_s)}}$ such that*

$$|f_s(u_{(m, i_s)})| = |f_s|_{r_{(m, i_s)}}, s = 1, 2, \dots, q.$$

Definition 2.2. The height of the function $f(z_{(m)})$ is defined by

$$H_f(r_{(m)}) = \log |f|_{r_{(m)}}.$$

If $f(z_{(m)}) \equiv 0$, then set $H_f(r_{(m)}) = -\infty$.

Let f be a non-zero holomorphic function in $D_{r_{(m)}}$ and

$$f = \sum_{|\gamma| \geq 0} a_\gamma z^\gamma, \quad |z_i| \leq r_i \text{ for } i = 1, \dots, m.$$

Write

$$f(z_{(m)}) = \sum_{k=0}^{\infty} f_{i,k}(\widehat{z_i}) z_i^k, \quad i = 1, 2, \dots, m.$$

Set

$$\begin{aligned} I_f(r_{(m)}) &= \left\{ (\gamma_1, \dots, \gamma_m) \in \mathbb{N}^m : |a_\gamma| r^\gamma = |f|_{r_{(m)}} \right\}, \\ n_{1i,f}(r_{(m)}) &= \max \left\{ \gamma_i : \exists (\gamma_1, \dots, \gamma_i, \dots, \gamma_m) \in I_f(r_{(m)}) \right\}, \\ n_{2i,f}(r_{(m)}) &= \min \left\{ \gamma_i : \exists (\gamma_1, \dots, \gamma_i, \dots, \gamma_m) \in I_f(r_{(m)}) \right\}, \\ n_{i,f}(0,0) &= \min \left\{ k : f_{i,k}(\widehat{z_i}) \neq 0 \right\}, \\ \nu_f(r_{(m)}) &= \sum_{i=1}^m (n_{1i,f}(r_{(m)}) - n_{2i,f}(r_{(m)})). \end{aligned}$$

Call $r_{(m)}$ a *critical point* if $\nu_f(r_{(m)}) \neq 0$.

For a fixed i ($i = 1, \dots, m$) we set for simplicity

$$n_{i,f}(0,0) = \ell, k_1 = n_{1i,f}(r_{(m)}), k_2 = n_{2i,f}(r_{(m)}).$$

Then there exist multi-indices $\gamma = (\gamma_1, \dots, \gamma_i, \dots, \gamma_m) \in I_f(r_{(m)})$ and $\mu = (\mu_1, \dots, \mu_i, \dots, \mu_m) \in I_f(r_{(m)})$ such that $\gamma_i = k_1, \mu_i = k_2$.

We consider the following holomorphic functions on $D_{r_{(m)}}$

$$f_\ell(z_{(m)}) = f_{i,\ell}(\widehat{z_i}) z_i^\ell, f_{k_1}(z_{(m)}) = f_{i,k_1}(\widehat{z_i}) z_i^{k_1}, f_{k_2}(z_{(m)}) = f_{i,k_2}(\widehat{z_i}) z_i^{k_2}.$$

The functions are not identically zero.

Set

$$\begin{aligned} U_{i,f,r_{(m)}} &= \{u = u_{(m)} \in D_{r_{(m)}} : |f_\ell(u)| = |f_\ell|_{r_{(m)}}, |f(u)| = |f|_{r_{(m)}}, \\ &|f_{k_1}(u)| = |f_{k_1}|_{r_{(m)}}, |f_{k_2}(u)| = |f_{k_2}|_{r_{(m)}}\}, \end{aligned}$$

where $i = 1, \dots, m$. By Lemma 2.1, $U_{i,f,r(m)}$ is a non-empty set. For each $u \in U_{i,f,r(m)}$, set

$$f_{i,u}(z) = f(u_1, \dots, u_{i-1}, z, u_{i+1}, \dots, u_m), \quad z \in D_{r_i}.$$

Theorem 2.3. *Let $f(z_{(m)})$ be a holomorphic function on $D_{r(m)}$. Assume that $f(z_{(m)})$ is not identically zero. Then for each $i = 1, \dots, m$, and for all $u \in U_{i,f,r(m)}$, we have*

- 1) $H_f(r(m)) = H_{f_{i,u}}(r_i)$,
- 2) $n_{1i,f}(r(m))$ is equal to the number of zeros of $f_{i,u}$ in D_{r_i} ,
- 3) $n_{1i,f}(r(m)) - n_{2i,f}(r(m))$ is equal to the number of zeros of $f_{i,u}$ on $D_{<r_i>}$.

For the proof, see [7, Theorem 3.1].

From Theorem 2.3 we see that $f(z_{(m)})$ has zeros on $D_{<r(m)>}$ if and only if $r(m)$ is a critical point.

For a an element of \mathbb{C}_p and f a holomorphic function on $D_{r(m)}$, which is not identically equal to a , define

$$n_{i,f}(a, r(m)) = n_{1i,f-a}(r(m)), \quad i = 1, \dots, m.$$

Fix real numbers ρ_1, \dots, ρ_m with $0 < \rho_i \leq r_i$, $i = 1, \dots, m$.

For each $x \in \mathbb{R}$, set $A_i(x) = (\rho_1, \dots, \rho_{i-1}, x, r_{i+1}, \dots, r_m)$, $i = 1, \dots, m$,
 $B_i(x) = (\rho_1, \dots, \rho_{i-1}, x, \rho_{i+1}, \dots, \rho_m)$, $i = 1, \dots, m$.

Define the counting function $N_f(a, r(m))$ by

$$N_f(a, r(m)) = \frac{1}{\ln p} \sum_{i=1}^m \int_{\rho_i}^{r_i} \frac{n_{i,f}(a, A_i(x))}{x} dx.$$

If $a=0$, then set $N_f(r(m)) = N_f(0, r(m))$.

Then

$$N_f(a, B_i(r_i)) = \frac{1}{\ln p} \int_{\rho_i}^{r_i} \frac{n_{i,f}(a, B_i(x))}{x} dx.$$

For each $i = 1, 2, \dots, m$, set

$$\begin{aligned} k_{1,i} &= n_{1i,f}(A_i(r_i)), k_{2,i} = n_{2i,f}(A_i(r_i)), \\ U_{i,f,A_i(r_i)}^i &= \{u^i = u_{(m)}^i \in D_{A_i(r_i)} : |f_\ell(u^i)| = |f_\ell|_{A_i(r_i)}, \\ &|f(u^i)| = |f|_{A_i(r_i)}, |f_{k_{1,i}}(u^i)| = |f_{k_{1,i}}|_{A_i(r_i)}, \\ &|f_{k_{2,i}}(u^i)| = |f_{k_{2,i}}|_{A_i(r_i)}\}, \\ \Gamma_i &= \{A_i(x) : A_i(x) \text{ is a critical point, } 0 < x \leq r_i\}. \end{aligned}$$

By Lemma 2.1 and Theorem 2.3, Γ_i is a finite set. Suppose that Γ_i , $i = 1, \dots, m$, contains n elements $A_i(x^j)$, $j = 1, \dots, n$. From this and Lemma 2.1 it follows that

$$\mathcal{U}_{i_f, A_i(r_i)}^i = \{u^i = u_{(m)}^i \in U_{i_f, A_i(r_i)}^i : \exists u_i^i(u^j) \in U_{i_f, A_i(x^j)}^i, j = 1, \dots, n\} \neq \emptyset,$$

$i = 1, \dots, m$.

Lemma 2.4.

1) Let f be a non-zero holomorphic function on $D_{r_{(m)}}$. Then for each $i = 1, 2, \dots, m$, and for all $u^i \in \mathcal{U}_{i_f, A_i(r_i)}^i$, we have

$$n_{f_i, u^i}(x) = n_{i, f} \circ A_i(x), \rho_i \leq x \leq r_i,$$

2) Let $f_s(z_{(m)}), s = 1, 2, \dots, q$, be q non-zero holomorphic functions on $D_{r_{(m)}}$. Then for each $i = 1, 2, \dots, m$, there exists $u^i \in \mathcal{U}_{i_{f_s}, A_i(r_i)}^i$ for all $s = 1, \dots, q$.

The result can be proved easily by using Lemma 2.1 and Theorem 2.3.

Theorem 2.5. Let f be a non-zero holomorphic function on $D_{r_{(m)}}$. Then

$$H_f(r_{(m)}) - H_f(\rho_{(m)}) = N_f(r_{(m)}).$$

The proof of Theorem 2.5 follows immediately from [7, Theorem 3.2].

Let f be a non-zero holomorphic function on $D_{r_{(m)}}$, $a = (a_1, \dots, a_m) \in D_{r_{(m)}}$, and

$$f = \sum_{|\gamma|=0}^{\infty} a_{\gamma} (z_1 - a_1)^{\gamma_1} \dots (z_m - a_m)^{\gamma_m}, \quad z_{(m)} \in D_{r_{(m)}}.$$

Set

$$v_f(a) = \min \{|\gamma| : a_{\gamma} \neq 0\}.$$

For each $i = 1, 2, \dots, m$, write

$$f(z_{(m)}) = \sum_{k=0}^{\infty} f_{i,k}(\widehat{z_i - a_i})(z_i - a_i)^k.$$

Set

$$g_{i,k}(z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_m) = f_{i,k}(\widehat{z_i - a_i}),$$

$$b_{i,k} = g_{i,k}(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_m).$$

Then

$$f_{i,a}(z) = \sum_{k=0}^{\infty} b_{i,k} (z_i - a_i)^k.$$

Set

$$v_{i,f}(a) = \begin{cases} \min \{k : b_{i,k} \neq 0\} & \text{if } f_{i,a}(z) \not\equiv 0 \\ +\infty & \text{if } f_{i,a}(z) \equiv 0, \end{cases}$$

$$ord_{i,f}(a) = \begin{cases} \min \{k : g_{i,k}(\widehat{z_i}) \neq 0\} \\ +\infty & \text{if } g_{i,k}(\widehat{z_i}) \equiv 0 \text{ for all } k. \end{cases}$$

If $f(a) = 0$, then a (resp., a_i) is a zero of $f(z_{(m)})$ (resp., $f_{i,a}(z)$). Then the numbers $v_f(a)$, $v_{i,f}(a)$, $ord_{i,f}(a)$ are called *multiplicity*, *i -th partial multiplicity*, *i -th partial order*, respectively, of a .

Set

$$v = (u^1, \dots, u^m), u^i \in \mathcal{U}_{i,f,A_i(r_i)}^i,$$

$$N_{f_v}(r_{(m)}) = N_{f_{1,u^1}}(r_1) + \dots + N_{f_{m,u^m}}(r_m),$$

$$V = \{v : N_{f_v}(r_{(m)}) = N_f(r_{(m)})\},$$

where $\bar{n}_{f_{i,u^i}}(r_i)$ be the number of distinct zeros of f_{i,u^i} . By Lemma 2.4 and [4], V is a non-empty set,

$$N_{f_v}(r_{(m)}) = \sum_{\rho_1 < |a| \leq r_1} (v(a) + \log r_1) + n_{f_{1,u^1}}(0, \rho_1)(\log r_1 - \log \rho_1)$$

$$+ \dots + \sum_{\rho_m < |a| \leq r_m} (v(a) + \log r_m) + n_{f_{m,u^m}}(0, \rho_m)(\log r_m - \log \rho_m), \quad (2.1)$$

where

$$\sum_{\rho_i < |a| \leq r_i} (v(a) + \log r_i)$$

is taken on all of zeros a of f_{i,u^i} (counting multiplicity) with $\rho_i < |a| \leq r_i$, $i = 1, 2, \dots, m$. Notice that, the sums in (2.1) are finite sums.

Denote by $\bar{N}_{f_v}(r_{(m)})$ the sum (2.1), where every zero a of the functions f_{i,u^i} , $i = 1, \dots, m$, is counted ignoring multiplicity. Set

$$\bar{N}_f(r_{(m)}) = \max_{v \in V} \bar{N}_{f_v}(r_{(m)}).$$

From Lemma 2.4 it follows that one can find $u^i \in \mathcal{U}_{i,f,A_i(r_i)}^i$ and $v = (u^1, \dots, u^m)$ such that $\bar{N}_f(r_{(m)}) = \bar{N}_{f_v}(r_{(m)})$.

If γ is a multi-index and f is a meromorphic function of m variables, then we denote by ∂_f^γ the partial derivative

$$\frac{\partial^{|\gamma|} f}{\partial z_1^{\gamma_1} \dots \partial z_m^{\gamma_m}}.$$

Theorem 2.6. *Let f be a non-zero entire function on \mathbb{C}_p^m and γ a multi-index with $|\gamma| > 0$. Then*

$$H_{\partial^\gamma f}(B_e(r_e)) - H_f(B_e(r_e)) \leq -|\gamma| \log r_e + O(1).$$

The proof of Theorem 2.6 follows immediately from [3, Lemma 4.1].

3. Unique range sets for p -adic meromorphic functions in several variables

Let $f = \frac{f_1}{f_2}$ be a meromorphic function on $D_{r(m)}$ (resp., \mathbb{C}_p^m), where f_1, f_2 be two holomorphic functions on $D_{r(m)}$ (resp., \mathbb{C}_p^m), have no common zeros, and $a \in \mathbb{C}_p$.

We set

$$H_f(r(m)) = \max_{1 \leq i \leq 2} H_{f_i}(r(m)),$$

and

$$N_f(a, r(m)) = N_{f_1 - af_2}(r(m)).$$

For a point $d \in \mathbb{C}_p$ we define the function $v_f^d : \mathbb{C}_p^m \rightarrow (\mathbb{N} \cup \{+\infty\})^m$ by $v_f^d(a(m)) = v_{f_1 - df_2}^0(a(m))$ and write $v_f^d(a(m)) = (v_{1,f}^d(a(m)), \dots, v_{m,f}^d(a(m)))$, and $v_f^\infty(a(m)) = v_{f_2}^0(a(m))$ and write $v_f^\infty(a(m)) = (v_{1,f}^\infty(a(m)), \dots, v_{m,f}^\infty(a(m)))$.

For a subset S of \mathbb{C}_p we set

$$E_i(f, S)$$

$$= \bigcup_{d \in S} \left\{ (q_i, a(m)) \in (\mathbb{N} \cup \{+\infty\}) \times \mathbb{C}_p^m \mid f(a(m)) - d = 0, v_{i,f}^d(a(m)) = q_i \right\},$$

$$E_i(f, S \cup \{\infty\})$$

$$= E_i(f, S) \bigcup \left\{ (q_i, a(m)) \in (\mathbb{N} \cup \{+\infty\}) \times \mathbb{C}_p^m \mid v_{i,f}^\infty(a(m)) = q_i \right\},$$

$i = 1, 2, \dots, m$.

$$\overline{E}_f(a) = \{z \in \mathbb{C}_p : f_1 - af_2 = 0 \text{ ignoring multiplicities}\},$$

$$\overline{E}_f(\infty) = \{z \in \mathbb{C}_p : f_2 = 0 \text{ ignoring multiplicities}\}.$$

Lemma 3.1. *Let $f = \frac{f_1}{f_2}$ be a non-constant meromorphic function on \mathbb{C}_p^m . Then there exists a multi-index $\gamma_1 = (0, \dots, 0, \gamma_{1e}, 0, \dots, 0)$ such that $\gamma_{1e} = 1$*

and $\partial_f^{\gamma_1} = \frac{\partial_{f_1}^{\gamma_1} \cdot f_2 - \partial_{f_2}^{\gamma_1} \cdot f_1}{f_2^2}$ and the Wronskians

$$W(f) = W(f_1, f_2) = \det \begin{pmatrix} f_1 & f_2 \\ \partial_{f_1}^{\gamma_1} & \partial_{f_2}^{\gamma_1} \end{pmatrix}$$

are not identically zero.

For the proof, see [3, Lemma 4.2].

Theorem 3.2. *Let f be a non-constant meromorphic function on \mathbb{C}_p^m and $a_i \in \mathbb{C}_p, i = 1, \dots, q$. Then*

$$(q - 1)H_f(B_e(r_e)) \leq \sum_{j=1}^q \overline{N}_f(a_j, B_e(r_e)) + \overline{N}_f(\infty, B_e(r_e)) - \log r_e + O(1).$$

Proof Set $G = \{G_{\beta_1} \dots G_{\beta_{q-1}}\}$, where $(\beta_1, \dots, \beta_{q-1})$ is taken on all different choices of $q-1$ numbers in the set $\{1, \dots, q+1\}$, and $G_i = f_1 - a_i f_2, i = 1, \dots, q$, and $G_{q+1} = f_2$. Set $H_G(B_e(r_e)) = \max_{(\beta_1 \dots \beta_{q-1})} H_{G_{\beta_1} \dots G_{\beta_{q-1}}}(B_e(r_e))$. We need the following

Lemma 3.3. *We have $H_G(B_e(r_e)) \geq (q - 1)H_f(B_e(r_e)) + O(1)$, where $O(1)$ does not depend on r_e .*

Proof We have

$$\begin{aligned} H_G(B_e(r_e)) &= \max_{(\beta_1, \dots, \beta_{q-1})} H_{G_{\beta_1} \dots G_{\beta_{q-1}}}(B_e(r_e)) \\ &= \max_{(\beta_1, \dots, \beta_{q-1})} \sum_{1 \leq i \leq q-1} H_{G_{\beta_i}}(B_e(r_e)). \end{aligned}$$

Assume that for a fixed r_e , the following inequalities hold

$$H_{G_{\beta_1}}(B_e(r_e)) \geq H_{G_{\beta_2}}(B_e(r_e)) \geq \dots \geq H_{G_{\beta_{q+1}}}(B_e(r_e)).$$

Then

$$H_G(B_e(r_e)) = H_{G_{\beta_1}}(B_e(r_e)) + H_{G_{\beta_2}}(B_e(r_e)) + \dots + H_{G_{\beta_{q-1}}}(B_e(r_e)). \quad (3.1)$$

Since a_1, \dots, a_q are distinct numbers in \mathbb{C}_p , then

$$f_i = b_{i_0} G_{\beta_q} + b_{i_1} G_{\beta_{q+1}}, \quad i = 1, 2,$$

where b_{i_0}, b_{i_1} are constants, which do not depend on r_e . It follows that

$$H_{f_i}(B_e(r_e)) \leq \max_{0 \leq j \leq 1} H_{G_{\beta_{q+j}}}(B_e(r_e)) + O(1).$$

Therefore, we obtain

$$H_{f_i}(B_e(r_e)) \leq H_{G_{\beta_j}}(B_e(r_e)) + O(1),$$

for $j = 1, \dots, q-1$ and $i = 1, 2$. Hence,

$$H_f(B_e(r_e)) = \max_{1 \leq i \leq 2} H_{f_i}(B_e(r_e)) \leq H_{G_{\beta_j}}(B_e(r_e)) + O(1), \quad (3.2)$$

for $j = 1, \dots, q-1$. Summarizing $(q-1)$ inequalities (3.2) and by (3.1), we have

$$H_G(B_e(r_e)) \geq (q-1)H_f(B_e(r_e)) + O(1).$$

Now we prove Theorem 3.2. Denote by $W(g_1, g_2)$ the Wronskian of two entire functions g_1, g_2 with respect to the γ_1 as in 3.1.

Since f is non-constant, we have $W(f_1, f_2) \neq 0$. Let (α_1, α_2) be distinct two numbers in $\{1, \dots, q+1\}$, and $(\beta_1, \dots, \beta_{q-1})$ be the rest. Note that the functions f_i can be represented as linear combinations of $G_{\alpha_1}, G_{\alpha_2}$. Then we have

$$W(G_{\alpha_1}, G_{\alpha_2}) = c_{(\alpha_1, \alpha_2)} W(f_1, f_2),$$

where $c_{(\alpha_1, \alpha_2)} = c$ is a constant, depending only on (α_1, α_2) . We denote

$$\begin{aligned} A &= A(\alpha_1, \alpha_2) = \frac{W(G_{\alpha_1}, G_{\alpha_2})}{G_{\alpha_1} G_{\alpha_2}} \\ &= \det \left(\begin{array}{cc} 1 & 1 \\ \frac{\partial_{G_{\alpha_1}}^{\gamma_1}}{G_{\alpha_1}} & \frac{\partial_{G_{\alpha_2}}^{\gamma_1}}{G_{\alpha_2}} \end{array} \right). \end{aligned}$$

Hence

$$\frac{G_1 \dots G_{q+1}}{W(f_1, f_2)} = \frac{c G_{\beta_1} \dots G_{\beta_{q-1}}}{A}. \quad (3.3)$$

Set $L_i = \frac{\partial_{G_{\alpha_i}}^{\gamma_1}}{G_{\alpha_i}}$, $i = 1, 2$. Then

$$\log |A|_{B_e(r_e)} \leq \max_{1 \leq i \leq 2} \log |L_i|_{B_e(r_e)}.$$

By Theorem 2.6

$$\log |L_i|_{B_e(r_e)} \leq -|\gamma_1| \log r_e + O(1).$$

Because $|\gamma_1| = 1$

$$\log |L_i|_{B_e(r_e)} \leq -\log r_e + O(1). \quad (3.4)$$

By (3.3), we obtain

$$\begin{aligned} & \sum_{i=1}^{q+1} H_{G_i}(B_e(r_e)) - H_W(B_e(r_e)) \\ &= H_{G_{\beta_1 \dots G_{\beta_{q-1}}}}(B_e(r_e)) - \log |A|_{B_e(r_e)} + O(1). \end{aligned}$$

From this and (3.4), we have

$$\begin{aligned} H_G(B_e(r_e)) &= \max_{(\beta_1, \dots, \beta_{q-1})} H_{G_{\beta_1 \dots G_{\beta_{q-1}}}}(B_e(r_e)) \\ &\leq \sum_{i=1}^{q+1} H_{G_i}(B_e(r_e)) - H_W(B_e(r_e)) - \log r_e + O(1). \end{aligned}$$

By Lemma 3.3

$$(q-1)H_f(B_e(r_e)) \leq \sum_{i=1}^{q+1} H_{G_i}(B_e(r_e)) - H_W(B_e(r_e)) - \log r_e + O(1).$$

Thus

$$(q-1)H_f(B_e(r_e)) + H_W(B_e(r_e)) \leq \sum_{j=1}^{q+1} H_{G_j}(B_e(r_e)) - \log r_e + O(1). \quad (3.5)$$

By Theorem 2.5

$$\begin{aligned} H_W(B_e(r_e)) &= N_W(B_e(r_e)) + O(1), \\ H_{G_j}(B_e(r_e)) &= N_{G_j}(B_e(r_e)) + O(1). \end{aligned}$$

Therefore and (3.5) we obtain

$$(q-1)H_f(B_e(r_e)) + N_W(B_e(r_e)) \leq \sum_{j=1}^{q+1} N_{G_j}(B_e(r_e)) - \log r_e + O(1). \quad (3.6)$$

For a fixed $B_e(r_e)$, we consider non-zero entire functions W, G_1, \dots, G_q on $D_{B_e(r_e)}$. From Lemma 2.4 it follows that one can find $u^e \in \mathcal{U}_{G_j, B_e(r_e)}^e$ and $u^e \in \mathcal{U}_{W, B_e(r_e)}^e, j = 1, \dots, q$, such that

$$N_W(B_e(r_e)) = N_{W_{e, u^e}}(r_e), N_{G_j}(B_e(r_e)) = N_{(G_j)_{e, u^e}}(r_e).$$

Now let $u_e^e(x)$ be a zero of G_j , having the e -th partial multiplicity equal to k , ($k \neq +\infty, k \geq 2$). Since $\gamma_1 = (0, \dots, 0, \gamma_{1e}, 0, \dots, 0)$ with $\gamma_{1e} = 1$, we have $v_{i, \delta_{G_j}^{\gamma_1}}(u_e^e(x)) = k - 1$ if $i = e$.

On the other hand

$$W(G_{\alpha_1}, G_{\alpha_2}) = c_{(\alpha_1, \alpha_2)} W,$$

where (α_1, α_2) are distinct two numbers in $\{1, \dots, q+1\}$. Therefore $u_e^e(x)$ is a zero of W having e -th partial multiplicity at least $k-1$.

Now we consider the function $F = \prod_{i=1}^q G_i$.

Because F is not a constant, F has zeros. Let $u_e^e(x)$ be a zero of F . By the hypothesis, a_1, \dots, a_q are distinct numbers, from this it follows that there exists only one function G_j such that $G_j(u_e^e(x)) = 0$. Therefore

$$\sum_{j=1}^q N_{(G_j)_{e,u^e}}(r_e) - N_{W_{e,u^e}}(r_e) \leq \sum_{j=1}^q \overline{N}_{(G_j)_{e,u^e}}(r_e).$$

Thus

$$\sum_{j=1}^q N_{G_j}(B_e(r_e)) - N_W(B_e(r_e)) \leq \sum_{j=1}^q \overline{N}_{(G_j)_{e,u^e}}(r_e) = \sum_{j=1}^q \overline{N}_{G_j}(B_e(r_e)).$$

From this and (3.6) we obtain Theorem 3.2.

Theorem 3.4. *Let f, g be two nonconstant meromorphic functions on \mathbb{C}_p^m such that $\overline{E}(f, a_i) = \overline{E}(g, a_i)$, $a_i \in \mathbb{C}_p \cup \{\infty\}$, $i = 1, 2, \dots, q$. If $q \geq 4$, then $f \equiv g$.*

Proof Assume, on the contrary, that $f \not\equiv g$. Set

$$\varphi = \frac{1}{f} - \frac{1}{g}.$$

Then, $\varphi \not\equiv 0$ and $H_\varphi(B_e(r_e)) \leq H_f(B_e(r_e)) + H_g(B_e(r_e))$.

Therefore, applying Theorem 3.2 to the function f and values a_1, \dots, a_q we have

$$\begin{aligned} (q-2)H_f(B_e(r_e)) &\leq \sum_{j=1}^q \overline{N}_f(a_j, B_e(r_e)) - \log r_e + O(1) \\ &\leq \overline{N}_\varphi(B_e(r_e)) - \log r_e + O(1). \end{aligned}$$

Similarly

$$(q-2)H_g(B_e(r_e)) \leq \overline{N}_\varphi(B_e(r_e)) - \log r_e + O(1).$$

Summing up these inequalities and using Theorem 2.5, we obtain

$$\begin{aligned} (q-2)(H_f(B_e(r_e)) + H_g(B_e(r_e))) &\leq 2(H_f(B_e(r_e)) \\ &\quad + H_g(B_e(r_e)) - 2\log r_e + O(1)). \end{aligned}$$

Therefore

$$(q - 4) \left(H_f \left(B_e(r_e) \right) + H_g \left(B_e(r_e) \right) + 2 \log r_e \right) \leq O(1).$$

It implies $q - 4 < 0$, a contradiction. Theorem 3.4 is proved.

Theorem 3.5. *Let f, g be two nonconstant meromorphic functions on \mathbb{C}_p^m such that $E_i(f, a_j) = E_i(g, a_j), i = 1, 2, \dots, m, a_j \in \mathbb{C}_p \cup \{\infty\}, j = 1, 2, 3$. Then $f \equiv g$.*

Proof We need the following

Lemma 3.6. *Let f, g be two non-zero entire functions on \mathbb{C}_p^m such that $v_f^0 = v_g^0$ on \mathbb{C}_p^m . Then $f = cg$ where c is a non-zero constant in \mathbb{C}_p .*

Proof Take $r_1, \dots, r_m > 0$ such that f, g have no zeros in $D_{<r(m)>}$. If f is a non-zero constant then so is g . Therefore $f = cg$. Assume that f is non-constant. Since $v_f^0 = v_g^0$, g is non-constant. Let $a = (a_1, \dots, a_m), b = (b_1, \dots, b_m)$ be two any elements of $D_{<r(m)>}$. Set

$$C_i(b_i) = (b_1, \dots, b_i, a_{i+1}, \dots, a_m), \quad i = 1, \dots, m.$$

By $v_f^0 = v_g^0, v_{i,f}(z(m)) = v_{i,g}(z(m)), i = 1, \dots, m$. Thus

$$f_{i,C_i(b_i)} = c_i g_{i,C_i(b_i)}, \text{ with } c_i = \frac{f(a)}{g(a)} = \frac{f(C_i(b_i))}{g(C_i(b_i))}, \text{ and } c_i = c_{i+1},$$

$i = 1, 2, \dots, m - 1$. From this we have

$$\frac{f(a)}{g(a)} = \frac{f(b)}{g(b)} \quad \text{for all } a, b \in D_{r_{<m>}}.$$

Set

$$c = \frac{f(a)}{g(a)}, \quad a \in D_{<r(m)>}, \quad h = f - cg.$$

Assume that h is not identically zero. Consider h, f, g in $D_{<r(m)>}$. By Lemma 2.2, there exists $u \in D_{<r(m)>}$ such that $h_{i,u}, f_{i,u}, g_{i,u}$ are not identically zero, $i = 1, 2, \dots, m$. We have $f_{i,u} = c' g_{i,u}, c' = \frac{f(u)}{g(u)}$. Therefore $c = c'$ and $h_{i,u} = f_{i,u} - c g_{i,u}$ identically zero. From this we obtain a contradiction. So, $f = cg$.

Now we prove Theorem 3.5. Write $f = \frac{f_1}{f_2}, g = \frac{g_1}{g_2}$, where f_1, f_2 are two holomorphic functions on \mathbb{C}_p^m , having no common zeros, and g_1, g_2 are too. Applying Lemma 3.6 to $f_1 + a_j f_2$ and $g_1 + a_j g_2, j = 1, 2, 3$ we have $f_1 + a_j f_2 = c_j (g_1 + a_j g_2), c_j \neq 0, j = 1, 2, 3$. From this we obtain $c_1 = c_2 = c_3$ and $f \equiv g$.

Theorem 3.7. *Let f, g be two nonconstant meromorphic functions on \mathbb{C}_p^m such that $\overline{E}(f, a_i) \subset \overline{E}(g, a_i), a_i \in \mathbb{C}_p \cup \{\infty\}, i = 1, 2, \dots, q$, and $E_i(f, b_j) = E_i(g, b_j), b_j \in \mathbb{C}_p \cup \{\infty\}, j = 1, 2$, and $a_i \neq b_j$ for all i, j . If $q \geq 4$, then $f \equiv g$.*

Proof Similarly as in the proof of Theorem 3.5, we have $f = cg$, $c \neq 0$. Assume, on the contrary, that $f \neq g$. Set

$$\varphi = \frac{1}{f} - \frac{1}{g}.$$

Then, $\varphi \neq 0$ and $H_\varphi(B_e(r_e)) \leq H_f(B_e(r_e)) + H_g(B_e(r_e))$.

Applying Theorem 3.2 to the function f and values a_1, \dots, a_q we have

$$\begin{aligned} (q-2)H_f(B_e(r_e)) &\leq \sum_{j=1}^q \overline{N}_f(a_j, B_e(r_e)) - \log r_e + O(1) \\ &\leq \overline{N}_\varphi(B_e(r_e)) - \log r_e + O(1). \end{aligned}$$

Using Theorem 2.5, we obtain

$$(q-2)(H_f(B_e(r_e))) \leq (H_f(B_e(r_e)) + H_g(B_e(r_e))) - 2 \log r_e + O(1).$$

By $H_f(B_e(r_e)) = H_g(B_e(r_e))$

$$(q-4)H_f(B_e(r_e)) + 2 \log r_e \leq O(1).$$

It implies $q-4 < 0$, a contradiction. Theorem 3.7 is proved.

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