East-West J. of Mathematics: Vol. 9, No 1 (2007) pp. 31- 37

MULTIPLICATIVE GENERALIZED DERIVATIONS WHICH ARE ADDITIVE

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Abstract

The purpose of this note is to prove the following. Suppose R is a ring having an idempotent element e (e $\neq 0$, e $\neq 1$) which satisfies some conditions. If g is any multiplicative generalized derivation of R, i.e. $g(xy) = g(x)y + xd(y)$, for all x, y in R and some derivation d of R, then g is additive.

1 Introduction

In [5] Martindale has asked the following question: When is a multiplicative mapping additive? He answered his question for a multiplicative isomorphism of a ring R under the existence of a family of idempotent elements in R which satisfies some conditions. In [2], Daif has given an answer to that question when the mapping is a multiplicative derivation on R.

In [3], Hvala has defined the notion of generalized derivation as follows: An additive mapping $g : R \to R$ is said to be a generalized derivation if there exists a derivation $d : R \to R$ such that

$$g(xy) = g(x)y + xd(y) \text{ for all } x, y \in R.$$

Also, he calls the maps of the form $x \to ax + xb$ where a, b are fixed elements in R by the inner generalized derivations. Hence the concept of a generalized

Key words: Ring, idempotent element, generalized derivation, Peirce decomposition.
2000 AMS Mathematics Subject Classification: 16W25, 16N60, 16U80.

derivation covers both the concepts of a derivation and a left centralizer (i.e., an additive map f satisfying $f(xy) = f(x)y$ for all $x, y \in R$). In [1, Remark 1] Brešar proved that: for a semiprime ring R, if g is a function from R to R and $d : R \to R$ is an additive mapping such that $g(xy) = g(x)y + xd(y)$ for all $x, y \in R$, then g is uniquely determined by d and moreover d must be a derivation.

In this note, We introduce the notion of the multiplicative generalized derivation of a ring R to be a mapping g of R into R such that $g(xy) = g(x)y + xd(y)$, for all $x, y \in R$, where d is a derivation from R into R. Parallel to the works of Martindale [5] and Daif [2], we ask the following question for a multiplicative generalized derivation, that is, when is a multiplicative generalized derivation additive? Under some conditions, we give an answer for this question.

As in [4], let e in R be an idempotent element so that $e \neq 1, e \neq 0$ (R need not have an identity). We will formally set $e_1 = e$ and $e_2 = 1 - e$. The two-sided Peirce decomposition of R relative to the idempotent e takes the form $R = e_1Re_1 \oplus e_1Re_2 \oplus e_2Re_1 \oplus e_2Re_2$. So letting $R_{mn} = e_mRe_n : m, n = 1, 2$, we may write $R = R_{11} \oplus R_{12} \oplus R_{21} \oplus R_{22}$. An element of the subring R_{mn} will be denoted by x_{mn}.

From the definition of g we note that $g(0) = g(00) = g(0)0 + 0d(0) = 0$, and also, $d(0) = 0$. Moreover, $d(e) = d(e^2) = d(e)e + ed(e)$. So we can express $d(e) = a_{11} + a_{12} + a_{21} + a_{22}$ and use the value of $d(e)$ to get that $a_{11} = a_{22}$, that is $a_{11} = 0 = a_{22}$. Consequently, we have

$$d(e) = a_{12} + a_{21}.$$

By the same manner $g(e) = g(e^2) = g(e)e + ed(e)$ and we can write $g(e) = b_{11} + b_{12} + b_{21} + b_{22}$ and using the value of $g(e)$ and $d(e)$ we get $b_{11} + b_{12} + b_{21} + b_{22} = b_{11} + b_{21} + a_{12}$, from this equality and since any element in a direct sum is written uniquely, we conclude that $b_{22} = 0$ and so,

$$g(e) = b_{11} + a_{12} + b_{21}.$$

In the sequel, and for simplifications, let f be the inner derivation of R determined by the element $a_{12} - a_{21}$, that is $f(x) = [x, a_{12} - a_{21}]$ for all x in R. Therefore,

$$f(e) = [e, a_{12} - a_{21}] = a_{12} + a_{21}.$$

Let $F(x) = (b_{11} + b_{21})x + x(a_{12} - a_{21})$ be the generalized inner derivation determined by the two elements $b_{11} + b_{21}$ and $a_{12} - a_{21}$, so we have,

$$F(e) = b_{11} + b_{21} + a_{12}$$

In the sequel, we will replace, without loss of generality, the derivation d by the derivation $D = d - f$ and the multiplicative generalized derivation g by the multiplicative generalized derivation $G = g - F$. This yields

$$D(e) = 0 \text{ and } G(e) = 0.$$

In our next proofs we will need the following lemma,

Lemma 1.1. *[2, Lemma 1] With the above notations, we have*

$$D(R_{mn}) \subset R_{mn}, m, n = 1, 2. \tag{1}$$

2 The main result.

We intend to prove the following

Theorem 2.1. *Let R be a ring containing an idempotent e which satisfies the following conditions,*

(T_1) $xRe = 0$ *implies* $x = 0$. (*and hence* $xR = 0$ *implies* $x = 0$.)

(T_2) $exeR(1-e) = 0$ *implies* $exe = 0$.

(T_3) $(1-e)xeR(1-e) = 0$ *implies* $(1-e)xe = 0$.

If g is any multiplicative generalized derivation of R, i.e. $g(xy) = g(x)y + xd(y)$, for all x, y in R and some derivation d of R, then g is additive.

Now we need several lemmas.

Lemma 2.2. $G(R_{1n}) \subset R_{1n}, n = 1, 2$; $G(R_{21}) \subset R_{11} + R_{21}$, $G(R_{11} + R_{21}) \subset R_{11} + R_{21}$ *and* $G(R_{22}) \subset R_{22} + R_{12}$. *Moreover, G is additive on R_{1n} and* $G(x_{11} + x_{12}) = G(x_{11}) + G(x_{12})$, *for every* $x_{11} \in R_{11}$ *and* $x_{12} \in R_{12}$.

Proof. Since $G(xy) = G(x)y + xD(y)$, for every $x, y \in R$, and it follows that for every $x_{1n} \in R_{1n}$, $n = 1, 2$, we have $G(x_{1n}) = G(ex_{1n}) = G(e)x_{1n} + eD(x_{1n}) = D(x_{1n})$, because $G(e) = 0$ and $D(R_{1n}) \subset R_{1n}$. So we have that $G \mid_{R_{1n}} = D \mid_{R_{1n}}$, $n = 1, 2$, and it follows that $G(R_{1n}) \subset R_{1n}$, $n = 1, 2$ and that G is additive on R_{1n}, $n = 1, 2$, since D is. Moreover, as a consequence, the same kind of arguments implies that if $x_{11} \in R_{11}$ and $x_{12} \in R_{12}$, then we have $G(x_{11}+x_{12}) = G(e(x_{11}+x_{12})) = G(e)(x_{11}+x_{12})+eD(x_{11}+x_{12}) = e[D(x_{11})+D(x_{12})] = D(x_{11}) + D(x_{12}) = G(x_{11}) + G(x_{12})$, by the above argument and Lemma 1.1, so we have $G(x_{11} + x_{12}) = G(x_{11}) + G(x_{12})$. Now let $x_{21} \in R_{21}$ and write $G(x_{21}) = a_{11} + a_{12} + a_{21} + a_{22}$, then $G(x_{21}) = G(x_{21}e) = G(x_{21})e = a_{11} + a_{21} \in R_{11} + R_{21}$, so $G(x_{21}) \in R_{11} + R_{21}$. If $y_{11} \in R_{11}$ and $y_{21} \in R_{21}$ then $G(y_{11}+y_{21}) = G[(y_{11}+y_{21})e] = G(y_{11}+y_{21})e+(y_{11}+y_{21})D(e) = G(y_{11}+y_{21})e \in R_{11} + R_{21}$. So, we get $G(R_{11} + R_{21}) \subset R_{11} + R_{21}$. Finally, let $x_{22} \in R_{22}$, write $G(x_{22}) = b_{11} + b_{12} + b_{21} + b_{22}$, then $0 = G(x_{22}e) = G(x_{22})e = b_{11} + b_{21}$, so $G(x_{22}) = b_{12} + b_{22} \in R_{12} + R_{22}$, so $G(x_{22}) \in R_{12} + R_{22}$ and the proof of the lemma is complete. ☐

Lemma 2.3. *For any $x_{11} \in R_{11}$, $z_{12} \in R_{12}$ and $x_{21} \in R_{21}$, we have*

$$G(x_{21} + x_{11}z_{12}) = G(x_{21}) + G(x_{11}z_{12}). \tag{2}$$

Proof. For any $u_{1n} \in R_{1n}$, where $n = 1, 2$, we have

$$[G(x_{21}) + G(x_{11}z_{12})]u_{1n} = G(x_{21})u_{1n} + G(x_{11}z_{12})u_{1n} = G(x_{21})u_{1n} =$$
$$G(x_{21}u_{1n}) - x_{21}D(u_{1n}) = G((x_{21} + x_{11}z_{12})u_{1n}) - x_{21}D(u_{1n}) =$$
$$G(x_{21}+x_{11}z_{12})u_{1n}+(x_{21}+x_{11}z_{12})D(u_{1n})-x_{21}D(u_{1n}) = G(x_{21}+x_{11}z_{12})u_{1n}.$$

So we have

$$\{G(x_{21} + x_{11}z_{12}) - G(x_{21}) - G(x_{11}z_{12})\}R_{1n} = (0). \tag{3}$$

Now, for any $u_{2n} \in R_{2n}$, where $n = 1, 2$, we have

$$[G(x_{21}) + G(x_{11}z_{12})]u_{2n} = G(x_{21})u_{2n} + G(x_{11}z_{12})u_{2n} = G(x_{11}z_{12})u_{2n} =$$
$$G(x_{11}z_{12}u_{2n}) - x_{11}z_{12}D(u_{2n}) = G((x_{21} + x_{11}z_{12})u_{2n}) - x_{11}z_{12}D(u_{2n}) =$$
$$G(x_{21}+x_{11}z_{12})u_{2n}+(x_{21}+x_{11}z_{12})D(u_{2n})-x_{11}z_{12}D(u_{2n}) = G(x_{21}+x_{11}z_{12})u_{2n}.$$

So we have

$$\{G(x_{21} + x_{11}z_{12}) - G(x_{21}) - G(x_{11}z_{12})\}R_{2n} = (0). \tag{4}$$

From equations (3) and (4) we get

$$\{G(x_{21} + x_{11}z_{12}) - G(x_{21}) - G(x_{11}z_{12})\}R = (0). \tag{5}$$

Using condition (T_1) in the above equation we get

$$G(x_{21} + x_{11}z_{12}) = G(x_{21}) + G(x_{11}z_{12}). \tag{6}$$

□

Lemma 2.4. *For any $x_{11} \in R_{11}$ and $x_{21} \in R_{21}$, we have*

$$G(x_{11} + x_{21}) = G(x_{11}) + G(x_{21}). \tag{7}$$

Proof. For any $u_{1n} \in R_{1n}$ and $z_{12} \in R_{12}$, where $n = 1, 2$, we have

$$\{G(x_{11} + x_{21}) - G(x_{11}) - G(x_{21})\}z_{12}u_{1n} = (0).$$

Which means

$$\{G(x_{11} + x_{21}) - G(x_{11}) - G(x_{21})\}z_{12}R_{1n} = (0). \tag{8}$$

Now, for any $u_{2n} \in R_{2n}$ and $z_{12} \in R_{12}$, where $n = 1, 2$, we have

$$G(x_{11}+x_{21})z_{12}u_{2n} = G((x_{11}+x_{21})z_{12}u_{2n}) - (x_{11}+x_{21})D(z_{12}u_{2n}) = G[(x_{11}z_{12}+x_{21})(u_{2n}+z_{12}u_{2n})] - (x_{11}+x_{21})D(z_{12}u_{2n}) = G(x_{11}z_{12}+x_{21})(u_{2n}+z_{12}u_{2n}) + (x_{11}z_{12}+x_{21})D(u_{2n}+z_{12}u_{2n}) - (x_{11}+x_{21})D(z_{12}u_{2n}) = G(x_{11}z_{12}+x_{21})(u_{2n}+z_{12}u_{2n}) - x_{11}D(z_{12})u_{2n} = G(x_{11}z_{12})(u_{2n}+z_{12}u_{2n}) + G(x_{21})(u_{2n}+z_{12}u_{2n}) - x_{11}D(z_{12})u_{2n} = G(x_{11}z_{12})u_{2n} + G(x_{21})z_{12}u_{2n} - x_{11}D(z_{12})u_{2n} = G(x_{11})z_{12}u_{2n} + G(x_{21})z_{12}u_{2n}.$$

So we have

$$\{G(x_{11}+x_{21}) - G(x_{11}) - G(x_{21})\}z_{12}u_{2n} = (0).$$

And so we get

$$\{G(x_{11}+x_{21}) - G(x_{11}) - G(x_{21})\}z_{12}R_{2n} = (0). \tag{9}$$

From equations (8) and (9) we get

$$\{G(x_{11}+x_{21}) - G(x_{11}) - G(x_{21})\}z_{12}R = (0). \tag{10}$$

Using condition (T_1) we have

$$\{G(x_{11}+x_{21}) - G(x_{11}) - G(x_{21})\}R_{12} = (0). \tag{11}$$

Using conditions (T_2), (T_3) and Lemma 2.2 we obtain

$$G(x_{11}+x_{21}) = G(x_{11}) + G(x_{21}). \tag{12}$$

□

Lemma 2.5. *For any $z_{12} \in R_{12}$ and x_{21}, $y_{21} \in R_{21}$, we have*

$$G(y_{21}+x_{21}z_{12}) = G(y_{21}) + G(x_{21}z_{12}). \tag{13}$$

Proof. For any $u_{1n} \in R_{1n}$, where $n = 1, 2$, we have

$$[G(y_{21}) + G(x_{21}z_{12})]u_{1n} = G(y_{21})u_{1n} + G(x_{21}z_{12})u_{1n} = G(y_{21})u_{1n} = G(y_{21}u_{1n}) - y_{21}D(u_{1n}) = G((y_{21}+x_{21}z_{12})u_{1n}) - y_{21}D(u_{1n}) = G(y_{21}+x_{21}z_{12})u_{1n} + (y_{21}+x_{21}z_{12})D(u_{1n}) - y_{21}D(u_{1n}) = G(y_{21}+x_{21}z_{12})u_{1n}.$$

So we have

$$\{G(y_{21}+x_{21}z_{12}) - G(y_{21}) - G(x_{21}z_{12})\}R_{1n} = (0). \tag{14}$$

Now, for any $u_{2n} \in R_{2n}$, where $n = 1, 2$, we have

$$[G(y_{21}) + G(x_{21}z_{12})]u_{2n} = G(y_{21})u_{2n} + G(x_{21}z_{12})u_{2n} = G(x_{21}z_{12})u_{2n} = G(x_{21}z_{12}u_{2n}) - x_{21}z_{12}D(u_{2n}) = G((y_{21}+x_{21}z_{12})u_{2n}) - x_{21}z_{12}D(u_{2n}) = G(y_{21}+x_{21}z_{12})u_{2n} + (y_{21}+x_{21}z_{12})D(u_{2n}) - x_{21}z_{12}D(u_{2n}) = G(y_{21}+x_{21}z_{12})u_{2n}.$$

So we have

$$\{G(y_{21} + x_{21}z_{12}) - G(y_{21}) - G(x_{21}z_{12})\}R_{2n} = (0). \tag{15}$$

From equations (14) and (15) we get

$$\{G(y_{21} + x_{21}z_{12}) - G(y_{21}) - G(x_{21}z_{12})\}R = (0). \tag{16}$$

Using condition (T_1) in the above equation we get

$$G(y_{21} + x_{21}z_{12}) = G(y_{21}) + G(x_{21}z_{12}). \tag{17}$$

□

Lemma 2.6. *G is additive on* R_{21}.

Proof. For any x_{21}, $y_{21} \in R_{21}$, $z_{12} \in R_{12}$ and $z_{2n} \in R_{2n}$ we have

$$G(x_{21} + y_{21})z_{12}z_{2n} = G((x_{21} + y_{21})z_{12}z_{2n}) - (x_{21} + y_{21})D(z_{12}z_{2n}) =$$
$$G(x_{21}z_{12}z_{2n} + y_{21}z_{12}z_{2n}) - (x_{21} + y_{21})D(z_{12}z_{2n}) =$$
$$G((x_{21}z_{12} + y_{21})(z_{2n} + z_{12}z_{2n})) - (x_{21} + y_{21})D(z_{12}z_{2n}) = G(x_{21}z_{12} +$$
$$y_{21})(z_{2n} + z_{12}z_{2n}) + (x_{21}z_{12} + y_{21})D(z_{2n} + z_{12}z_{2n}) - (x_{21} + y_{21})D(z_{12}z_{2n}) =$$
$$G(x_{21}z_{12} + y_{21})(z_{2n} + z_{12}z_{2n}) - x_{21}D(z_{12})z_{2n} =$$
$$G(x_{21}z_{12})z_{2n} + G(y_{21})z_{2n} + G(x_{21}z_{12})z_{12}z_{2n} + G(y_{21})z_{12}z_{2n} - x_{21}D(z_{12})z_{2n} =$$
$$G(x_{21}z_{12})z_{2n} + G(y_{21})z_{12}z_{2n} - x_{21}D(z_{12})z_{2n} = (G(x_{21}) + G(y_{21}))z_{12}z_{2n}.$$

So we have,

$$[G(x_{21} + y_{21}) - G(x_{21}) - G(y_{21})]R_{12}R_{2n} = (0) \tag{18}$$

Also, it is clear that

$$[G(x_{21} + y_{21}) - G(x_{21}) - G(y_{21})]R_{12}R_{1n} = (0) \tag{19}$$

where $n = 1, 2$. From (18) and (19) we get

$$[G(x_{21} + y_{21}) - G(x_{21}) - G(y_{21})]R_{12}R = (0) \tag{20}$$

By condition (T_1) we have

$$[G(x_{21} + y_{21}) - G(x_{21}) - G(y_{21})]R_{12} = (0) \tag{21}$$

Using conditions (T_2), (T_3) and Lemma 2.2 we get

$$G(x_{21} + y_{21}) = G(x_{21}) + G(y_{21}). \tag{22}$$

□

Lemma 2.7. *G is additive on* $R_{11} + R_{21} = Re$.

Proof. Consider the arbitrary elements x_{11}, y_{11} in R_{11} and x_{21}, y_{21} in R_{21}. So, Lemmas 2.2, 2.4 and 2.6 give $G((x_{11}+x_{21})+(y_{11}+y_{21})) = G((x_{11}+y_{11})+(x_{21}+y_{21})) = G(x_{11}+y_{11})+G(x_{21}+y_{21}) = G(x_{11})+G(y_{11})+G(x_{21})+G(y_{21}) = (G(x_{11})+G(x_{21}))+(G(y_{11})+G(y_{21})) = G(x_{11}+x_{21})+G(y_{11}+y_{21})$. Thus G is additive on $R_{11}+R_{21}$ which as required. □

Now we are in a position to prove the main theorem,

Proof of Theorem 2.1. Let x and y be any elements of R. Consider $G(x)+G(y)$. Take an element t in $Re = R_{11}+R_{21}$. Thus, xt and yt are elements of Re. According to Lemma 2.7, we can obtain $(G(x)+G(y))t = G(x)t+G(y)t = G(xt)+G(yt)-(x+y)D(t) = G(xt+yt)-(x+y)D(t) = G((x+y)t)-(x+y)D(t) = G(x+y)t+(x+y)D(t)-(x+y)D(t) = G(x+y)t$. Thus, $(G(x)+G(y))t = G(x+y)t$. Since t is an arbitrary element in Re, we obtain $(G(x)+G(y)-G(x+y))Re = 0$. By condition (T_1), we get

$$G(x+y) = G(x)+G(y).$$

Which shows that the multiplicative generalized derivation G, and also g, is additive. □

Acknowledgment

The authors thank the referee for his/her helpful suggestions. The second author would like to thank Profesor Miguel Ferrero for many helpful comments.

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