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MULTIHOMOMORPHISMS FROM $(\mathbb{Z}, +)$ INTO CERTAIN HYPERGROUPS

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Abstract

By a *multihomomorphism* from a hypergroup $(H, \circ)$ into a hypergroup $(H', \circ')$ we mean a multi-valued function f from H into H' such that $f(x \circ y) = f(x) \circ' f(y)$ for all $x, y \in H$ and f is called *surjective* if $f(H) = H'$. Denote by $\mathrm{MHom}((H, \circ), (H', \circ'))$ and $\mathrm{SMHom}((H, \circ), (H', \circ'))$ the set of all multihomomorphisms and the set of all surjective multihomomorphisms from $(H, \circ)$ into $(H', \circ')$, respectively. Characterizations of the elements of $\mathrm{MHom}((\mathbb{Z}, +), (\mathbb{Z}, +))$, $\mathrm{SMHom}((\mathbb{Z}, +), (\mathbb{Z}, +))$, $\mathrm{MHom}((\mathbb{Z}, \circ_n), (\mathbb{Z}, +))$ and $\mathrm{SMHom}((\mathbb{Z}, \circ_n), (\mathbb{Z}, +))$ have been given where n is a positive integer and $\circ_n$ is the hyperoperation on $\mathbb{Z}$ defined by $x \circ_n y = x + y + n\mathbb{Z}$. It has also been shown that $|\mathrm{MHom}((\mathbb{Z}, +), (\mathbb{Z}, +))| = \aleph_0 = |\mathrm{SMHom}((\mathbb{Z}, +), (\mathbb{Z}, +))|$ and $|\mathrm{MHom}((\mathbb{Z}, \circ_n), (\mathbb{Z}, +))| = 2^{\aleph_0} = |\mathrm{SMHom}((\mathbb{Z}, \circ_n), (\mathbb{Z}, +))|$. In this paper, characterizations of the elements of $\mathrm{MHom}((\mathbb{Z}, +), (\mathbb{Z}, \circ_n))$ and $\mathrm{SMHom}((\mathbb{Z}, +), (\mathbb{Z}, \circ_n))$ are provided. We also show that $|\mathrm{MHom}((\mathbb{Z}, +), (\mathbb{Z}, \circ_n))| = \sum_{\substack{k \in \mathbb{Z}^+ \\ k \mid n}} k$ and $|\mathrm{SMHom}((\mathbb{Z}, +), (\mathbb{Z}, \circ_n))| = n$.

1 Introduction

A *multi-valued function* from a nonempty set X into a nonempty set Y is a function $f : X \to \mathcal{P}(Y) \setminus \{\emptyset\}$ where $\mathcal{P}(Y)$ is the power set of Y, and for $A \subseteq X$, let $f(A) = \bigcup_{a \in A} f(a)$.

A *hyperoperation* $\circ$ on a nonempty set H is a function $\circ : H \times H \to \mathcal{P}(H) \setminus \{\emptyset\}$.

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The value of $(x,y)\in H\times H$ under $\circ$ is denoted by $x\circ y$. For $A,B\subseteq H$ and $x\in H$, let

$$A\circ B=\bigcup_{\substack{a\in A\\ b\in B}} a\circ b,\ \ x\circ A=\{x\}\circ A \text{ and } A\circ x=A\circ\{x\}.$$

The system $(H,\circ)$ is called a *hypergroup* if

$$x\circ(y\circ z)=(x\circ y)\circ z \text{ and } x\circ H=H=H\circ x \text{ for all } x,y,z\in H.$$

If N is a normal subgroup of a group G and $\circ_N$ is a hyperoperation on G defined by $x\circ_N y=xyN$ for all $x,y\in G$, then $(G,\circ_N)$ is a hypergroup ([1], page 11). It is clearly seen that for all $x_1,x_2,\dots,x_k\in G$ with $k>1$, $x_1\circ_N x_2\circ_N\cdots\circ_N x_k=x_1x_2\cdots x_kN$. Observe that if $N=\{e\}$ where e is the identity of G, then $(G,\circ_N)$ is the group G.

The cardinality of a set X is denoted by $|X|$.

Let $\mathbb{Z}$ be the set of integers, $\mathbb{Z}^+=\{x\in\mathbb{Z}\mid x>0\}$ and $\mathbb{Z}^-=\{x\in\mathbb{Z}\mid x<0\}$. For $a,b\in\mathbb{Z}$, not both 0, let (a,b) be the g.c.d. of a and b. It is clealy seen that $a\mathbb{Z}+b\mathbb{Z}=(a,b)\mathbb{Z}$. Recall that the Euler φ-function is defined by $\varphi(1)=1$ and for $k\in\mathbb{Z}^+$ with $k>1$, $\varphi(k)$ is the number of positive integers less than k and relatively prime to k. Then

$$\varphi(k)=|\{a\in\{1,2,\dots,k\}\mid (a,k)=1\}| \text{ for all } k\in\mathbb{Z}^+.$$

It is known that for $n\in\mathbb{Z}^+$, $\sum_{\substack{k\in\mathbb{Z}^+\\ k|n}}\varphi(k)=n$ ([3], page 191).

Let n be a positive integer and let $\circ_n$ stand for $\circ_{n\mathbb{Z}}$ in the group $(\mathbb{Z},+)$, that is, $(\mathbb{Z},\circ_n)$ is the hypergroup with the hyperoperation $\circ_n$ defined by

$$x\circ_n y=x+y+n\mathbb{Z} \text{ for all } x,y\in\mathbb{Z}.$$

A *multihomomorphism* f from a hypergroup $(H,\circ)$ into a hypergroup $(H',\circ')$ is a multi-valued function f from H into H' such that

$$f(x\circ y)=f(x)\circ' f(y)\ (=\bigcup_{\substack{s\in f(x)\\ t\in f(y)}} s\circ' t) \text{ for all } x,y\in H.$$

and f is called *surjective* if $f(H)=H'$. The set of all multihomomorphisms and the set of all surjective multihomomorphisms from $(H,\circ)$ into $(H',\circ')$ are denoted by $\mathrm{MHom}((H,\circ),(H',\circ'))$ and $\mathrm{SMHom}((H,\circ),(H',\circ'))$, respectively. Set $\mathrm{MHom}(H,\circ):=\mathrm{MHom}((H,\circ),(H,\circ))$ and $\mathrm{SMHom}(H,\circ):=\mathrm{SMHom}((H,\circ),(H,\circ))$.

In [5], the authors characterized the elements of $\mathrm{MHom}(\mathbb{Z},+)$ and determined $|\mathrm{MHom}(\mathbb{Z},+)|$:

Theorem 1.1. ([5]). *For a multi-valued function f from $\mathbb{Z}$ into itself, $f \in MHom(\mathbb{Z}, +)$ if and only if there exist a subsemigroup H of $(\mathbb{Z}, +)$ containing 0 and an element $a \in \mathbb{Z}$ such that*

$$f(x) = xa + H \text{ for all } x \in \mathbb{Z}.$$

Theorem 1.2. ([5]). $|MHom(\mathbb{Z}, +)| = \aleph_0$.

In [2], the authors used Theorem 1.1 to characterize the elements of SMHom($\mathbb{Z}$,+). Also, $|\text{SMHom}(\mathbb{Z}, +)|$ was determined.

Theorem 1.3. ([2]). *For a multi-valued function f from $\mathbb{Z}$ into itself, $f \in SMHom(\mathbb{Z}, +)$ if and only if there exist a subsemigroup H of $(\mathbb{Z}, +)$ containing 0 and $a \in \mathbb{Z}$ such that*

$$f(x) = xa + H \text{ for all } x \in \mathbb{Z},$$
$$(a, h) = 1 \text{ for some } h \in H \text{ and}$$
$$H = \mathbb{Z} \text{ whenever } a = 0.$$

Theorem 1.4. ([2]). $|SMHom(\mathbb{Z}, +)| = \aleph_0$.

We characterized the elements of $\text{MHom}((\mathbb{Z}, \circ_n), (\mathbb{Z}, +))$ and $\text{SMHom}((\mathbb{Z}, \circ_n), (\mathbb{Z}, +))$ in [4]. Also, the cardinalities of these sets were provided.

Theorem 1.5. ([4]). *For a multi-valued function f from $\mathbb{Z}$ into itself, $f \in MHom((\mathbb{Z}, \circ_n), (\mathbb{Z}, +))$ if and only if one of the following two conditions holds.*

(i) *There is a subsemigroup H of $(\mathbb{Z}, +)$ containing 0 such that*

$$f(x + n\mathbb{Z}) = H \quad \text{for all } x \in \mathbb{Z} \text{ and}$$
$$f(x) + f(y) = H \quad \text{for all } x, y \in \mathbb{Z}.$$

(ii) *There are $l, a \in \mathbb{Z}$ such that $l \neq 0$, $\frac{l}{(l, n)} \mid a$,*

$$f(x + n\mathbb{Z}) = xa + l\mathbb{Z} \quad \text{for all } x \in \mathbb{Z} \text{ and}$$
$$f(x) + f(y) = f(x) + f(y) + l\mathbb{Z} \quad \text{for all } x, y \in \mathbb{Z}.$$

Theorem 1.6. ([4]). *For a multi-valued function f from $\mathbb{Z}$ into itself, $f \in SMHom((\mathbb{Z}, \circ_n), (\mathbb{Z}, +))$ if and only if one of the following two conditions holds.*

(i)
$$f(x + n\mathbb{Z}) = \mathbb{Z} \quad \text{for all } x \in \mathbb{Z} \text{ and}$$
$$f(x) + f(y) = \mathbb{Z} \quad \text{for all } x, y \in \mathbb{Z}.$$

(ii) *There are $l, a \in \mathbb{Z}$ such that $l \neq 0$, $l \mid n$, $(a, l) = 1$,*

$$f(x + n\mathbb{Z}) = xa + l\mathbb{Z} \quad \text{for all } x \in \mathbb{Z} \text{ and}$$
$$f(x) + f(y) = f(x) + f(y) + l\mathbb{Z} \quad \text{for all } x, y \in \mathbb{Z}.$$

Also, it was shown in [4] that $\mathrm{MHom}((\mathbb{Z},\circ_n),(\mathbb{Z},+))$ and $\mathrm{SMHom}((\mathbb{Z},\circ_n),(\mathbb{Z},+))$ are uncountably infinite.

Theorem 1.7. ([4]). $|MHom((\mathbb{Z},\circ_n),(\mathbb{Z},+))| = |SMHom((\mathbb{Z},\circ_n),(\mathbb{Z},+))| = 2^{\aleph_0}$.

This paper is a continuation of the works mentioned above. We characterize the elements of $\mathrm{MHom}((\mathbb{Z},+)),(\mathbb{Z},\circ_n))$ and $\mathrm{SMHom}((\mathbb{Z},+)),(\mathbb{Z},\circ_n))$. It is also shown that these sets are finite. We show precisely from our characterizations that $|\mathrm{MHom}((\mathbb{Z},+)),(\mathbb{Z},\circ_n))| = \sum_{\substack{k\in\mathbb{Z}^+\\ k\mid n}} k$ and $|\mathrm{SMHom}((\mathbb{Z},+)),(\mathbb{Z},\circ_n))| = n$.

In the remainder of this paper, n is a positive integer and $\circ_n$ is the hyperoperation defined on $\mathbb{Z}$ as above.

2 Main Results

The following result was given in [5].

Lemma 2.1. ([5]). *If H is a subsemigroup of $(\mathbb{Z},+)$ such that $H\cap\mathbb{Z}^+\neq\emptyset$ and $H\cap\mathbb{Z}^-\neq\emptyset$, then $H = k\mathbb{Z}$ for some $k\in\mathbb{Z}\setminus\{0\}$.*

The following two lemmas are also needed.

Lemma 2.2. *Let G be a group with identity e. If $f\in MHom(G,(\mathbb{Z},\circ_n))$, then $f(e) = k\mathbb{Z}$ for some $k\in\mathbb{Z}\setminus\{0\}$ with $k\mid n$.*

Proof. Let $f\in\mathrm{MHom}(G,(\mathbb{Z},\circ_n))$. Then

$$\begin{aligned} f(e) = f(ee) = f(e)\circ_n f(e) &= f(e)+f(e)+n\mathbb{Z} \\ &\supseteq f(e)+f(e) \end{aligned}$$

which implies that $f(e)$ is a subsemigroup of $(\mathbb{Z},+)$. Let $a\in f(e)$. It is immediate from the above equalities that $a+a+n\mathbb{Z} = 2a+n\mathbb{Z}\subseteq f(e)$. It follows that $f(e)\cap\mathbb{Z}^+\neq\emptyset$ and $f(e)\cap\mathbb{Z}^-\neq\emptyset$. By Lemma 2.1, $f(e) = k\mathbb{Z}$ for some $k\in\mathbb{Z}\setminus\{0\}$. This implies that

$$k\mathbb{Z} = f(e) = f(e)+f(e)+n\mathbb{Z} = k\mathbb{Z}+k\mathbb{Z}+n\mathbb{Z} = k\mathbb{Z}+n\mathbb{Z} = (k,n)\mathbb{Z}.$$

Consequently, $k = \pm(k,n)$, so $k\mid n$. □

Lemma 2.3. *Let G be a group with identity e and $f\in MHom(G,(\mathbb{Z},\circ_n))$. Then for every $x\in G$, there exists $a\in f(x)$ such that*

$$f(x^m) = ma + f(e) \text{ for all } m\in\mathbb{Z}.$$

Proof. By Lemma 2.2, $f(e) = k\mathbb{Z}$ for some $k \in \mathbb{Z} \setminus \{0\}$ with $k \mid n$. Let $x \in G$ be given. Then

$$f(x) = f(xe) = f(x) + f(e) + n\mathbb{Z} = f(x) + k\mathbb{Z} + n\mathbb{Z} = f(x) + k\mathbb{Z}, \qquad (1)$$

$$\begin{aligned} f(x^{-1}) = f(x^{-1}e) = f(x^{-1}) + f(e) + n\mathbb{Z} &= f(x^{-1}) + k\mathbb{Z} + n\mathbb{Z} \\ &= f(x^{-1}) + k\mathbb{Z}. \qquad (2) \end{aligned}$$

Since $k\mathbb{Z} + n\mathbb{Z} = k\mathbb{Z}$, it follows from (1) and (2) that

$$f(x) + n\mathbb{Z} \quad = f(x) + k\mathbb{Z} = f(x), \qquad (3)$$

$$f(x^{-1}) + n\mathbb{Z} = f(x^{-1}) + k\mathbb{Z} = f(x^{-1}), \qquad (4)$$

respectively. It follows from (4) that

$$\begin{aligned} k\mathbb{Z} = f(e) = f(xx^{-1}) &= f(x) + f(x^{-1}) + n\mathbb{Z} \\ &= f(x) + (f(x^{-1}) + n\mathbb{Z}) = f(x) + f(x^{-1}), \end{aligned}$$

so $0 = a + b$ for some $a \in f(x)$ and $b \in f(x^{-1})$. Thus $b = -a \in f(x^{-1})$. Since

$$f(x) - a \subseteq f(x) + f(x^{-1}) = k\mathbb{Z}$$

and

$$a + f(x^{-1}) \subseteq f(x) + f(x^{-1}) = k\mathbb{Z},$$

it follows that

$$f(x) \subseteq a + k\mathbb{Z} \text{ and } f(x^{-1}) \subseteq -a + k\mathbb{Z}. \qquad (5)$$

From (1), (2) and (5), we have

$$f(x) \subseteq a + k\mathbb{Z} \subseteq f(x) + k\mathbb{Z} = f(x)$$

$$f(x^{-1}) \subseteq -a + k\mathbb{Z} \subseteq f(x^{-1}) + k\mathbb{Z} = f(x^{-1}).$$

Hence

$$f(x) = a + k\mathbb{Z} \text{ and } f(x^{-1}) = -a + k\mathbb{Z}. \qquad (6)$$

Note that $f(x^0) = f(e) = 0a + f(e)$. If $m \in \mathbb{Z}^+$ and $m > 1$, then

$$\begin{aligned} f(x^m) &= f(x) \circ_n f(x) \circ_n \cdots \circ_n f(x) && (m \text{ copies}) \\ &= \underbrace{f(x) + \cdots + f(x)}_{m \text{ copies}} + n\mathbb{Z} \\ &= (f(x) + n\mathbb{Z}) + \cdots + (f(x) + n\mathbb{Z}) && (m \text{ copies}) \\ &= f(x) + \cdots + f(x) && \text{from (3)} \\ &= (a + k\mathbb{Z}) + \cdots + (a + k\mathbb{Z}) && \text{from (6)} \\ &= ma + k\mathbb{Z} \\ &= ma + f(e) \end{aligned}$$

and

$$\begin{aligned}
f(x^{-m}) &= f(x^{-1}) \circ_n f(x^{-1}) \circ_n \cdots \circ_n f(x^{-1}) && (m \text{ copies})\\
&= \underbrace{f(x^{-1}) + \cdots + f(x^{-1})}_{m \text{ copies}} + n\mathbb{Z}\\
&= (f(x^{-1}) + n\mathbb{Z}) + \cdots + (f(x^{-1}) + n\mathbb{Z}) && (m \text{ copies})\\
&= f(x^{-1}) + \cdots + f(x^{-1}) && \text{from (4)}\\
&= (-a + k\mathbb{Z}) + \cdots + (-a + k\mathbb{Z}) && \text{from (6)}\\
&= -ma + k\mathbb{Z}\\
&= -ma + f(e).
\end{aligned}$$

Therefore the proof is complete. □

Theorem 2.4. *For a multi-valued function f from $\mathbb{Z}$ into itself, $f \in MHom((\mathbb{Z},+),(\mathbb{Z},\circ_n))$ if and only if there are $a,k \in \mathbb{Z}$ such that $k \neq 0$, $k \mid n$ and*

$$f(x) = xa + k\mathbb{Z} \quad \textit{for all } x \in \mathbb{Z}.$$

Proof. If $f \in \mathrm{MHom}((\mathbb{Z},+),(\mathbb{Z},\circ_n))$, then by Lemma 2.3, there exists $a \in f(1)$ such that

$$f(x) = xa + f(0) \text{ for all } x \in \mathbb{Z}.$$

By Lemma 2.2, $f(0) = k\mathbb{Z}$ for some $k \in \mathbb{Z} \setminus \{0\}$ with $k \mid n$. Hence

$$f(x) = xa + k\mathbb{Z} \text{ for all } x \in \mathbb{Z}.$$

For the converse, assume that there are $a,k \in \mathbb{Z}$ such that $k \neq 0$, $k \mid n$ and

$$f(x) = xa + k\mathbb{Z} \text{ for all } x \in \mathbb{Z}.$$

Since $k \mid n$, we have $k\mathbb{Z} + n\mathbb{Z} = k\mathbb{Z}$. If x, $y \in \mathbb{Z}$, then

$$\begin{aligned}
f(x+y) &= (x+y)a + k\mathbb{Z}\\
&= xa + ya + k\mathbb{Z}\\
&= xa + ya + k\mathbb{Z} + n\mathbb{Z}\\
&= (xa + k\mathbb{Z}) + (ya + k\mathbb{Z}) + n\mathbb{Z}\\
&= f(x) + f(y) + n\mathbb{Z}\\
&= f(x) \circ_n f(y).
\end{aligned}$$

This implies that $f \in \mathrm{MHom}((\mathbb{Z},+),(\mathbb{Z},\circ_n))$. □

Theorem 2.5. *For a multi-valued function f from $\mathbb{Z}$ into itself, $f \in$ SMHom $((\mathbb{Z},+),(\mathbb{Z},\circ_n))$ if and only if there are $a,k \in \mathbb{Z}$ such that $k \neq 0$, $k \mid n$, $(a,k)=1$ and*

$$f(x) = xa + k\mathbb{Z} \text{ for all } x \in \mathbb{Z}.$$

Proof. Assume that $f \in \mathrm{SMHom}((\mathbb{Z},+),(\mathbb{Z},\circ_n))$. Then $f \in \mathrm{MHom}((\mathbb{Z},+),(\mathbb{Z},\circ_n))$ and $f(\mathbb{Z}) = \mathbb{Z}$. From Theorem 2.4, there are $a,k \in \mathbb{Z}$ such that $k \neq 0$, $k \mid n$ and

$$f(x) = xa + k\mathbb{Z} \text{ for all } x \in \mathbb{Z}.$$

Since $f(\mathbb{Z}) = \mathbb{Z}$, it follows that

$$\mathbb{Z} = f(\mathbb{Z}) = a\mathbb{Z} + k\mathbb{Z} = (a,k)\mathbb{Z}.$$

This implies that $(a,k)=1$.

The converse is obtained directly from Theorem 2.4 and the fact that $(a,k)=1$ implies $f(\mathbb{Z}) = a\mathbb{Z} + k\mathbb{Z} = (a,k)\mathbb{Z} = \mathbb{Z}$. □

For $k \in \mathbb{Z}\setminus\{0\}$ with $k \mid n$ and $a \in \mathbb{Z}$, let $F_{k,a} \in \mathrm{MHom}((\mathbb{Z},+),(\mathbb{Z},\circ_n))$ be defined by

$$F_{k,a}(x) = xa + k\mathbb{Z} \text{ for all } x \in \mathbb{Z}.$$

To determine the number of the elements in $\mathrm{MHom}((\mathbb{Z},+),(\mathbb{Z},\circ_n))$ and SMHom $((\mathbb{Z},+),(\mathbb{Z},\circ_n))$, the following lemma is needed.

Lemma 2.6. *Let $k,\, l \in \mathbb{Z}\setminus\{0\}$ with $k \mid n$ and $l \mid n$ and $a,\, b \in \mathbb{Z}$. Then $F_{k,a} = F_{l,b}$ if and only if $l = \pm k$ and $b \equiv a \mod |k|$.*

Proof. Assume that $F_{k,a} = F_{l,b}$. Then

$$xa + k\mathbb{Z} = F_{k,a}(x) = F_{l,b}(x) = xb + l\mathbb{Z} \text{ for all } x \in \mathbb{Z}.$$

In particular, $k\mathbb{Z} = 0a + k\mathbb{Z} = 0b + l\mathbb{Z} = l\mathbb{Z}$ which implies that $l = \pm k$. Thus $k\mathbb{Z} = l\mathbb{Z} = |k|\mathbb{Z}$. Hence

$$a + |k|\mathbb{Z} = 1a + k\mathbb{Z} = 1b + l\mathbb{Z} = b + |k|\mathbb{Z},$$

so $b - a \in |k|\mathbb{Z}$. Hence $b \equiv a \mod |k|$.

Conversely, assume that $l = \pm k$ and $b \equiv a \mod |k|$. Then $l\mathbb{Z} = k\mathbb{Z}$ and $b - a \in |k|\mathbb{Z}$, so

$$\text{for all } x \in \mathbb{Z}, xb - xa = x(b-a) \in x(|k|\mathbb{Z}) \subseteq |k|\mathbb{Z} = k\mathbb{Z}.$$

It follows that

$$\text{for all } x \in \mathbb{Z}, F_{k,a}(x) = xa + k\mathbb{Z} = xb + k\mathbb{Z} = xb + l\mathbb{Z} = F_{l,a}(x).$$

Therefore we have $F_{k,a} = F_{l,b}$. □

Theorem 2.7. $|MHom((\mathbb{Z},+),(\mathbb{Z},\circ_n))| = \sum_{\substack{k\in\mathbb{Z}^+\\ k|n}} k$ *and* $|SMHom((\mathbb{Z},+),(\mathbb{Z},\circ_n))| = n.$

Proof. From Theorem 2.4 and Theorem 2.5, we have

$$\mathrm{MHom}((\mathbb{Z},+),(\mathbb{Z},\circ_n)) = \{F_{k,a} \mid k \in \mathbb{Z}\setminus\{0\},\ a \in \mathbb{Z} \text{ and } k \mid n\} \tag{1}$$

and

$$\mathrm{SMHom}((\mathbb{Z},+),(\mathbb{Z},\circ_n)) = \{F_{k,a} \mid k \in \mathbb{Z}\setminus\{0\},\ a \in \mathbb{Z},\ k \mid n \text{ and } (a,k)=1\} \tag{2}$$

respectively. Thus (1), (2) and Lemma 2.6 yield the following equalities

$$\mathrm{MHom}((\mathbb{Z},+),(\mathbb{Z},\circ_n)) = \{F_{k,a} \mid k \in \mathbb{Z}^+,\ k \mid n \text{ and } a \in \{0,1,\ldots,k-1\}\} \tag{3}$$

and

$$\mathrm{SMHom}((\mathbb{Z},+),(\mathbb{Z},\circ_n)) = \{F_{k,a} \mid k \in \mathbb{Z}^+,\ k \mid n,\ a \in \{0,1,\ldots,k-1\} \text{ and } (a,k)=1\}. \tag{4}$$

By (3), (4) and Lemma 2.6, we have

$$|\mathrm{MHom}((\mathbb{Z},+),(\mathbb{Z},\circ_n))| = \sum_{\substack{k\in\mathbb{Z}^+\\ k\mid n}} k,$$

$$|\mathrm{SMHom}((\mathbb{Z},+),(\mathbb{Z},\circ_n))| = \sum_{\substack{k\in\mathbb{Z}^+\\ k\mid n}} \varphi(k) = n.$$

□

Example. If p is a prime and $m \in \mathbb{Z}^+$, then by Theorem 2.7,

$$|\mathrm{MHom}((\mathbb{Z},+),(\mathbb{Z},\circ_{p^m}))| = 1 + p + \cdots + p^m = \frac{p^{m+1}-1}{p-1},$$
$$|\mathrm{SMHom}((\mathbb{Z},+),(\mathbb{Z},\circ_{p^m}))| = p^m.$$

It follows that the number of nonsurjective multihomomorphisms from $(\mathbb{Z},+)$ into $(\mathbb{Z},\circ_{p^m})$ is $1+p+\cdots+p^{m-1} (= \frac{p^m-1}{p-1})$.

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