East-West J. of Mathematics: Vol. 10, No 2 (2008) pp. 133-139

SKEW POLYNOMIAL RINGS OVER 2-PRIMAL NOETHERIAN RINGS

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Abstract

Let R be a ring and σ an automorphism of R and δ a σ-derivation of R. We say that R is a δ-ring if $a\delta(a) \in P(R)$ implies $a \in P(R)$, where $P(R)$ is the prime radical of R.
We prove that $R[x;\sigma,\delta]$ is a 2-primal Noetherian ring if R is a Noetherian ring, which moreover an algebra over the field of rational numbers, σ and δ are such that R is a δ-ring and $\sigma(P) = P$, P being any minimal prime ideal of R. We use this to prove that if R is a Noetherian $\sigma(*)$-ring (i.e. $a\sigma(a) \in P(R)$ implies $a \in P(R)$), δ a σ-derivation of R such that R is a δ-ring, then $R[x;\sigma,\delta]$ is a 2-primal Noetherian ring.

1 Introduction

We begin with the following question:

Question (2) of Bhat [4]: If R is Noetherian ring, which is also an algebra over the field of rational numbers, σ an automorphism of R and δ a σ-derivation of R. Is $R[x;\sigma,\delta]$ 2-primal?

In this paper an affirmative answer to this quetion is given in case R is a δ-ring.

We follow the notation as in Bhat [4], but to make the paper self contained, we have the following:

A ring R always means an associative ring. The field of rational numbers is denoted by $\mathbb{Q}$. The set of prime ideals and the set of minimal prime ideals of R are denoted by $Spec(R)$ and $MinSpec(R)$ respectively. $P(R)$ and $N(R)$

Key words: 2-primal, Minimal prime, prime radical, nil radical, automorphism, derivation.
2000 AMS Mathematics Subject Classification: Primary 16XX, Secondary 16S36, 16N40, 16P40, 16W20, 16W25.

denote the prime radical and the set of nilpotent elements of R, respectively. Let I and J be any two ideals of a ring R. Then $I \subset J$ means that I is strictly contained in J. Let I be an ideal of a ring R such that $\sigma^m(I) = I$ for some integer $m \geq 1$, we denote $\cap_{i=1}^{m}\sigma^i(I)$ by I^0.

This article concerns the study of skew polynomial rings in terms of 2-primal rings. 2-primal rings have been studied in recent years and the 2-primal property is being studied for various types of rings. In [15], G. Marks discusses the 2-primal property of $R[x;\sigma,\delta]$, where R is a local ring, σ is an automorphism of R and δ is a σ-derivation of R.

Recall that a σ-derivation of R is an additive map $\delta : R \to R$ such that $\delta(ab) = \delta(a)\sigma(b) + a\delta(b)$, for all $a, b \in R$. In case σ is the identity map, δ is called just a derivation of R. For example for any endomorphism τ of a ring R and for any $a \in R$, $\varrho : R \to R$ defined as $\varrho(r) = ra - a\tau(r)$ is a τ-derivation of R.

Let σ be an endomorphism of a ring R and $\delta : R \to R$ any map.
Let $\phi : R \to M_2(R)$ be a homomorphism defined by

$$\phi(r) = \begin{pmatrix} \sigma(r) & 0 \\ \delta(r) & r \end{pmatrix}, \text{ for all } r \in R.$$

Then δ is a σ-derivation of R.

Also let $R = K[x]$, K a field. Then the formal derivative $\frac{d}{dx}$ is a derivation of R.

Minimal prime ideals of 2-primal rings have been discussed by Kim and Kwak in [12] and Shin in [17]. 2-primal near rings have been discussed by Argac and Groenewald in [1]. Recall that a ring R is called 2-primal if the set of nilpotent elements of R coincides with the prime radical of R (G. Marks [15]), or equivalently if its radical contains every nilpotent element of R, or if P(R) is a completely semiprime ideal of R. An ideal I of a ring R is called completely semiprime if $a^2 \in I$ implies $a \in I$ for $a \in R$.

We also note that a reduced ring (i.e. a ring with no nonzero nilpotent elements) is 2-primal and a commutative ring is also 2-primal. For further details on 2-primal rings, we refer the reader to [3, 4, 12, 14, 15].

Recall that $R[x;\sigma,\delta]$ is the skew polynomial ring with coefficients in R in which multiplication is subject to the relation $ax = x\sigma(a) + \delta(a)$ for all $a \in R$. We denote $R[x;\sigma,\delta]$ by $O(R)$. In case σ is the identity map, we denote the ring of differential operators $R[x;\delta]$ by $D(R)$, if δ is the zero map, we denote $R[x;\sigma]$ by $S(R)$.

Recall that in Krempa [13], a ring R is called σ-rigid if there exists an endomorphism σ of R with the property that $a\sigma(a) = 0$ implies a = 0 for $a \in R$. In [14], Kwak defines a $\sigma(*)$-ring R to be a ring if $a\sigma(a) \in P(R)$ implies $a \in P(R)$ for $a \in R$ and establishes a relation between a 2-primal ring and a $\sigma(*)$-ring. The property is also extended to $S(R)$.

Example 1.1 *Let* $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$*, where F is a field. Then P(R)* $= \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$ *Let* $\sigma : R \to R$ *be defined by* $\sigma\Big(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\Big) = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}$*. Then it can be seen that* σ *is an endomorphism of R and R is a* $\sigma(*)$*-ring.*

We note that if R is a ring and σ an automorphism of R such that R is a $\sigma(*)$-ring, then R is 2-primal. For let $a \in R$ be such that $a^2 \in P(R)$. Then $a\sigma(a)\sigma(a\sigma(a)) = a\sigma(a)\sigma(a)\sigma^2(a) \in \sigma(P(R)) = P(R)$. Therefore $a\sigma(a) \in P(R)$ and so $a \in P(R)$. Thus $P(R)$ is a completely semiprime ideal of R and hence R is 2-primal.

In Theorem (12) of [14], Kwak has proved that if R is a $\sigma(*)$-ring such that $\sigma(P(R)) = P(R)$, then $R[x;\sigma]$ is 2-primal if and only if $P(R)[x;\sigma] = P(R[x;\sigma])$.

Hong, Kim and Kwak have proved in Corollary (2.8) of [11] that if R is a 2-primal ring and every simple singular left R-module is p-injective, then every prime ideal of R is maximal. In particular, every prime factor ring of R is a simple domain.

It is known (Theorem (1.2) of Bhat [3]) that if R is 2-primal Noetherian $\mathbb{Q}$-algebra and δ is a derivation of R, then $D(R)$ is 2-primal. We also note that if R is a Noetherian ring, then even $R[x]$ need not be 2-primal.

Example 1.2 *Let* $R = M_2(\mathbb{Q})$*, the set of* 2×2 *matrices over* $\mathbb{Q}$*. Then* $R[x]$ *is a prime ring with non-zero nilpotent elements and, so can not be 2-primal.*

Let now R be a 2-primal ring. Is $O(R)$ also a 2-primal ring? This question was attacked by the author and towards this the following has been proved in Bhat [4]:

Let R be a ring, σ be an automorphism of R and δ be a σ-derivation of R. We say that R is a δ-ring if $a\delta(a) \in P(R)$ implies $a \in P(R)$. We note that a ring with identity is not a δ-ring. Then:

1. (Theorem 2 of Bhat [4]): Let R be a 2-primal Noetherian ring. Then $S(R)$ is 2-primal Noetherian.

2. (Theorem 6 of Bhat [4]): Let R be a Noetherian $\mathbb{Q}$-algebra. Let σ be an automorphism of R and δ a σ-derivation of R such that R is a δ-ring, $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$; $\sigma(P) = P$ for all $P \in MinSpec(R)$ and $\delta(P(R)) \subseteq P(R)$. Then $O(R)$ is 2-primal Noetherian.

3. (Theorem 7 of Bhat [4]): Let R be a Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ be a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$ and R is a δ-ring. Then $R[x;\sigma,\delta]$ is 2-primal Noetherian.

In this paper we prove (2) and (3) above even without the condition that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$. These results are proved in Theorems (2.10) and (2.12) respectively.

Ore-extensions including skew-polynomial rings and differential operator rings have been of interest to many authors. See [2, 4, 5, 6, 7, 10, 13, 14, 15].

2 Skew polynomial ring $O(R)$

Recall that an ideal I of a ring R is called σ-invariant if $\sigma(I) = I$. Also I is called completely prime if $ab \in I$ implies $a \in I$ or $b \in I$ for $a, b \in R$. We also note that in a right Noetherian ring R, $MinSpec(R)$ is finite (Theorem (2.4) of Goodearl and Warfield [9]), and for any $P \in MinSpec(R)$, $\sigma^t(P) \in MinSpec(R)$ for all integers $t \geq 1$. Let $MinSpec(R) = \{P_1, P_2, ..., P_n\}$. Let $\sigma^{m_i}(P_i) = P_i$, for some positive integers m_i, $1 \leq i \leq n$, and $u = m_1.m_2...m_n$. Then $\sigma^u(P_i) = P_i$ for all $P_i \in MinSpec(R)$. We use same u henceforth, and as mentioned in introduction above, we denote $\cap_{i=1}^{u}\sigma^i(P)$ by P^0, P being any minimal prime ideal of R.

Definition 2.1 *Let R be a ring. Let σ be an automorphism of R and δ a σ-derivation of R. We say that R is a δ-ring if $a\,\delta(a) \in P(R)$ implies $a \in P(R)$.*

Recall that an ideal I of a ring R is called δ-invariant if $\delta(I) \subseteq I$. If an ideal I of R is σ-invariant and δ-invariant, then $O(I)$ is an ideal of $O(R)$ as for any $a \in I$, $\sigma^j(a) \in I$ and $\delta^j(a) \in I$ for all positive integers j.

Gabriel proved in Lemma (3.4) of [8] that if R is a Noetherian $\mathbb{Q}$-algebra and δ is a derivation of R, then $\delta(P) \subseteq P$, for all $P \in MinSpec(R)$. The author generalized this for a σ-derivation δ of R in [4] and proved the following:

Theorem 2.2 *(***Theorem 3 of Bhat [4]):** *Let R be a Noetherian $\mathbb{Q}$-algebra. Let σ be an automorphism of R and δ a σ-derivation of R such that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$. Then:*

1. *$P_1 \in MinSpec(R)$ such that $\sigma(P_1) = P_1$ implies $O(P_1) \in MinSpec(O(R))$.*
2. *$P \in MinSpec(O(R))$ such that $\sigma(P \cap R) = P \cap R$ implies $P \cap R \in MinSpec(R)$.*

We now prove the above result without the condition that $\sigma(\delta(a)) = \delta(\sigma(a))$, for all $a \in R$. Towards this we have the following:

Theorem 2.3 *Let R be a Noetherian $\mathbb{Q}$-algebra. Let σ be an automorphism of R and δ a σ-derivation of R. Then:*

1. *$P_1 \in MinSpec(R)$ such that $\sigma(P_1) = P_1$ implies $O(P_1) \in MinSpec(O(R))$.*
2. *$P \in MinSpec(O(R))$ such that $\sigma(P \cap R) = P \cap R$ implies $P \cap R \in MinSpec(R)$.*

Proof **(1)** Let $P_1 \in MinSpec(R)$ with $\sigma(P_1) = P_1$. Let $T = \{a \in P_1$ such that $\delta^k(a) \in P_1$, for all positive integers $k\}$. The it can be seen that $T \in Spec(R)$. Also $\delta(T) \subseteq T$. Now $T \subseteq P_1$, and P_1 being a minimal prime ideal of R implies that $T = P_1$. Hence $\delta(P_1) \subseteq P_1$.
Now on the same lines as in Theorem (2.22) of Goodearl and Warfield [9], it can be easily seen that $O(P_1) \in Spec(O(R))$. Suppose that $O(P_1) \notin MinSpec(O(R))$, and $P_2 \subset O(P_1)$ is a minimal prime ideal of $O(R)$. Then we have $P_2 = O(P_2 \cap R) \subset O(P_1) \in MinSpec(O(R))$. Therefore $P_2 \cap R \subset P_1$, which is a contradiction as $P_2 \cap R \in Spec(R)$. Hence $O(P_1) \in MinSpec(O(R))$.

(2) Let $P \in MinSpec(O(R))$ with $\sigma(P \cap R) = P \cap R$. Then on the same lines as in Theorem (2.22) of Goodearl and Warfield [9], it can be seen that $P \cap R \in Spec(R)$ and $O(P \cap R) \in Spec(O(R))$. Therefore $O(P \cap R) = P$. We now show that $P \cap R \in MinSpec(R)$. Suppose that $U \subset P \cap R$, and $U \in MinSpec(R)$. Then $O(U) \subset O(P \cap R) = P$. But $O(U) \in Spec(O(R))$ and, $O(U) \subset P$, which is not possible. Thus we have $P \cap R \in MinSpec(R)$. □

Recall that in Proposition (1.11) of Shin [17], it has been proved that a ring R is 2-primal if and only if each minimal prime ideal of R is a completely prime ideal.

Proposition 2.4 *Let R be a 2-primal ring. Let σ be an automorphism of R and δ a σ-derivation of R such that $\delta(P(R)) \subseteq P(R)$. If $P \in MinSpec(R)$ is such that $\sigma(P) = P$, then $\delta(P) \subseteq P$.*

Proof See Proposition (3) of Bhat [4]. □

Theorem 2.5 *Let R be a ring. Let σ be an automorphism of R and δ a σ-derivation of R such that R is a δ-ring and $\delta(P(R)) \subseteq P(R)$. Then R is 2-primal.*

Proof See Theorem (4) of Bhat [4] □

Proposition 2.6 *Let R be a ring. Let σ be an automorphism of R and δ a σ-derivation of R. Then:*

1. *For any completely prime ideal P of R with $\sigma(P) = P$ and $\delta(P) \subseteq P$, O(P) is a completely prime ideal of O(R).*
2. *For any completely prime ideal U of O(R), $U \cap R$ is a completely prime ideal of R.*

Proof See Proposition (4) of Bhat [4] □

Corollary 2.7 *Let R be a ring and σ an automorphism of R. Then:*

1. *For any completely prime ideal P of R with $\sigma(P) = P$, $S(P)$ is a completely prime ideal of $S(R)$.*

2. *For any completely prime ideal U of $S(R)$, $U \cap R$ is a completely prime ideal of R.*

Corollary 2.8 *Let R be a ring, σ an automorphism of R and δ a σ-derivation of R such that R is moreover a δ-ring and $\delta(P(R)) \subseteq P(R)$. Let $P \in MinSpec(R)$ be such that $\sigma(P) = P$. Then $O(P)$ is a completely prime ideal of $O(R)$.*

Theorem 2.9 *Let R be a ring, σ an automorphism of R and δ a σ-derivation of R such that R is a δ-ring and $\delta(P(R)) \subseteq P(R)$ and $\sigma(P) = P$ for all $P \in MinSpec(R)$. Then $O(R)$ is 2-primal if and only if $O(P(R)) = P(O(R))$.*

Proof See Theorem (5) of Bhat [4] □

Theorem 2.10 *Let R be a Noetherian $\mathbb{Q}$-algebra, σ an automorphism of R and δ a σ-derivation of R such that R is a δ-ring, $\sigma(P) = P$ for all $P \in MinSpec(R)$ and $\delta(P(R)) \subseteq P(R)$. Then $O(R)$ is 2-primal.*

Proof Let $P_1 \in MinSpec(R)$. Then it is given that $\sigma(P_1) = P_1$, and therefore Theorem (2.3) implies that $O(P_1) \in MinSpec(O(R))$. Similarly for any $P \in MinSpec(O(R))$ such that $\sigma(P \cap R) = P \cap R$ Theorem (2.3) implies that $P \cap R \in MinSpec(R)$. Therefore, $O(P(R)) = P(O(R))$, and now the result is obvious by using Theorem (2.9). □

Corollary 2.11 *Let R be a Noetherian $\mathbb{Q}$-algebra, σ an automorphism of R and δ a σ-derivation of R such that R is a δ-ring and $\sigma(P) = P$ for all $P \in MinSpec(R)$. Then $O(R)$ is 2-primal.*

Proof Let $P_1 \in MinSpec(R)$ with $\sigma(P_1) = P_1$. Then as in the proof of Theorem (2.3) $\delta(P_1) \subseteq P_1$, and therefore $\delta(P(R)) \subseteq P(R)$. Now the rest is obvious using Theorem (2.10). □

Theorem 2.12 *Let R be a Noetherian ring, which is also an algebra over $\mathbb{Q}$. Let σ be an automorphism of R such that R is a $\sigma(*)$-ring and δ be a σ-derivation of R such that R is a δ-ring. Then $R[x; \sigma, \delta]$ is 2-primal Noetherian.*

Proof We show that $\sigma(U) = U$ for all $U \in MinSpec(R)$. Suppose $U = U_1$ is a minimal prime ideal of R such that $\sigma(U) \neq U$. Let $U_2, U_3, ..., U_n$ be the other minimal primes of R. Now $\sigma(U)$ is also a minimal prime ideal of R. Renumber so that $\sigma(U) = U_n$. Let $a \in \cap_{i=1}^{n-1} U_i$. Then $\sigma(a) \in U_n$, and so $a\sigma(a) \in \cap_{i=1}^{n} U_i = P(R)$. Therefore $a \in P(R)$, and thus $\cap_{i=1}^{n-1} U_i \subseteq U_n$, which implies that $U_i \subseteq U_n$ for some $i \neq n$, which is impossible. Hence $\sigma(U) = U$. Now the rest is obvious. □

We now have the following question:

Question 2.13 *If R is a Noetherian $\mathbb{Q}$-algebra (even commutative), σ is an automorphism of R and δ is a σ-derivation of R. Is $O(R)$ 2-primal?*

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