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FIXED POINT THEOREM FOR COUNTABLE FAMILY OF MAPS THAT SATISFY A GENERAL CONTRACTIVE CONDITION DEPENDENT ON ANOTHER FUNCTION

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Abstract

In this paper, we prove the fixed theorem for a countable family of maps that satisfy a general contractive condition dependent on another function.

1 Introduction

Let (X, d) be a complete metric space and let $\mathcal{F} = \{\mathcal{T}_\alpha : \alpha \in \mathcal{I}\}$ be a family of maps which map X into itself. A point $u \in X$ is a common fixed point of $\mathcal{F}$ iff $u = T_\alpha(u)$ for each $T_\alpha \in \mathcal{F}$. In [3], Ćirić proved the following result.

Theorem 1.1. (Ćirić) *Let (X, d) be a complete metric space and let $\{S_n : n = 0, 1, 2, ...\}$ be a sequence of maps which map X into itself. If for some $q \in (0, 1)$*

$$d(S_0x, S_ny) \leqslant q \max\Big\{d(x, y), d(x, S_0x), d(y, S_ny), \frac{1}{2}\big(d(x, S_ny) + d(y, S_0x)\big)\Big\}$$

holds for each $n = 1, 2, ...$ and all $x, y \in X$, then $\{S_n : n = 0, 1, 2, ..\}$ has a unique fixed point.

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Applying above theorem for $\mathcal{F}$ is a singleton, we can get the following corollary.

Corollary 1.2. ([2]) *Let S be a X complete space and let $S : X \to X$ be a map. If for some $q \in (0,1)$*

$$d(Sx, Sy) \leqslant q\max\{d(x,y), d(x,Sx), d(y,Sy), \frac{1}{2}\big(d(x,Sy) + d(Sx,y)\big)\} \qquad (1)$$

holds for every $x, y \in X$, then S has a unique fixed point.

Recently, A. Beiranvand, S. Moradi,... (see[1]) have provided the result on the existence of fixed points for new contractive mappings. We recall some concepts.

Definition 1.3. ([1]) Let (X,d) be a metric and $T, S : X \to X$ be two functions. A mapping S is called to be a *T-contraction* if there exists $q \in (0,1)$ such that

$$d(TSx, TSy) \leqslant qd(Tx, Ty), \ \forall x, y \in X.$$

Clearly, if we choose $Tx = x$ for all $x \in X$ then *T-contraction* mapping becomes to a contraction. Note that, one can give an example which states that the map S is a $T-$contraction but T is not a contraction (see[1]). We recall the concept of generalized contraction maps.

Definition 1.4. ([2],[3]) Let (X,d) be a metric and $S : X \to X$ be a function. A mapping S is said to be a *generalized contraction* if there exists $q \in (0,1)$ such that

$$d(Sx, Sy) \leqslant q\max\{d(x,y), d(x,Sx), d(y,Sy), \frac{1}{2}\big(d(x,Sy) + d(Sx,y)\big)\}, \ \forall x, y \in X.$$

In [2], authors give an example that states that the map S is a generalized contraction, but S is not a contraction. By the ideas of combining the Definition 1.3 and Definition 1.4, we have the following concept.

Definition 1.5. Let (X,d) be a metric and $T, S : X \to X$ be two functions. A mapping S is called a *T-generalized contraction* if there exists $q \in (0,1)$ such that

$$\begin{aligned} d(TSx, TSy) \leqslant q\max\Big\{&d(Tx,Ty), d(Tx,TSx), d(Ty,TSy), \\ &\frac{1}{2}\big(d(Tx,TSy) + d(TSx,Ty)\big)\Big\}, \ \forall x, y \in X. \end{aligned} \qquad (2)$$

Definition 1.6. ([1]) Let (X,d) be a metric. A mapping $T : X \to X$ is called *sequentially convergent* if for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergent, then $\{y_n\}$ is also convergent.

2 The main results

The aim of this work is to prove the following result.

Theorem 2.1. *Let (X, d) be a complete metric space and $T : X \to X$ be an one-to-one, continuous and sequentially convergent mapping. If for some $q \in (0, 1)$*

$$d(TS_0x, TS_ny) \leqslant q \max\Big\{d(Tx, Ty), d(Tx, TS_0x), d(Ty, TS_ny), \frac{1}{2}\big(d(Tx, TS_ny) + d(Ty, TS_0x)\big)\Big\} \tag{3}$$

holds for each $n = 1, 2, ...$ and all $x, y \in X$, then $\{S_n : n = 0, 1, 2, ..\}$ has a unique fixed point.

Remark 2.2. By the above theorem and taking $Tx = x, \forall x \in X$, we obtain Theorem 1.1.

Next, applying Theorem 2.1 for the family $\mathcal{F} = \{S_n : n = 0, 1, 2, ...\}$ with $S_n = S$ for all n, we can get the following result.

Corollary 2.3. *Let X a complete metric space and $T : X \to X$ be an one-to-one, continuous and sequentially convergent mapping. Then every T-generalized contraction continuous function $S : X \to X$, S has a unique fixed point.*

The following example is due to [4]. It shows that the Corollary 2.3 is stronger than Corollary 1.2.

Example 2.4. Let $X = [1, +\infty)$ be a subset of reals with the usual metric. Define $S : X \to X$ by

$$Sx = 4\sqrt{x}, \forall x \in X.$$

It is easy to see that $a = 16$ is unique of S. If (1) holds for some $q \in (0, 1)$ then

$$d(Sx, Sy) < \max\Big\{d(x, y), d(x, Sx), d(y, Sy), \frac{1}{2}\big(d(x, Sy) + d(y, Sx)\big)\Big\},$$

for every $x, y \in X$. But by taking $x = 1, y = 4$ we have

$$d(Sx, Sy) = \max\Big\{d(x, y), d(x, Sx), d(y, Sy), \frac{1}{2}\big(d(x, Sy) + d(y, Sx)\big)\Big\} = 4.$$

We get a contradiction. So, we cannot apply Corollary 2.1 for the map S. However, S will satisfy Corollary 2.3 if we choose $T(x) = \ln(ex)$. Indeed, obviously T is one-to-one, continuous and sequentially convergent and

$$\begin{aligned} d(TSx, TSy) &= |\ln(e4\sqrt{x}) - \ln(e4\sqrt{y})| = \frac{1}{2}\Big|\ln\frac{x}{y}\Big| \\ &= \frac{1}{2}|\ln(ex) - \ln(ey)| = \frac{1}{2}d(Tx, Ty) \\ &\leqslant \frac{1}{2}\max\Big\{d(Tx, Ty), d(Tx, TSx), d(Ty, TSy), \frac{1}{2}\big(d(Tx, TSy) + d(TSx, Ty)\big)\Big\} \end{aligned}$$

for every $x, y \in X$.

We need following lemma for the proof of Theorem 2.1. It is a generalization of the result of [3].

Lemma 2.5. *Let* (X, d) *is a metric space,* $T : X \to X$*, which is one-to-one and* $S_0, S : X \to X$ *be two maps on* X*. If*

$$d(TS_0x, TSy) \leqslant q \max\Big\{d(Tx, Ty), d(Tx, TS_0x), d(Ty, TSy), d(Tx, TSy), d(Ty, TS_0x)\Big\} \tag{4}$$

holds for some q, $0 < q < 1$ *and every* $x, y \in X$ *and* $\{x \in X : S_0(x) = x\}$ *is a non empty set, then* $\{x \in X : S_0(x) = x\}$ *is a singleton and*

$$\{x \in X : S_0(x) = x\} = \{x \in X : S(x) = x\}.$$

Proof Let $a \in \{x \in X : S_0x = x\}$ be any fixed point. Then, by (4)

$$\begin{aligned} d(Ta, TSa) = d(TS_0a, TSa) &\leqslant q \max\Big\{d(Ta, Ta), d(Ta, TS_0a), d(Ta, TSa), \\ &\qquad d(Ta, TSa), d(Ta, TS_0a)\Big\} \\ &\leqslant q \max\{d(Ta, TSa), 0\} = qd(Ta, TSa). \end{aligned}$$

Since $q \in (0, 1)$, we have $d(Ta, TSa) = 0$. It implies that $Ta = TSa$. By the fact that T is one-to-one, we get that $Sa = a$. Hence $a \in \{x \in X : Sx = x\}$. Next, let $a' \in \{x \in X : S_0x = x\}$ be arbitrary. Then $a' \in \{x \in X : Sx = x\}$ and by (4),

$$\begin{aligned} d(Ta, Ta') = d(TS_0a, TSa') &\leqslant q \max\{d(Ta, Ta'), d(Ta, TS_0a), d(Ta', TSa'), \\ &\qquad d(Ta, TSa'), d(Ta', TS_0a')\} \\ &= q \max\{0, d(Ta, Ta')\} = qd(Ta, Ta'). \end{aligned}$$

It follows that $d(Ta, Ta') = 0$. Since T is one-to-one, we have $a = a'$. Therefore

$$\{x \in X : S_0(x) = x\} = \{a\} = \{x \in X : S(x) = x\}.$$

□

Proof of Theorem 2.1 Fix $x_0 \in X$. Consider the sequence $\{x_n\}$ define by $x_0, x_1 = S_0x_0, x_2 = S_1x_1, x_3 = S_0x_2, x_4 = S_2x_3, ..., x_{2n-1} = S_0x_{2n-2}, x_{2n} = S_nx_{2n-1}, \ldots$. For each $n = 0, 1, 2, ...$, we set $y_n = Tx_n$. We claim that $\{y_n\}$ is a Cauchy sequence. Indeed, for each $n = 1, 2, ...$, we have

$$\begin{aligned}
d(y_{2n}, y_{2n-1}) &= d(Tx_{2n}, Tx_{2n-1}) = d(TS_0x_{2n-2}, TS_nx_{2n-1})\\
&\leqslant q\max\Big\{d(Tx_{2n-2}, Tx_{2n-1}), d(Tx_{2n-2}, TS_0x_{2n-2}), d(Tx_{2n-1}, TS_nx_{2n-1}),\\
&\qquad \frac{1}{2}\big(d(Tx_{2n-2}, TS_nx_{2n-1}) + d(Tx_{2n-1}, TS_0x_{2n-2})\big)\Big\}\\
&= q\max\Big\{d(y_{2n-2}, y_{2n-1}), d(y_{2n-2}, y_{2n-1}), d(y_{2n-1}, y_{2n}),\\
&\qquad \frac{1}{2}\big(d(y_{2n-2}, y_{2n}) + d(y_{2n-1}, y_{2n-1})\big)\Big\}\\
&= q\max\Big\{d(y_{2n-2}, y_{2n-1}), d(y_{2n-1}, y_{2n}), \frac{1}{2}d(y_{2n-2}, y_{2n})\Big\}.
\end{aligned}$$

Since $q \in (0, 1)$, we infer that

$$d(y_{2n}, y_{2n-1}) \leqslant q\max\Big\{d(y_{2n-2}, y_{2n-1}), \frac{1}{2}d(y_{2n-2}, y_{2n})\Big\}.$$

We now show that

$$d(y_{2n}, y_{2n-1}) \leqslant qd(y_{2n-2}, y_{2n-1}).$$

Suppose that $\frac{1}{2}d(y_{2n-2}, y_{2n}) > d(y_{2n-2}, y_{2n-1})$. Then

$$2d(y_{2n-2}, y_{2n-1}) < d(y_{2n-2}, y_{2n}) \leqslant d(y_{2n-2}, y_{2n-1}) + d(y_{2n-1}, y_{2n}).$$

Hence $d(y_{2n-2}, y_{2n-1}) < d(y_{2n-1}, y_{2n})$. We obtain

$$\begin{aligned}
d(y_{2n}, y_{2n-1}) &\leqslant q\max\Big\{d(y_{2n-2}, y_{2n-1}), \frac{1}{2}d(y_{2n-2}, y_{2n})\Big\}\\
&\leqslant q\max\Big\{d(y_{2n-2}, y_{2n-1}, \frac{1}{2}\big(d(y_{2n-2}, y_{2n-1}) + d(y_{2n-1}, y_{2n})\big)\Big\}\\
&< q\max\big\{d(y_{2n-1}, y_{2n}), \frac{1}{2}\big(d(y_{2n-1}, y_{2n}) + d(y_{2n-1}, y_{2n})\big\}\\
&= qd(y_{2n-1}, y_{2n}).
\end{aligned}$$

Since $q \in (0, 1)$, we get a contracdition. Thus $\frac{1}{2}d(y_{2n-2}, y_{2n}) \leqslant d(y_{2n-2}, y_{2n-1})$. It follows that

$$d(y_{2n}, y_{2n-1}) \leqslant q\max\Big\{d(y_{2n-2}, y_{2n-1}), \frac{1}{2}d(y_{2n-2}, y_{2n})\Big\} = qd(y_{2n-2}, y_{2n-1}).$$

By the same way, we get that

$$d(y_{2n-2}, y_{2n-1}) \leqslant qd(y_{2n-3}, y_{2n-2}), \forall n = 1, 2, ...$$

It implies that

$$d(y_{2n+1}, y_{2n}) \leqslant qd(y_{2n-2}, y_{2n-1}) \leqslant q^2 d(y_{2n-3}, y_{2n-2}) \leqslant ... \leqslant q^{2n-1} d(y_0, y_1).$$

By an elementary computation, we can take

$$d(y_k, y_{k+p}) \leqslant d(y_k, y_{k+1}) + ... + d(y_{k+p-1}, y_{k+p}) \leqslant q^k \frac{d(y_0, y_1)}{1-q}, \ \forall p \in \mathbb{N}.$$

Therefore, $\{y_n\}$ is a Cauchy sequence. The claim is proved.

Since X is complete, we have $\lim\limits_{n\to\infty} y_n = Tx_n = u \in X$. By the fact that T is a continuous and sequentially convergent mapping, we infer

$$\lim_{n\to\infty} x_n = a \in X \ \text{ and } \ Ta = u.$$

We need to show that $S_n a = a$ for all $n = 0, 1, 2, \ldots$. By the triangle inequality and (3), we have

$$\begin{aligned} d(Ta,TS_0a) &\leqslant d(Ta, y_{2n}) + d(y_{2n}, TS_0 a) = d(u, y_{2n}) + (Tx_{2n}, TS_0 a) \\ &= d(u, y_{2n}) + d(TS_n x_{2n-1}, TS_0 a) \\ &\leqslant d(u, y_{2n}) + q \max\Big\{ d(Tx_{2n-1}, Ta), d(Ta, TS_0 a), d(Tx_{2n-1}, TS_n x_{2n-1}), \\ &\qquad \frac{1}{2}\big(d(Ta, TS_n x_{2n-1}) + d(Tx_{2n-1}, TS_0 a)\big)\Big\} \\ &= d(u, y_{2n}) + q \max\Big\{ d(y_{2n-1}, u), d(Ta, TS_0 a), d(y_{2n-1}, y_{2n}) \\ &\qquad \frac{1}{2}\big(d(u, y_{2n}) + d(y_{2n-1}, TS_0 a)\big)\Big\}. \end{aligned}$$

Since

$$\frac{1}{2} d(y_{2n-1}, TS_0 a) \leqslant d(y_{2n-1}, TS_0 a) \leqslant d(y_{2n-1}, Ta) + d(Ta, TS_0 a),$$

we have

$$\begin{aligned} d(Ta, TS_0 a) &= d(u, y_{2n}) + q \max\Big\{ d(y_{2n-1}, u), d(Ta, TS_0 a), d(y_{2n-1}, y_{2n}) \\ &\qquad \frac{1}{2}\big(d(u, y_{2n}) + d(y_{2n-1}, TS_0 a)\big)\Big\} \\ &\leqslant d(u,y_{2n}) + q\big(d(y_{2n-1}, u) + d(Ta, TS_0 a) + d(y_{2n-1}, y_{2n}) + d(u, y_{2n})\big). \end{aligned}$$

It follows that

$$d(Ta, TS_0 a) \leqslant \frac{1}{1-q}\Big((1+q)d(u, y_{2n}) + qd(y_{2n-1}, u) + qd(y_{2n-1}, y_{2n})\Big), \ \forall n.$$

Combining this and the fact that $\lim\limits_{n\to\infty} y_n = u$, we get $d(Ta, TS_0 a) = 0$. Since T is one-in-one, we must have $S_0 a = a$. By (3), it follows that $d(TS_0 x, TS_n y) \leqslant$

$q \max \Big\{ d(Tx, Ty), d(Tx, TS_0x), d(Ty, TS_ny), d(Tx, TS_ny), d(Ty, TS_0x) \Big\}$. Applying Lemma 2.5, we can conclude that $S_na = a$ for all $n = 1, 2, ...$ and a is unique fixed point of $\{S_n : n = 0, 1, 2, ...\}$, and our proof is now completed. □

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