East-West J. of Mathematics: Vol. 10, No 2 (2008) pp. 111-117

ONγ-LABELING THE ALMOST-BIPARTITE GRAPH $K_{m,n} + e$

S. I. El-Zanati, W. A. O'Hanlon and **E. R. Spicer**

Mathematics Department
Illinois State University
Normal, Illinois 61790-4520, U.S.A.
E-mail: saad@ilstu.edu

Mathematics Department
Illinois Central College
East Peoria, IL 61635-0001, U.S.A.

Pensacola Junior College
Pensacola, FL 32507, U.S.A

Abstract

Let G be a graph with n edges and let k be a positive integer. A *G-design of order k* is a G-decomposition of K_k. A *labeling* of G is an assignment of non-negative integers to the vertices of G. Some labelings of such a G lead to cyclic G-designs of order $2nx+1$. Until recently, such labelings were restricted to bipartite graphs only. An *almost-bipartite* graph is a non-bipartite graph with the property that the removal of a particular single edge renders the graph bipartite. A graph labeling of an almost-bipartite graph G with n edges that yields cyclic G-designs of order $2nx+1$ was recently introduced by Blinco, El-Zanati, and Vanden Eynden. They called such a labeling a γ-labeling. In this note, we survey almost-bipartite graphs that admit γ-labelings and find necessary and sufficient conditions for a γ-labeling of $K_{m,n}+e$.

1 Introduction

If a and b are integers we denote $\{a, a+1, \ldots, b\}$ by $[a, b]$ (if $a > b$, $[a, b] = \emptyset$). Let $\mathbb{N}$ denote the set of nonnegative integers and $\mathbb{Z}_n$ the group of integers modulo n. For a graph G, let $V(G)$ and $E(G)$ denote the vertex set of G and

Key words: G-design, almost-bipartite graph,γ-labeling.
2000 AMS Mathematics Subject Classification: 05C69, 05C75

the edge set of G, respectively. The *order* and the *size* of a graph G are $|V(G)|$ and $|E(G)|$, respectively.

Let $V(K_k) = \mathbb{Z}_k$ and let G be a subgraph of K_k. By *clicking* G, we mean applying the isomorphism $i \to i+1$ to $V(G)$. Let H and G be graphs such that G is a subgraph of H. A *G-decomposition* of H is a set $\Delta = \{G_1, G_2, \ldots, G_t\}$ of pairwise edge-disjoint subgraphs of H each of which is isomorphic to G and such that $E(H) = \bigcup_{i=1}^{t} E(G_i)$. A G-decomposition of K_k is also known as a *(K_k, G)-design*. A (K_k, G)-design Δ is *cyclic* if clicking is a permutation of Δ. For a comprehensive source on graph decompositions we refer the reader to [5]. For recent surveys on G-designs, see [1] and [6].

Let $V(K_k) = \{0, 1, \ldots, k-1\}$. The *length* of an edge $\{i, j\}$ in K_k is $\min\{|i-j|, k-|i-j|\}$. Note that clicking an edge does not change its length. Also note that if k is odd, then K_k consists of k edges of length i for $i = 1, 2, \ldots, \frac{k-1}{2}$.

For any graph G, a one-to-one function $f : V(G) \to \mathbb{N}$ is called a *labeling* (or a *valuation*) of G. In [12], Rosa introduced a hierarchy of labelings. We add a few items to this hierarchy. Let G be a graph with n edges and no isolated vertices and let f be a labeling of G. Let $f(V(G)) = \{f(u) : u \in V(G)\}$. Define a function $\bar{f} : E(G) \to \mathbb{Z}^+$ by $\bar{f}(e) = |f(u) - f(v)|$, where $e = \{u, v\} \in E(G)$. We will refer to $\bar{f}(e)$ as the *label* of e. Let $\bar{E}(G) = \{\bar{f}(e) : e \in E(G)\}$. Consider the following conditions:

- $\ell 1$: $f(V(G)) \subseteq [0, 2n]$,
- $\ell 2$: $f(V(G)) \subseteq [0, n]$,
- $\ell 3$: $\bar{E}(G) = \{x_1, x_2, \ldots, x_n\}$, where for each $i \in [1, n]$ either $x_i = i$ or $x_i = 2n + 1 - i$,
- $\ell 4$: $\bar{E}(G) = [1, n]$.

If in addition G is bipartite with bipartition $\{A, B\}$ of $V(G)$ (with every edge in G having one endvertex in A and the other in B) consider also

- $\ell 5$: for each $\{a, b\} \in E(G)$ with $a \in A$ and $b \in B$, we have $f(a) < f(b)$,
- $\ell 6$: there exists an integer λ (called the *boundary value* of f) such that $f(a) \leq \lambda$ for all $a \in A$ and $f(b) > \lambda$ for all $b \in B$.

Then a labeling satisfying the conditions:

- $\ell 1, \ell 3$**:** is called a *ρ-labeling*;
- $\ell 1, \ell 4$**:** is called a *σ-labeling*;
- $\ell 2, \ell 4$**:** is called a *β-labeling*.

A β-labeling is necessarily a σ-labeling which in turn is a ρ-labeling. If G is bipartite and a ρ, σ or β-labeling of G also satisfies ($\ell 5$), then the labeling is *ordered* and is denoted by ρ^+, σ^+ or β^+, respectively. If in addition ($\ell 6$) is satisfied, the labeling is *uniformly-ordered* and is denoted by ρ^{++}, σ^{++} or β^{++}, respectively.

A β-labeling is better known as a *graceful* labeling and a uniformly-ordered β-labeling is an α-*labeling* as introduced in [12]. Labelings of the types above are called *Rosa-type* because of Rosa's original article [12] on the topic. For a survey of Rosa-type labelings and their graph decomposition applications, see [9]. A dynamic survey on general graph labelings is maintained by Gallian [11].

Labelings are critical to the study of cyclic graph decompositions as seen in the following two results from Rosa [12] and El-Zanati, Vanden Eynden and Punnim [10], respectively.

Theorem 1 *Let G be a graph with n edges. There exists a cyclic G-decomposition of K_{2n+1} if and only if G has a ρ-labeling.*

Theorem 2 *Let G be a graph with n edges that has a ρ^+-labeling. Then there exists a cyclic G-decomposition of K_{2nx+1} for all positive integers x.*

If G with n edges is not bipartite, then the best that could be obtained until recently from a Rosa-type labeling was a cyclic G-decomposition of K_{2n+1}. A non-bipartite graph G is *almost-bipartite* if G contains an edge e whose removal renders the remaining graph bipartite (for example, odd cycles are almost-bipartite). In [4], Blinco, El-Zanati, and Vanden Eynden introduced a variation of a ρ-labeling of an almost-bipartite graph G of size n that yields cyclic G-decompositions of K_{2nx+1}. They called this labeling a γ-labeling.

In this note, we survey the known γ-labelings of almost-bipartite graphs and show that the class of almost-bipartite graphs obtained from $K_{m,n}$ by adding an edge joining distinct vertices in the n-vertex part of $V(K_{m,n})$ has a γ-labeling if and only if $n \geq 3$.

2 Definition of γ-labeling and Some Known Results

Let G be a graph with n edges and h a labeling of the vertices of G. We call h a γ-*labeling* of G if the following conditions hold.

g1: The function h is a ρ-labeling of G.

g2: The graph G is tripartite with vertex tripartition A, B, C with $C = \{c\}$ and $\hat{b} \in B$ such that $\{\hat{b}, c\}$ is the unique edge joining an element of B to c.

g3: If $\{a, v\}$ is an edge of G with $a \in A$, then $h(a) < h(v)$.

g4: We have $h(c) - h(\hat{b}) = n$.

Note that if a non-bipartite graph G has a γ-labeling, then it is almost-bipartite as defined earlier. In this case, removing the edge $\{c, \hat{b}\}$ from G produces a bipartite graph. Figure 1 shows γ-labelings of C_5 and of C_7, respectively.

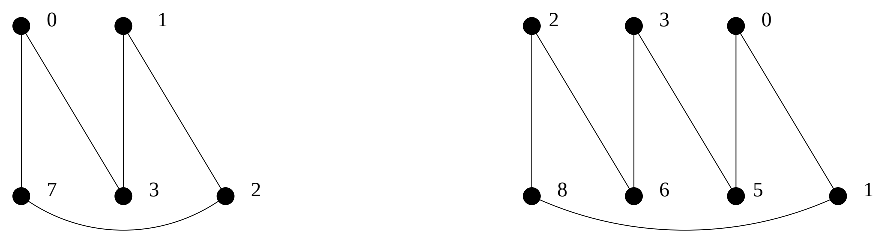

Figure 1: γ-labelings of C_5 and of C_7.

Theorem 3 *Let G be a graph with n edges having a γ-labeling. Then G divides K_{2nx+1} cyclically for all positive integers x.*

We illustrate how Theorem 3 works. See [4] for a complete proof. Let G have n edges and let h be a γ-labeling for G, with A, B, C, c, and $\hat{b}$ as in the above definition. Let $B_1, B_2, \ldots, B_x$ be x vertex-disjoint copies of B, and let $c_1, c_2, \ldots, c_x$ be x new vertices. The vertex in B_i corresponding to $b \in B$ will be called b_i. Let $B^* = \bigcup_1^x B_i$ and $C^* = \{c_1, c_2, \ldots, c_x\}$. We define a new graph G^* with vertex set $A \bigcup B^* \bigcup C^*$ and edges $\{a, v_i\}$, $1 \leq i \leq x$, whenever $a \in A$ and $\{a, v\}$ is an edge of G, and $\{\hat{b_i}, c_i\}$, $1 \leq i \leq x$. Clearly G^* has nx edges and G divides G^*. Define a labeling h^* on G^* by

$$h^*(v) = \begin{cases} h(v) & v \in A, \\ h(b) + (i-1)2n & v = b_i \in B_i, \\ h(c) + (x-i)2n & v = c_i. \end{cases}$$

The labeling h^* is a ρ-labeling of G^* and thus the result follows by Theorem 1.

We illustrate the previous result with an example where $G = C_5$ and $x = 3$. Figure 2 shows a A γ-labeling of C_5 and the three starters in a cyclic C_5-decomposition of K_{31}.

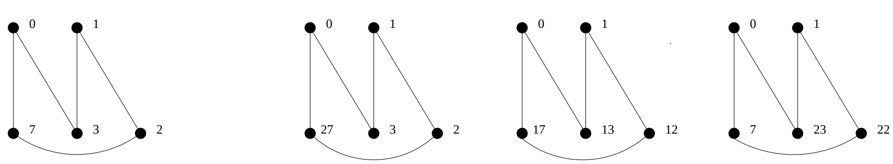

Figure 2: A γ-labeling of C_5 and the three starters in a cyclic (K_{31}, C_5)-design.

Blinco, El-Zanati, and Vanden Eynden [4] showed that odd cycles other than C_3 admit γ-labelings. These results were extended in [8].

Theorem 4 *Every 2-regular graph with exactly one odd component, except for C_3 and $C_3 \cup C_4$, admits a γ-labeling.*

In his PhD dissertation [3], Blinco showed that if $m \geq 5$ is odd, then $C_m + e$ admits a γ-labeling. In [2], it is shown that if $C_{2m} + e$ is almost-bipartite, then it admits a γ-labeling if and only if $m > 2$. In [7], it is shown that $P_m + e$, where $m \geq 4$ and e is an edge joining two vertices in the same part in the bipartition of $V(P_m)$ admits a γ-labeling.

3 γ-labelings of $K_{m,n} + e$

Theorem 5 *Let G denote the almost-bipartite graph obtained from $K_{m,n}$ by adding an edge e between two vertices in the part of $V(K_{m,n})$ of size n. Then G has a γ-labeling if and only if $n > 2$.*

Proof. We show sufficiency first. Assume $n \geq 3$. Let $U = \{(u_1, u_2, \ldots, u_m\}$ and $V = \{v_1, v_2, \ldots, v_n\}$ be the vertex sets in the bipartition of $K_{m,n}$ and let $e = \{v_2, v_n\}$. Define a labeling f of G as follows:

$$f(x) = \begin{cases} (i-1)n & \text{if } x = u_i,\ 1 \leq i \leq m, \\ (m-1)n + i & \text{if } x = v_j,\ 1 \leq j \leq n-1, \\ (2m-1)n + 3 & \text{if } x = v_n. \end{cases}$$

Figure 3 shows such labelings of $K_{2,4}+e$ and of $K_{3,5}+e$. We will show that f is a γ-labeling of G. Clearly G is almost-bipartite with tripartition (A, B, C) where $A = U$, $B = V \setminus \{v_n\}$ and $C = \{v_n\}$. If we let $\hat{b} = v_2$ and $c = v_n$, then condition **g2** is satisfied. Moreover, since $f(c) - f(\hat{b}) = (2m-1)n + 3 - ((m-1)n + 2) = mn + 1$, condition **g4** is satisfied. Also note that since $f(u_i) < f(v_j)$, for every edge $\{u_i, v_j\}$, condition **g3** is satisfied. Thus it remains to show that the induced labeling is a ρ-labeling.

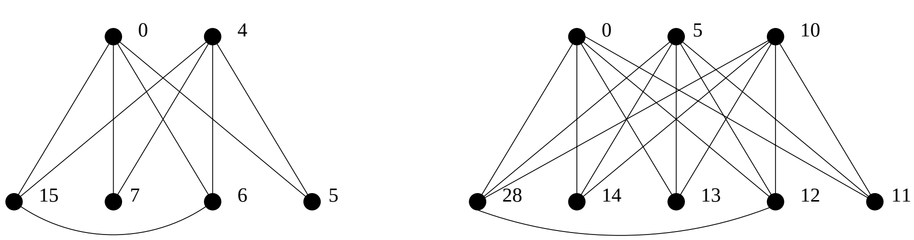

Figure 3: γ-labelings of $K_{2,4} + e$ and of $K_{3,5} + e$.

For $1 \le i \le m$ and $1 \le j \le n-1$, the edge $\{u_i, v_j\}$ has label $(m-1)n + j - (m-i)n = (i-1)n + j$, which is also its length. While the edge $\{u-i, v_n\}$ has label $(2m-1)n + 3 - (m-i)n = mn + (i-1)n + 3$, which is edge length $(m-i+1)n$. Thus in $G - v_n$, the set of labels of edges incident with u_i is $\{(i-1)n+j : 1 \le j \le n-1\} = [(i-1)n+1, (i-1)n+n-1]$. These edge labels are distinct for different i's and their union is $\bigcup_{i=1}^{m}[(i-1)n+1, (i-1)n+n-1] = [0, mn] \setminus \{(i)n : 1 \le i \le m\}$. Finally, the set of labels of edges incident with v_n is $\{(m+i-1)n+3 : 1 \le i \le m\} \bigcup \{mn+1\}$. But these labels correspond precisely to edge lengths $\{(i)n : 1 \le i \le m\} \bigcup \{mn+1\}$. Thus f is a ρ-labeling and hence a γ-labeling.

Now let $n = 2$ and let U and V be as before. In a γ-labeling of $K_{m,2}+e$, the edge e must have label $2m+1$. Without loss of generality, assume $\hat{b} = v_1$ (and thus $c = v_2$). Let u_i be the vertex in A incident with the edge f of length 1. Note that $f(v_2) - f(u_i) = f(v_1) + 2m + 1 - f(u_i)$. Thus if the label of f is 1, u_i must be adjacent to v_1, while if the label on f is $2m+2$, then u_i must be adjacent to v_2. In the first case $f(v_2) - f(u_i) = 2m+2$, while in the second case $f(v_1) - f(u_i) = 2m+1$. Either way we have two edges with length $2m+1$. □

In light of Theorem 3, we have the following corollary.

Corollary 6 *Let G denote the almost-bipartite graph obtained from $K_{m,n}$ by adding an edge e between two vertices in the part of $V(K_{m,n})$ of size n and let $k = mn+1$. Then there exists a cyclic (K_{2kx+1}, G)-design for every positive integer x.*

Although the graph G in Corollary 6 does not admit a γ-labeling when $n = 2$, we expect that a cyclic (K_{2kx+1}, G)-design will exist in that case too. However, this design cannot be found using a γ-labeling of G.

References

[1] P. Adams, D. Bryant, and M. Buchanan, A survey of the existence of G-designs, *J. Comb. Designs*, to appear.

[2] G. Blair, D. Bowman, S. I. El-Zanati, S. Hlad, M. Priban, and K. Sebesta, On cyclic $(C_m + e)$-designs, submitted.

[3] A. Blinco, Graph decompositions, theta graphs and related labelling techniques, Ph.D. Thesis, Department of Mathematics, The University of Queensland, 2003.

[4] A. Blinco, S.I. El-Zanati, and C. Vanden Eynden, On the cyclic decomposition of complete graphs into almost-bipartite graphs, *Discrete Math.* **284** (2004), 71–81.

[5] J. Bosák, *Decompositions of Graphs*, Kluwer Academic Publishers Group, Dordrecht, 1990.

[6] D. Bryant and S. El-Zanati, Graph Decompositions, in The CRC Handbook of Combinatorial Designs, 2nd Edition, (J. H. Dinitz and C. J. Colbourn Eds) CRC Press, Boca Raton, FL, 477-486 (2007).

[7] R. C. Bunge, S. I. El-Zanati, W. A. O'Hanlon, and C. Vanden Eynden, On γ-labeling the almost-bipartite graph $P_m + e$, *Ars Combinatoria*, to appear.

[8] R.C. Bunge, S.I. El-Zanati, and C. Vanden Eynden, On γ-labelings of 2-regular almost-bipartite graphs, in preparation.

[9] S.I. El-Zanati and C. Vanden Eynden, On Rosa-type labelings and cyclic graph decompositions, *Mathematica Slovaca*, to appear.

[10] S.I. El-Zanati, C. Vanden Eynden and N. Punnim, On the cyclic decomposition of complete graphs into bipartite graphs, *Australas. J. Combin.* **24** (2001), 209–219.

[11] J.A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.*, Dynamic Survey 6, 148 pp.

[12] A. Rosa, On certain valuations of the vertices of a graph, in: Théorie des graphes, journées internationales d'études, Rome 1966 (Dunod, Paris, 1967), 349–355.