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On Epi-Projective Modules

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Abstract

In this paper, firstly we show that for lifting modules M and N, M is N-projective if and only if M is epi-N-projective and im-small N-projective. Secondly we show that for any weakly supplemented module N, if $M \oplus N$ is small epi-N-projective then M is N-projective.

1 Preliminaries

Throughout this paper R is a ring with identity and all modules considered are unitary right R-modules.

A submodule S of a module M is called a *small* submodule, if $M \neq K + S$ for any proper submodule K of M. In this case we write $S \ll M$. Let M be a module and let N and K be submodules of M with $K \subseteq N$. K is called a *co-essential* submodule of N in M if $N/K \ll M/K$ and we write $K \subseteq_c N$ in M. Let X be a submodule of M. X is called a *co-closed* submodule in M if X does not have a proper co-essential submodule in M. X' is called a *co-closure* of X in M if X' is a co-closed submodule of M with $X' \subseteq_c X$ in M. $K <_\oplus N$ means that K is a direct summand of N. Let $M = M_1 \oplus M_2$ and let $\varphi : M_1 \to M_2$ be a homomorphism. Put $\langle M_1 \stackrel{\varphi}{\to} M_2\rangle = \{m_1 - \varphi(m_1) \mid m_1 \in M_1\}$. Then this is a submodule of M which is called *the graph* with respect to $M_1 \stackrel{\varphi}{\to} M_2$. Note that $M = M_1 \oplus M_2 = \langle M_1 \stackrel{\varphi}{\to} M_2\rangle \oplus M_2$.

A module M is said to be a *lifting* module if, for any submodule X, there exists a direct summand X^* of M such that $X^* \subseteq_c X$ in M.

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Let X be a submodule of a module M. A submodule Y of M is called a *supplement* of X in M if $M = X + Y$ and $X \cap Y \ll Y$. Note that a supplement Y of X in M is co-closed in M. A module M is *supplemented* ($\oplus$-*supplemented*) if, for any submodule X of M, there exists a submodule (direct summand) Y of M such that Y is a supplement of X in M. A module M is called *amply supplemented* if, X contains a supplement of Y in M whenever $M = X + Y$. A module M is said to be *weakly supplemented* if for any submodule X of M, there exists a submodule Y of M such that $M = X + Y$ and $X \cap Y \ll M$. We see that M is an amply supplemented module if and only if M is a weakly supplemented module and any submodule of M has a co-closure in M (cf. [3, Lemma 1.7]).

Let M and N be modules. M is called *(epi-)N-projective* if, for any submodule A of N, every homomorphism (epimorphism) $f : M \to N/A$ can be lifted to a homomorphism $g : M \to N$. M is called *quasi-projective* (*epi-projective*) if it is (epi-)M-projective. A module M is called *small epi-N-projective* if, for any small submodule A of N, every epimorphism $f : M \to N/A$ can be lifted to a homomorphism $g : M \to N$. If M is small epi-N-projective, then M need not be epi-N-projective (Example 3.1). A module M is called *im-small N-projective* if, for any submodule A of N, any homomorphism $f : M \longrightarrow N/A$ with $f(M) \ll N/A$ can be lifted to a homomorphism $g : M \longrightarrow N$. In the study of discrete modules and lifting modules, these projectivities are important (cf. [1], [2]).

Let M be any module. Consider the following conditions:

(D_2) If $A \leq M$ such that M/A is isomorphic to a direct summand of M, then A is a direct summand of M.

(D_3) If M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, then $M_1 \cap M_2$ is a direct summand of M.

Then the module M is called *discrete* if it is lifting and satisfies the condition (D_2) and it is called *quasi-discrete* if it is lifting and satisfies the condition (D_3). Since (D_2) implies (D_3), every discrete module is quasi-discrete. It is easy to see that any epi-projective module M satisfies the condition (D_2) (see, [1, 4.24(4)]). But the converse is not true (see, Example 2.3).

In this paper, we show the following:

(1) Let M and N be lifting modules. Then M is N-projective if and only if M is epi-N-projective and im-small N-projective.

(2) Let N be weakly supplemented. If $M \oplus N$ is small epi-N-projective, then M is N-projective.

For undefined terminologies, the reader is referred to [1], [7] and [8].

Lemma 1.1 *Let $X' \subseteq X \subseteq M$. If $M = X' + Y$ and $X \cap Y \ll M$, then $X' \subseteq_c X$ in M.*

Proof By [5, Lemma 1.4]. □

Lemma 1.2 (cf. [6, Lemma 1.7]) *Let $f : M \to N$ be an epimorphism with* $\ker f \ll M$*. If X is co-closed in M, then $f(X)$ is co-closed in N.*

2 Epi-projective Modules

We recall the definition of relative epi-projectivity.

Definition Let M and N be modules. M is called *epi-N-projective* if, for any submodule A of N, every epimorphism $f : M \to N/A$ can be lifted to a homomorphism $g : M \to N$. In particular, M is called *epi-projective* if M is epi-M-projective. Note that epi-projective modules are well known as pseudo-projective modules.

Lemma 2.1 *If M is epi-projective, then the following are equivalent:*
(1) M is discrete,
(2) M is quasi-discrete,
(3) M is lifting,
(4) M is $\oplus$-supplemented.

Proof It is enough to show that (4) $\Rightarrow$ (1): Let A be any submodule of M. Since M is $\oplus$-supplemented, then $M = A + B$ and $A \cap B \ll B$ for some direct summand B of M. Assume that $M = B \oplus B'$. By [1, 4.24(2)], B' is B-projective. Then by [1, 4.12], $M = A' \oplus B$ for some submodule A' of M with $A' \leq A$. It is easy to show that $A/A' \ll M/A'$. Thus M is lifting. On the other hand, M is a (D_2)-module by [1, 4.24(4)]. □ Recall that a module M is called *hollow* if every proper submodule of M is small in M.

Lemma 2.2 (see, [1] and [2]) *Let M be a hollow epi-projective module. Then M is quasi-projective.*

Proof Let A be a submodule of M and $f : M \to M/A$ be a nonzero homomorphism. If $f(M) = M/A$, then f can be lifted to a homomorphism from M to M since M is epi-projective. So assume that $f(M) \neq M/A$. Let $\gamma = \pi - f$, where $\pi : M \to M/A$ is the natural epimorphism. Since $f(M) \neq M/A$ and M/A is hollow, $\gamma(M) = M/A$. Then by epi-projectivity assumption γ can be lifted to a homomorphism $g : M \to M$. Now $1 - g$ lifts f. □

We note that discrete modules need not be epi-projective as the following example shows:

Example 2.3 Let R be an incomplete rank one discrete valuation ring and let K be its quotient field. The R-module K_R is indecomposable and discrete and hence hollow, but is not quasi-projective. By Lemma 2.2, K_R cannot be epi-projective.

By the same argument of the proof of Lemma 2.2, we obtain the following:

Lemma 2.4 *If M is epi-projective then M is im-small M-projective.*

Now we discuss the relationship between relative epi-projectivity and relative projectivity of modules.

Proposition 2.5 *Let M and N be lifting modules. Then M is N-projective if and only if M is epi-N-projective and im-small N-projective.*

Proof "Only if" part is clear. "If" part: Let $g : N \to X$ be an epimorphism and let $f : M \to X$ be a homomorphism. Since M and N are lifting, we may assume $\ker f \ll M$ and $\ker g \ll N$. As $X = g(N)$ is amply supplemented, there exist a co-closure K of $f(M)$ in X and a supplement K' of $f(M)$ in X. Then $X = f(M)+K' = K+K'$ and so $f(M) = K+(f(M)\cap K')$. As $f(M)$ is amply supplemented, there exists a co-closure S of $f(M)\cap K'$ in $f(M)$. Since M is lifting, there exists a decomposition $M = L \oplus M'$ with $L \subseteq_c f^{-1}(S)$ in M. Hence $f(L) \subseteq_c S$ in $f(M)$ by [1, 3.2(7)] and so $f(L) = S$. Then

$$f(M) = f(L) + f(M') = S + f(M') \qquad \cdots (*).$$

By $f^{-1}(S) \cap M' \ll M'$, $S \cap f(M') = f(f^{-1}(S) \cap M') \ll f(M)$. Now we prove $f(M')$ is co-closed in X. Let $A \subseteq_c f(M')$ in X. As $S \subseteq f(M) \cap K' \ll K' \subseteq X$, $S \ll X \quad \cdots (**)$. By $(*)$, $f(M) = S + f(M') = S + A$ and so $X = f(M)+K' = S+f(M')+K' = f(M')+K'$. Since $f(M')\cap S \ll f(M)$, by Lemma 1.1, $A \subseteq_c f(M')$ in $f(M)$. As $f(M')$ is co-closed in $f(M)$ (by Lemma 1.2) $A = f(M')$. Thus $f(M')$ is co-closed in X.

Since N is lifting, there exists a decomposition $N = T \oplus N'$ with $T \subseteq_c g^{-1}(f(M'))$ in N. Hence $g(T) \subseteq_c f(M')$ in X and so $g(T) = f(M')$. Since M' is epi-T-projective, there exists a homomorphism $\varphi_1 : M' \to T$ such that $(g|_T) \circ \varphi_1 = f|_{M'}$. On the other hand, by $(**)$, there exists a homomorphism $\varphi_2 : L \to N$ such that $g \circ \varphi_2 = f|_L$ since L is im-small N-projective.

Thus f is extended to $\varphi = \varphi_1 + \varphi_2 : M = L \oplus M' \to N$. Therefore M is N-projective. □

The following is due to L. Ganesan and N. Vanaja [2]. As a corollary of Lemma 2.4 and Proposition 2.5, we obtain this result.

Corollary 2.6 *Let M be a lifting module. Then M is epi-projective if and only if M is quasi-projective.*

There exist modules M and N such that M is epi-N-projective but M is not im-small N-projective.

Example 2.7 Let S and S' be simple modules with $S \not\simeq S'$. Let M and K be uniserial modules with the following conditions:

(i) $M \cap K = S$,
(ii) $M \supset S \supset 0$, $K \supset K_1 \supset K_2 \supset S \supset 0$,
(iii) $M/S \simeq S$, $K/K_1 \simeq S'$, $K_1/K_2 \simeq S$, $K_2/S \simeq S'$.

Put $N = M + K$. (Using path algebra, we can see that there exist such modules M, N.)
(1) First we show "M is epi-N-projective." Let $f : M \to X$ and $g : N \to X$ be epimorphisms. Since f is an epimorphism, $X \simeq M$ or $X \simeq M/S$. Assume that $X \simeq M$. Since $N/\ker g \simeq X \simeq M$ is uniserial with length 2, we see $\ker g = M + K_2$ (essentially, factor modules of N are 0, N, $N/S \simeq M/S \oplus K/S$, $N/M \simeq K/S$, $N/(M+K_2) \simeq K/K_2$ and $N/(M+K_1) \simeq K/K_1$). Now $M/S \simeq S \not\simeq S' \simeq K/K_1$ imply $X \simeq M \not\simeq K/K_2 \simeq N/\ker g$, a contradiction. Hence we see $X \simeq M/S$, that is, $\ker g = K$. Since $g|_M : M \overset{i}{\hookrightarrow} N = M + K \overset{g}{\to} X$ is an epimorphism, there exist a homomorphism $h : M \to M$ such that $(g|_M) \circ h = f$ and so $g \circ h = f$. Thus M is epi-N-projective.

(2) Next we show "M is not im-small N-projective." Let $f : M \to X$ be a homomorphism with $\ker f = S$ and let $g : N \to X$ be an epimorphism with $\ker g = M + K_2$. Then $S \simeq f(M) \ll X$. Since the tops of K and M are not isomorphic, f can not be lifted. Thus M is not im-small N-projective.

Therefore, by (1) and (2), M is epi-N-projective, but not im-small N-projective.

The following example shows that N is not lifting in Example 2.7.

Example 2.8 Let M and K be uniserial modules with

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_s = M, \text{ and } 0 = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_t = K,$$

where $s < t$. Let $0 \neq M \cap K$ be a proper submodule of M and put $N = M + K$. Then N is not lifting.

Proof (1) Since K is a proper submodule of N and $N = M + K$, we see M is not small in N.
(2) Now we show that "any nonzero submodule of M is not a direct summand of N." Let L be a nonzero proper submodule of M. Since $L <_\oplus N$ imply $L <_\oplus M$, L is not a direct summand of N. Next we assume that M is a direct summand of N. Put $N = M \oplus T$. Let $p : N = M \oplus T \to M$ be the projection. By $M \cap K \neq 0$, $M_1 = K_1 \subseteq M \cap K$ and so $T \cap K = 0$. Thus $p|_K : K \to p_M(K)$ is an isomorphism and hence $t \leq s$, a contradiction. So any nonzero submodule of M is not a direct summand of N.

By (1) and (2), N is not lifting. □

In [4], Keskin studied the T-modules. Recall that a module M is called a *T-module* if $M/A \cong M/B$ where A is a co-closed submodule of M and B is any submodule of M implies that B is a co-closed submodule of M. In the following proposition we show that any amply supplemented epi-projective module is a T-module.

Proposition 2.9 *If M is an amply supplemented epi-projective module, then M is a T-module.*

Proof Let M be amply supplemented and epi-projective. Let A and K be submodules of M where K is co-closed in M and $f : M/K \longrightarrow M/A$ be any isomorphism. By [3, Proposition 1.5], there exists a submodule B of A such that $A/B \ll M/B$ and B is co-closed in M. Let $\pi : M/B \longrightarrow M/A$ be the epimorphism with the kernel A/B, $\pi_K : M \longrightarrow M/K$, $\pi_A : M \longrightarrow M/A$ and $\pi_B : M \longrightarrow M/B$ the natural epimorphisms. Since M is epi-projective, there exists a homomorphism $\gamma : M \longrightarrow M$ such that $\pi_A\gamma = f\pi_K$. Now we have the homomorphism $g = \pi_B\gamma : M \longrightarrow M/B$. Clearly, $\pi g = f\pi_K$. Therefore by [4, Proposition 2.11], M is a T-module. □

Note that Example 2.3 shows that any amply supplemented T-module need not be epi-projective.

3 Small Epi-N-projective Modules

In this section, we study the relative small epi-projectivity. Let us recall the definition of relative small epi-projectivity.

Definition A module M is called *small epi-N-projective* if, for any small submodule A of N, every epimorphism $f : M \to N/A$ can be lifted to a homomorphism $g : M \to N$.

Obviously any epi-N-projective module is small epi-N-projective, but the converse is not true in general, as the following shows:

Example 3.1 Let S be a small submodule of $G = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and let $\pi_S : G \to G/S$ be a canonical epimorphism. As $J(\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}) = 0$, $S = 0$. So π_S is an isomorphism. So G is small epi-projective. Let $p : G = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ be the projection and put $K = 2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Assume that G is epi-projective. Then there exists a homomorphism $h : G \to G$ such that $\pi \circ h = p$, where $\pi : G \to G/K$ is a canonical epimorphism. For any $0 \neq x \in \mathbb{Z}/2\mathbb{Z}$, we see

$$x = p(x) = \pi \circ h(x) = \pi(x) = 0.$$

This is a contradiction. Therefore G is not epi-projective.

Note that by the same proof as Lemma 2.2, any small epi-projective hollow module is quasi-projective. The following proposition gives a characterization of small epi-N-projectivity.

Proposition 3.2 *Let M and N be two modules and $X = M \oplus N$ amply supplemented. The following are equivalent:*

(1) M is small epi-N-projective,

(2) For any supplement K of N in X with $X = M + K$, $X = N \oplus K$.

Proof (1)$\Longrightarrow$ (2) : Let K be a supplement of N in X with $X = M + K$. Let $\pi : N \longrightarrow N/(K \cap N)$ be the natural epimorphism, $\pi_M : M \longrightarrow X/K$ the epimorphism with $\pi_M(m) = m + K$ $(m \in M)$ and $\alpha : X/K \longrightarrow N/(K \cap N)$ the obvious isomorphism. Then we have the epimorphism $\alpha\pi_M : M \longrightarrow N/(K \cap N)$. By hypothesis, there exists a homomorphism $\psi : M \longrightarrow N$ such that $\pi\psi = \alpha\pi_M$. Now X=$\langle M \overset{\psi}{\to} N\rangle \oplus N$. Since K is a supplement of N in X, $K = \langle M \overset{\psi}{\to} N\rangle$. Therefore $X = N \oplus K$.
(2)$\Longrightarrow$(1) : Let A be a small submodule of N, $f : M \longrightarrow N/A$ an epimorphism and $\pi : N \longrightarrow N/A$ the natural epimorphism. Let $H = \{m + n \mid f(m) = -\pi(n), m \in M, n \in N\}$. Obviously, $X = N + H = M + H$ and $A \leq H$. Since $N \cap H = A$, $N \cap H \ll N$. Let H' be a supplement of N in X contained in H. Now $[(N \cap H) + H']/H' = H/H' \ll X/H'$. Then $X = M + H'$. By hypothesis, $X = N \oplus H'$. Now let $\psi : N \oplus H' \longrightarrow N$ be the projection. Then the restriction $\psi|_M : M \longrightarrow N$ is the desired homomorphism. □

Proposition 3.3 *Let R be a right perfect ring. Let M and N be modules with $X = M \oplus N$. The following are equivalent:*

(1) M is N-projective,

(2) M is small epi-L-projective for every $L \leq N$.

Proof (1)$\Longrightarrow$(2) : Clear.
(2)$\Longrightarrow$(1) : Assume M is small epi-L-projective for every $L \leq N$. Let $X = M \oplus N = A + N$ for any submodule A of X. Let K be a supplement of N in X which is contained in A, namely $X = K + N$, $K \cap N \ll K \leq A$. Assume that $L = N \cap (K + M)$. By hypothesis, M is small epi-L-projective. Let $X' = M \oplus L$. It is easy to see that $X' = K + M = K + L$. Since $K \cap L = K \cap N \ll K$, K is a supplement of L in X'. Therefore $X' = K \oplus L$ by Propsition 3.2. Hence $X = K \oplus N$. Thus by [1, 4.12], M is N-projective. □

Lemma 3.4 *Let M be small epi-N-projective and put $M = M' \oplus M''$. Then M' is small epi-N-projective.*

*Proof*Let f be any epimorphism from M' to N/X with $X \ll N$ and $g : N \longrightarrow N/X$ be the canonical epimorphism. Consider the projection map $\alpha : M \longrightarrow M'$. Since M is small epi-N-projective, there exists a homomorphism $h : M \longrightarrow N$ such that $g \circ h = f \circ \alpha$. Therefore, $g \circ (h|_{M'}) = f$. □

Lemma 3.5 *Let N be a $\oplus$-supplemented module. Then M is epi-N-projective if and only if M is small epi-N'-projective for any direct summand N' of N.*

Proof “Only if” part is clear.
“If” part : Let $g : N \to N/K$ be a canonical epimorphism and let $f : M \to N/K$ be an epimorphism. Since N is $\oplus$-supplemented, there exists a direct summand N' of N such that $N = \ker g + N'$ and $\ker g \cap N' \ll N'$. Since M is small epi-N'-projective, there exists a homomorphism $h : M \to N'$ such that

$\phi \circ h = \alpha \circ f$, where $\phi : N' \to N'/(N' \cap \ker g)$ is the natural epimorphism and $\alpha : N/K \to N'/(K \cap N')$ is the obvious isomorphism. Thus f is lifted to h. □

Proposition 3.6 *If $M \oplus N$ is small epi-N-projective then M is small epi-N'-projective for any direct summand N' of N.*

Proof Let $g : N' \to N'/X$ be the canonical epimorphism with $X \ll N'$ and $f : M \to N'/X$ be an epimorphism. Define $g^* = g + 1_{N''} : N = N' \oplus N'' \to N'/X \oplus N''$ by $(n', n'') \mapsto (g(n'), n'')$ and $f^* = f + 1_{N''} : M \oplus N'' \to N'/X \oplus N''$ by $(m, n'') \mapsto (f(m), n'')$. By $\ker g^* = X \ll N'$ and Lemma 3.4, there exists a homomorphism $h : M \oplus N'' \to N$ such that $g^* \circ h = f^*$. Let p' and p'' be the projections : $N = N' \oplus N'' \to N'$, $N = N' \oplus N'' \to N''$, respectively. Then, for $m \in M$,

$$(f(m), 0) = f^*(m, 0) = g^* h(m, 0) = g^*(n', n'') = (g(n'), n'')$$

where $h(m, 0) = (n', n'')$. Thus $n'' = 0$ and $f(m) = g(n')$. Put $\varphi = p' \circ (h|_M)$. Then $f(m) = g(p' h(m, 0)) = g\varphi(m, 0)$. Thus f is lifted to φ. Therefore M is small epi-N'-projective. □

Proposition 3.7 *Let N be weakly supplemented. If $M \oplus N$ is small epi-N-projective, then M is N-projective.*

Proof Let $\pi : N \to N/K$ be the canonical epimorphism and let $f : M \to N/K$ be a homomorphism. Since N is weakly supplemented, there exists a weak supplement L of K in N and so $N = L + K$ and $L \cap K \ll N$. Define $g : N = L + K \to N/K \oplus K/(L \cap K)$ by $g(l + k) = (\pi(l), \nu(k))$ for $l \in L$ and $k \in K$, where $\nu : K \to K/(L \cap K)$ is the canonical epimorphism. Then g is an epimorphism. Hence $\varphi = f - g \ : \ M \oplus N \to N/K \oplus K/(L \cap K)$ is an epimorphism. Since $M \oplus N$ is small epi-N-projective, there exists a homomorphism $\psi : M \oplus N \to N$ such that $g \circ \psi = \varphi$. As $\psi(M) \subseteq L$,

$$f(m) = \varphi(m) = g\psi(m) = \pi\psi(m).$$

Thus $f = \pi \circ (\psi|_M)$. □

Corollary 3.8 *Let M be a lifting module. Then the following are equivalent:*

(1) M is epi-projective,

(2) M is small epi-M'-projective for any direct summand M' of M,

(3) $M \oplus M$ is small epi-M-projective,

(4) M is quasi-projective.

Corollary 3.9 *The following are equivalent for a ring R:*

(1) R is semisimple,

(2) For every simple right R-module M, $M \oplus R$ is small epi-R-projective and R_R is weakly supplemented,

(3) Every right R-module satisfies (D_2),
(4) Every 2-generated right R-module satisfies (D_2),
(5) The direct sum of two right R-module with (D_2) satisfies (D_2),
(6) The direct sum of two quasi-projective right R-module satisfies (D_2),
(7) Every right R-module is epi-projective,
(8) Every 2-generated right R-module is epi-projective,
(9) The direct sum of two epi-projective right R-module is epi-projective,
(10) The direct sum of two quasi-projective right R-module is epi-projective.

Proof (1)⇒(2) is clear.
(2)⇒(1) : Let M be a simple right R-module. By hypothesis, $M \oplus R$ is small epi-R-projective. By Proposition 3.7, M is R-projective. Therefore M is F-projective for every free right R-module F. Hence M is projective and so R is semisimple. (1)⇔(3)⇔(4)⇔(5)⇔(6) by [9, Theorem 9]. (1)⇔(7)⇔(8)⇔(9)⇔(10) are clear because every epi-projective module satisfies (D_2). □

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