East-West J. of Mathematics: Vol. 10, No 1 (2008) pp. 1-14

GENERATING ALL EFFICIENT EXTREME SOLUTIONS IN MULTIPLE OBJECTIVE LINEAR PROGRAMMING PROBLEM AND ITS APPLICATION TO MULTIPLICATIVE PROGRAMMING

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Abstract

In this paper we proposed a new algorithm for generating the all efficient extreme solutions and all efficient extreme edges of a multiple objective linear programming problem (VP). The algorithm has been implemented and numerical examples are shown. As an application we solve the linear multiplicative programming associated with the problem (VP).

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This paper is partially supported by the National Basic Program on Natural Science, Vietnam.
Key words: multiple objective linear programming, linear multiplicative programming, efficient solution.
2000 AMS Mathematics Subject Classification: Primary: 90 C29; Secondary: 90 C26

1 Introduction

We consider the multiple objective linear programming problem

$$\text{MIN } Cx, \text{ s.t. } x \in M, \tag{VP}$$

where $M \subset \mathbb{R}^n$ is a nonempty polyhedral convex set and C is a $(p \times n)$ matrix with $p \geq 2$ rows $c^1, \cdots, c^p$. The problem (VP) has p linear objective functions $f_j(x) = \langle c^j, x \rangle$, $j = 1, 2, \cdots, p$. Generally, these objective functions conflict with one another over the decision set

$$M = \{x \in R^n \; : \; Ax = h, \; x \geq 0\}, \tag{1}$$

where A is a $(m \times n)$ matrix with m rows $a^1, \cdots, a^m$, $m < n$, A is of full row rank and $h = (h_1, \cdots, h_m)^T \in \mathbb{R}^m$.

The problem (VP) arises from various applications in engineering, economics, network planning, production planning etc. (see., e.g, [7], [20], [22], [25]). For instance, for the perfect economical production plan, one wants to simultaneously minimize the cost and maximize the quality. This example illustrates a natural feature of this problem, namely, that typically the different objectives contradict each other.

Various solution concepts for problem (VP) have been proposed. The concept of an efficient solution is commonly used. In particular, a point $x^0 \in M$ is said to be an *efficient solution* for problem (VP) if there exists no $x \in M$ such that $Cx^0 \geq Cx$ and $Cx^0 \neq Cx$. Let M_E denote the set of all efficient solutions of problem (VP). Many algorithms have been proposed to generate either all of the efficient set M_E, or a representative portion thereof, without any input from decision maker; see, e.g., [1, 2, 10, 11, 15, 21, 24] and references therein. For a survey of these and related results see [4].

It is well known that M_E consists of a union of faces of M. While M_E is also always a connected set, generally, it is a complicated nonconvex subset of the boundary of M [16]. Let M_{ex} denote the set of all extreme points of M. The set of all efficient extreme solutions $M_E \cap M_{ex}$ is a finite, discrete set and is smaller than all of M_E. Therefore, it ought to be more computationally practical to generate the set $M_E \cap M_{ex}$ and to present it to the decision maker without overwhelming him or her than M_E [4].

In this paper, we present a quite easy algorithm for generating all efficient extreme solutions $M_E \cap M_{ex}$ and all unbounded efficient edges in problem (VP). As an application we solve the linear multiplicative programming associated with the problem (VP).

2 Efficient Condition

Assume henceforth that the decision M is a nonempty, nondegenerate polyhedral convex set. In the case of degeneracy, one can use the right hand side perturbation method of Charnes (see, e.g., Chapter 10 [8]; Chapter 6 [13]) to reduce the nondegeneracy. For two vectors $y^1 = (y_1^1, ..., y_p^1)$, $y^2 = (y_1^2, ..., y_p^2) \in \mathbb{R}^p$, $y^1 \geq y^2$ denotes $y_j^1 \geq y_j^2$ for $j = 1, ..., p$ and $y^1 \gg y^2$ denotes $y_j^1 > y_j^2$ for $j = 1, ..., p$. As usual, $\mathbb{R}_+^p$ denotes the nonnegative orthant of $\mathbb{R}^p$.

A key result in all the sequel is the following characterization of efficiency which can be found in many places (see, e.g., [16],[21], [23])

Theorem 2.1. *A point $x^0 \in M$ is an efficient solution of the problem (VP) if and only if there are real numbers $\lambda_1 > 0, \cdots, \lambda_p > 0$ such that x^0 is an optimal solution of the linear programming problem*

$$\min\Big\{\Big\langle \sum_{j=1}^{p} \lambda_j c^j, x\Big\rangle \ : \ x \in M\Big\}. \tag{$LP1$}$$

It is well known that a set $\Gamma \subset \mathbb{R}^n$ is a face of M if and only if Γ equals the optimal solution set to the problem

$$\min\{\langle \alpha, x\rangle \ : \ x \in M\},$$

for some $\alpha \in \mathbb{R}^n \setminus \{0\}$. The following result is directly deduced from this fact and Theorem 2.1.

Proposition 2.1. *Let x^0 be a relative interior point of a face $\Gamma \subseteq M$. If x^0 is an efficient solution to the problem (VP) then every point of Γ is efficient solution (i.e., $\Gamma \subset M_E$).*

Let x^0 be a point of M and let

$$I(x^0) = \{j \in \{1, \cdots, n\} \ : \ x_j^0 = 0\}. \tag{2}$$

Denote $e^j = (0, \cdots, 0, \underbrace{1}_{j^{th}}, 0, \cdots, 0)^T$. Below is an optimality condition that will be important in helping to develop our algorithm.

Theorem 2.2. *A feasible solution $x^0 \in M$ is an efficient solution to the problem (VP) if and only if there are real numbers $\lambda_j > 0, j = 1, \cdots, p$, $v_j \geq 0$, $j \in I(x^0)$, $u_1, \cdots, u_m$ such that*

$$\sum_{j=1}^{p} \lambda_j c^j + \sum_{i=1}^{m} u_i a^i - \sum_{j \in I(x^0)} v_j e^j = 0. \tag{3}$$

Proof. By Theorem 2.1, a point $x^0 \in M$ is an efficient solution of the problem (VP) if and only if there are real numbers $\lambda_1 > 0, \cdots, \lambda_p > 0$ such that x^0 is an optimal solution of the linear programming problem $(LP1)$. Since the problem $(LP1)$ is a linear programming problem, any $x^0 \in M$ is regular. By Kuhn-Tucker Theorem, a point $x^0 \in M$ is an optimal solution of $(LP1)$, i.e

x^0 is an efficient solution of (VP), if and only if there exists real numbers u_i, $i = 1, \cdots, m$ and $v_j \geq 0$, $j = 1, \cdots, n$ such that

$$\sum_{j=1}^{p} \lambda_j c^j + \sum_{i=1}^{m} u_i a^i - \sum_{j=1}^{n} v_j e^j = 0 \tag{4}$$

and

$$v_j x_j = 0,\ j = 1, \cdots, n. \tag{5}$$

From (5) and (2), we have

$$v_j = 0 \ \ \forall j \notin I(x^0).$$

Combine this fact and (4), we can rewrite (4) as follows

$$\sum_{j=1}^{p} \lambda_j c^j + \sum_{i=1}^{m} u_i a^i - \sum_{j \in I(x^0)} v_j e^j = 0.$$

The proof is straight forward. □

3 Determination of Efficient Extreme Solutions and Unbounded Efficient Edges

It is well known that the efficient solution set M_E is pathwise connected [16]. Hence, according to standard scheme (see [1], [2], [10],[15] etc.), to generate the set of all efficient extreme solutions and all unbounded efficient edges in the problem (VP) we just need to present the procedure that generates all the efficient extreme solutions adjacent to a given efficient extreme solution $x^0 \in M$ and all unbounded efficient edges emanating from x^0.

The mail tool for our algorithm is Theorem 2.2 that provides a condition for a point $x^0 \in M$ to be an efficient solution for multiple objective linear programming problem (VP). This condition is closed to the efficient condition which presented in [15] in terms of normal cones. Combining this condition and pivot technique of the simplex procedure we will introduce a simple algorithm for generating all efficient extreme solutions and all unbounded efficient edges in the problem (VP). Notice that this algorithm does not require the assumption "M is nonempty polyhedral convex set in $\mathbb{R}^n$ of dimension n" as the algorithm is proposed in [15].

Denote by $A_1, A_2, \cdots, A_n$ the columns of matrix A. Let x^0 be a given extreme point of M. Let $J_0 = \{j \in \{1, 2, \cdots, n\} \ : \ x^0_j > 0\}$. It is clear that $J_0 = \{1, \cdots, n\} \setminus I(x^0)$. We have $|J_0| = m$ and the set of m linearly independent vectors $\{A_j \; : \; j \in J_0\}$ are grouped together to form $m \times m$ basic matrix B [8, 9]. The variables x_j, $j \in J_0$, are said to be basic variables and the vectors A_j, $j \in J_0$, are said to be basic vectors. For each $k \notin J_0$, we denote by $z^k = (z_1^k, \cdots, z_n^k)^T \in \mathbb{R}^n$,

$$z_j^k = \begin{cases} -z_{jk} & \text{if } j \in J_0 \\ 0 & \text{if } j \notin J_0, j \neq k \\ 1 & \text{if } j = k, \end{cases} \tag{6}$$

where the real numbers z_{jk}, $j \in J_0$, satisfy the system of linear equations

$$A_k = \sum_{j \in J_0} z_{jk} A_j.$$

The next result is easily obtained by the theory linear programming and Proposition 2.1, but we give a proof here for the convenience of the reader.

Proposition 3.1. *Let x^0 be an efficient extreme solution to the problem (VP). Assume that $k \notin J_0$ and $\hat{x} := x^0 + \varepsilon z^k$ where z^k defined by (6) and ε is a small enough positive number. Then*

i) If $\hat{x}$ is an efficient solution to the problem (VP) and $z_{jk} \leq 0$ for all $j \in J_0$ then $\Gamma(k) = \{x = x^0 + tz^k \; : \; t \geq 0\}$ is an unbounded efficient edge of M emanating from x^0.

ii) If $\hat{x}$ is an efficient solution to the problem (VP) and there is at least $j_0 \in J_0$ such that $z_{j_0 k} > 0$ then $[x^0, x^1]$ is an efficient edge to problem (VP) where $x^1 = (x_1^1, \cdots, x_n^1) \in M$ is an efficient extreme solution adjacent to the efficient extreme solution x^0,

$$x_j^1 = \begin{cases} x_j^0 - \theta_0 z_{jk} & \textit{if } j \in J_0 \setminus \{r\} \\ \theta_0 & \textit{if } j = k \\ 0 & \textit{otherwise,} \end{cases} \tag{7}$$

and

$$\theta_0 = \min\{\frac{x_j^0}{z_{jk}} \; : \; z_{jk} > 0, \; j \in J_0\} = \frac{x_r^0}{z_{rk}}.$$

Proof. Consider a given efficient extreme solution $x^0 = (x_1^0, \cdots, x_n^0)^T$ and vector z^k defined by (6) with $k \notin J_0$. For every $\theta > 0$, let

$$x(\theta) = x^0 + \theta z^k.$$

Compute directly, we have

$$Ax(\theta) = h. \tag{8}$$

i) If $z_{jk} \leq 0$ for all $j \in J_0$ then $x(\theta) \geq 0$ for all $\theta \geq 0$. Combining this fact and (8) once can see that $\Gamma(k) = \{x = x^0 + tz^k \; : \; t \geq 0\}$ is an unbounded edge of the feasible solution set M emanating from x^0. The point $\hat{x} = x^0 + \varepsilon z^k$ is a relative interior point of this ray $\Gamma(k)$. Since $\hat{x} \in M_E$, we invoke Proposition 2.1 to deduce that $\Gamma(k) \subset M_E$.

ii) In case there is $j_0 \in J_0$ such that $z_{j_0 k} > 0$, we have

$$x(\theta) = x^0 + \theta z^k \geq 0, \ \ \forall 0 \leq \theta \leq \theta_0, \tag{9}$$

where

$$\theta_0 = \min\{\frac{x_j^0}{z_{jk}} \ : \ z_{jk} > 0, \ j \in J_0\} = \frac{x_r^0}{z_{rk}}.$$

From (8) and (9) we have

$$[x^0, x^1] := \{x^0 + \theta z^k \ : \ 0 \leq \theta \leq \theta_0\}$$

is a finite edge of the feasible solution set M emanating from x^0 and x^1 is determined by (7). By virtue of Proposition 2.1 we have $[x^0, x^1] \subset M_E$ because $\hat{x} \in M_E$ is a relative interior point of this edge. □

Proposition 3.1 give us a way for finding all efficient edges emanating from a given efficient extreme solution x^0. In view of Theorem 2.2, in order to decide whether a point $\hat{x} \in M$ is an efficient solution to problem (VP) we have to verify whether the following system has a solution

$$\begin{gathered}\sum_{j=1}^{p} \lambda_j c^j + \sum_{i=1}^{m} u_i a^i - \sum_{j \in I(\hat{x})} v_j e^j = 0 \\ \lambda_j > 0, j = 1, \cdots, p \\ v_j \geq 0, j \in I(\hat{x}),\end{gathered} \tag{10}$$

where $I(\hat{x}) = \{j \in \{1, \cdots, n\} : \hat{x}_j = 0\}$.

Note that if $(\lambda, u, v) \in R^{p+m+|I(\hat{x})|}$ is a solution of the system (10) then $(t\lambda, tu, t\mu)$ is also a solution of this system for an arbitrary real number $t > 0$. Therefore, instead of checking the consistency of the system (10), one can use Phase I of simplex algorithm to check the consistency of the following system

$$\begin{gathered}\sum_{j=1}^{p} \lambda_j c^j + \sum_{i=1}^{m} (\bar{u}_i - \bar{\bar{u}}_i) a^i - \sum_{j \in I(\hat{x})} v_j e^j = 0 \\ \lambda_j \geq 1, j = 1, \cdots, p; \quad v_j \geq 0, j \in I(\hat{x}), \\ \bar{u}_i, \bar{\bar{u}}_i \geq 0, \ i = 1, \cdots, m.\end{gathered} \tag{11}$$

Let x^0 be a given efficient extreme solution. The procedure for determining all the efficient extreme solutions adjacent to x^0 and all the unbounded efficient edges emanating from x^0 can be described as follows.

Procedure EFFICIENCY(x^0)

Step 1. Determine the set $J_0 = \{j \in \{1, 2, \cdots, n\} \ : \ x_j^0 > 0\}$. Let ε be a small enough positive real number.

Step 2.
for each $k \notin \{1, \cdots, n\} \setminus J_0$ **do**

- $\diamond$ Solve the system of linear equations

$$A_k = \sum_{j \in J_0} z_{jk} A_j$$

  to determine the coefficients $z_{jk},\ j \in J_0$.
- $\diamond$ Let $\hat{x} := x^0 + \varepsilon z^k$, where z^k defined by (6).
- $\diamond$ **if** $\hat{x} \in M_E$ (*i.e., the system* (11) *has a solution*) **then**

  if $z_{jk} \leq 0 \ \forall j \in J_0$ **then**

  $\Gamma = \{x^0 + \theta z^k \ : \theta \geq 0\}$ is an unbounded efficient edge.
  Store the result.

  else

  $[x^0, x^1]$ is a finite efficient edge where x^1 determined by (7).
  Store x^1 if it has not been stored before

  end if

  end if

end for.

Note that the efficient solution set of (VP) is pathwise connected. Hence, by applying the above procedure for each new efficient extreme solution uncovered in the process, we obtain all efficient extreme solutions and all unbounded efficient edges of the problem (VP).

Remark 3.1 It is well known that the data associated to a given extreme point $x^0 \in M$ can be performed in the simplex tableau. Without loss of generality and for the convenience, assume that $J_0 = \{1, ..., m\}$ and the basis $B = (A_1, ..., A_m)$ is the unit matrix I_m. Then we have

$$A_k = \sum_{j \in J_0} z_{jk} A_j = (z_{1k}, ..., z_{mk})^T,$$

where $(z_{1k}, ..., z_{mk})^T$ stands for the transpose of the vector $(z_{1k}, ..., z_{mk})$. The following Tableau 1 is the simplex tableau associated to x^0.

Tableau 1

B	basic variables	A_1	A_2	...	A_m	A_{m+1}	A_{m+2}	...	A_n
A_1	x_1	1	0	...	0	$z_{1\ m+1}$	$z_{1\ m+2}$	...	$z_{1\ n}$
A_2	x_2	0	1	...	0	$z_{2\ m+1}$	$z_{2\ m+2}$	...	$z_{2\ n}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	x_m	0	0	...	1	$z_{m\ m+1}$	$z_{m\ m+2}$	...	$z_{m\ n}$

Then the Procedure EFFICIENCY (x^0) has become much simpler by familiar simplex pivot technique [8, 9, 13] (See illustrated example in Section 4).

Remark 3.2 Several methods for finding an initial efficient extreme solution for (VP) have proposed (see, e.g., [3], [12], [15], [21]). Here we have made use of Benson's test [3]. This procedure determines whether the problem (VP) has efficient solutions and to find an initial extreme solution if it exists. Namely, solve

$$\begin{aligned} &\min\{-z^T C\bar{x} + u^T b\} \\ \text{s.t. } \ & z^T C - u^T A + w^T = -e^T C \\ & w, z \geq 0, \end{aligned}$$

where $\bar{x}$ is any feasible solution. If no optimal solution exists, the problem (VP) has $M_E = \emptyset$. Otherwise, let $(\bar{z}, \bar{u}, \bar{w})$ be an optimal solution and we obtain an efficient extreme solution to the problem (VP) by solving

$$\min\Big\{\Big\langle \sum_{j=1}^{p} \bar{\lambda}_j c^j, x \Big\rangle \ : \ x \in M\Big\},$$

where $\bar{\lambda} = (\bar{\lambda}_1, \cdots, \bar{\lambda}_p) = (\bar{z} + e)$ and e is the vector in $\mathbb{R}^p$ whose entries are each equal to one.

4 Computational Results

We begin with the following simple example to illustrate our algorithm. Consider the multiple objective linear programming

$$\text{MIN}\{Cx \ : \ Ax = b, x \geq 0\},$$

where

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} -2 & -1 & 1 & 0 & 0 \\ -1 & -2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} -2 \\ -2 \\ 6 \end{pmatrix}.$$

In this example, $m = 3$, $n = 5$ and $p = 2$. Choose $\varepsilon = 0.1$. Using Benson's test, we obtain the first efficient extreme solution

$$x^0 = \Big(\frac{2}{3}, \frac{2}{3}, 0, 0, \frac{14}{3}\Big).$$

It is clear that

$$J_0 = \{j \in \{1, \cdots, 5\} : x_j^0 > 0\} = \{1, 2, 5\} \text{ and } \{1, \cdots, 5\} \setminus J_0 = \{3, 4\}.$$

The data associated to this efficient extreme solution is shown in Tableau 2.

Tableau 2

A_J	basic variables	A_1	A_2	A_3	A_4	A_5
A_1	$\frac{2}{3}$	1	0	$-\frac{2}{3}$	$\frac{1}{3}$	0
A_2	$\frac{2}{3}$	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	0
A_5	$\frac{14}{3}$	0	0	$\frac{1}{3}$	$\frac{1}{3}$	1

• Consider $k = 3$. We have $\theta_0 = 2,\ r = 2$ and

$$z^3 = \Big(\frac{2}{3}, -\frac{1}{3}, 1, 0, -\frac{1}{3}\Big).$$

The system (11) associated to $\hat{x} = x^0 + \varepsilon z^3$ has a solution. That means $\hat{x} \in M_E$. We have a new efficient extreme solution $x^1 = x^0 + \theta_0 z^3 = (2, 0, 2, 0, 4)$ which adjacent x^0. The data associated to efficient extreme solution x^1 is shown in Tableau 3.

Tableau 3

A_J	basic variables	A_1	A_2	A_3	A_4	A_5
A_1	2	1	2	0	-1	0
A_3	2	0	3	1	-2	0
A_5	4	0	-1	0	1	1

• Consider $k = 4$. We have $\theta_0 = 2,\ r = 1$ and

$$z^4 = \Big(-\frac{1}{3}, \frac{2}{3}, 0, 1, -\frac{1}{3}\Big).$$

The system (11) associated to $\hat{x} = x^0 + \varepsilon z^4$ has a solution. So, we have $\hat{x} \in M_E$ and obtain a new efficient extreme solution $x^2 = x^0 + \theta_0 z^4 = (0, 2, 0, 2, 4)$ which adjacent x^0. The data associated to efficient extreme solution x^2 is shown in Tableau 4.

Tableau 4

A_J	basic variables	A_1	A_2	A_3	A_4	A_5
A_4	2	3	0	-2	1	0
A_2	2	2	1	-1	0	0
A_5	4	-1	0	1	0	1

Repeat using Procedure EFFICIENCY(x^1) and EFFICIENCY(x^2) where x^1 and x^2 are two efficient vertices have just determined above. At last, we have obtained 3 efficient extreme solution $x^0 = (\frac{2}{3}, \frac{2}{3}, 0, 0, 4\frac{2}{3})$; $x^1 = (2, 0, 2, 0, 4)$; $x^2 = (0, 2, 0, 2, 4)$ for this problem.

In order to obtain a preliminary evaluation of the performance of the proposed algorithm, we built a test software using C^{++} programming language that implements the algorithm.

The following example introduced by Yu and Zeleny [24], and also considered in [1,2,15]. The problem is stated as follows.

$$\text{MIN}\{Cx \;:\; Ax \le b,\; x \ge 0\},$$

where

$$C = \begin{pmatrix} -3 & 7 & -4 & -1 & 0 & 1 & 1 & -8 \\ -2 & -5 & -1 & 1 & -6 & -8 & -3 & 2 \\ -5 & 2 & -5 & 0 & -6 & -7 & -2 & -6 \\ 0 & -4 & 1 & 1 & 3 & 0 & 0 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & 3 & -4 & 1 & -1 & 1 & 2 & 4 \\ 5 & 2 & 4 & -1 & 3 & 7 & 2 & 7 \\ 0 & 4 & -1 & -1 & -3 & 0 & 0 & 1 \\ -3 & -4 & 8 & 2 & 3 & -4 & 5 & -1 \\ 12 & 8 & -1 & 4 & 0 & 1 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ 8 & -12 & -3 & 4 & -1 & 0 & 0 & 0 \\ 15 & -6 & 13 & 1 & 0 & 0 & -1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 40 \\ 84 \\ 18 \\ 100 \\ 40 \\ -12 \\ 30 \\ 100 \end{pmatrix}.$$

Note that each vertex is nondegenerate. We computed and have obtained 29 efficient extreme solutions in 0.031 seconds. This numerical result coincides the result reported in [2] and [15].

Below we present our computational experimentation with the algorithm. For each triple (p, m, n), the algorithm was run on 20 randomly generated test problems having form similar to Yu and Zeleny problem. The elements of constraint matrix A, the right-hand-side vector b and the objective function coefficient matrix C were randomly generated integers belonging to the discrete uniform distribution in the intervals $[-12,\ 15]$, $[-12,\ 100]$ and $[-7,\ 8]$, respectively. Test problems are executed on IBM-PC, chip Intel Celeron PIV 1.7 GHz, RAM 640 MB, C^{++} programming language, Microsoft Visual C^{++} compiler. Numerical results are summarized in Table 5.

In Table 5, it can be observed that computational requirements increase with constraints size (i.e. $m \times n$ size). Another observation is that the number of objectives have a significant effect on number of efficient points, therefore, effect on computational time.

Table 5. Computational Results

p	m	n	$NEPs$	$TIME$
3	17	20	8	0.08
3	20	20	6	0.068
3	25	30	8	0.248
5	17	20	110	1.308
5	20	20	106	1.324
5	25	30	40	1.467
7	17	20	209	3.292
7	20	20	160	2.166
7	25	30	176	7.178
8	17	20	553	14.471
9	17	20	665	23.769
10	17	20	372	7.033

NEPs: Average number of efficient vertices
TIME: Average CPU-Time in seconds.

5 An Application

As an application of above proposed algorithm, we consider the linear multiplicative programming

$$\min\{\prod_{j=1}^{p}\langle c^j, x\rangle \ : \ x \in M\}, \tag{LMP}$$

where M is the polyhedral convex set defined by (1), $p \geq 2$ is an integer, and for each $j = 1, \cdots, p$, vector $c^j \in \mathbb{R}^n$ satisfies

$$\langle c^j, x\rangle > 0 \ \text{ for all } x \in M. \tag{12}$$

It is well known that the problem (LMP) is difficult global optimization problem and it has been shown to be NP-hard, even when $p = 2$ [17]. This problem have some important applications in engineering, finance, economics, and other fields (see, e.g., [6]). In recent years, due to the requirement of the practical applications, a resurgence of interest in problem (LMP) occurred (see. e.g., [5, 6, 14, 18]). In this section, we solve the problem (LMP) based on the relationships between this problem and associated multiple objective linear programming problem.

First, we show the existence of solution of the problem (LMP).

Proposition 5.1 *The problem (LMP) always has an optimal solution.*

Proof. It is clear that it is sufficient to treat the case in which M is unbounded. Let C denote the $p \times n$ matrix whose j^{th} row equals c^j, $j = 1, 2, ..., p$. Let Y be defined by

$$Y = \{y \in \mathbb{R}^p | y = Cx, \text{ for some } x \in M\}.$$

It follows readily from definitions that the problem (LMP) is equivalent to the following problem

$$\min\{g(y) = \prod_{j=1}^{p} y_j \ : \ y \in Y\}. \tag{LMP_Y}$$

Therefore instead of showing the existence of solution of the problem (LMP) we show the existence of solution of the problem (LMP_Y). It can be shown that the set Y is a nonempty polyhedral convex set in $\mathbb{R}^p$, see, e.g., [19]. Denote the set extreme points of Y by $V(Y)$ and the set of extreme directions of Y by $R(Y)$. Then

$$Y = \text{conv}V(Y) + \text{cone}R(Y), \tag{13}$$

where $\text{conv}V(Y)$ is the convex hull of $V(Y)$ and $\text{cone}R(Y)$ is the cone generated by $R(Y)$ [19]. Taking account of the assumption (12) the set Y must be contained in $\text{int}\mathbb{R}^p_+ = \{u \in \mathbb{R}^p | u \gg 0\}$. It implies that

$$\text{cone}R(Y) \subset \text{int}\mathbb{R}^p_+ \cup \{0\}. \tag{14}$$

Since $\text{conv}V(Y)$ is a compact set, there is $y^0 \in \text{conv}V(Y)$ such that

$$g(\hat{y}) \geq g(y^0), \text{ for all } \hat{y} \in \text{conv}V(Y). \tag{15}$$

We claim that y^0 must be a global optimal solution for problem (LMP_Y). Indeed, for any $y \in Y$, it follows from (13) and (14) that

$$y = \bar{y} + v \geq \bar{y}, \tag{16}$$

where $\bar{y} \in \text{conv}V(Y)$ and $v \in \text{cone}R(Y)$. Furthermore, it is easily seen that the objective function $g(y) = \prod_{j=1}^{p} y_j$ of problem (LMP_Y) is increasing on $\text{int}\mathbb{R}^p_+$, i.e., if $y^1 \geq y^2 \gg 0$ implies that $g(y^1) \geq g(y^2)$. Combining (15), (16) and this fact gives

$$g(y) \geq g(\bar{y}) \geq g(y^0).$$

In other words, y^0 is a minimal optimal solution of problem (LPM_Y). The proof is complete. □

The multiple objective linear programming problem (VP) associated with the linear multiplicative programming problem (LMP) may be written as

$$\text{MIN}\{Cx, \ x \in M\},$$

where C is the $p \times n$ matrix whose j^{th} row equals c^j, $j = 1, ..., p$.

The next proposition tells us the relationships between problem (VP) and problem (LMP). It is obtained from the definitions and the fact that the objective function $h(x) = \prod_{j=1}^{p} \langle c^j, x\rangle$ of problem (LMP) is a quasiconcave function [5].

Proposition 5.2. *The problem (LMP) has at least one global optimal solution that belongs to the efficient extreme solution set $M_E \cap M_{ex}$ of the problem (VP).*

As a consequence of Proposition 5.2, one can find a global optimal solution to the problem (LMP) by evaluating the objective function $h(x) = \prod_{j=1}^{p} \langle c^j, x\rangle$ at each efficient extreme solution of problem (VP). More precise, we have the following procedure

Procedure SOLVE(LMP)
Step 1. Determine the set of all efficient extreme solution $M_E \cap M_{ex}$ for the multiple objective linear programming problem which associates the problem (LMP) (Section 3).

Step 2. Determine the set

$$S^* = \{x^* \in M_E \cap M_{ex} \ : \ h(x^*) \leq h(x) \ \forall \, x \in M_E \cap M_{ex}\}$$

and terminate the procedure: each $x^* \in S^*$ is a global optimal solution to (LMP).

References

[1] P. Armand and C. Malivert, *Determination of the Efficient Set in Multiobjective Linear Programming*, Journal of Optimization Theory and Applications, Vol. 70, 1991, pp. 467-489.

[2] P. Armand, *Finding all Maximal Efficient Faces in Multiobjective Linear Programming*, Mathematical Programming, Vol. 61, 1993, pp. 357-375.

[3] H.P. Benson, *Finding an Initial Efficient Extreme Point for a Linear Multiple Objective Program*, Journal of the Operational Research Society, Vol. 32, 1981, pp. 495-498.

[4] H.P. Benson, *An Outer Approximation Algorithm for Generating All Efficient Extreme Points in the Outcome Set of a Multiple Objective Linear Programming Problem*, Journal of Global Optimization, Vol. 13, 1998, pp. 1-24.

[5] H.P. Benson and G. M. Boger, *Multiplicative Programming Problems: Analysis and Efficient Point Search Heuristic*, Journal of Optimization Theory and Applications, Vol. 94, 1997, pp. 487 - 510.

[6] Benson, H. P., and Boger, G. M., *Outcome-Space Cutting-Plane Algorithm for Linear Multiplicative Programming*, Journal of Optimization Theory and Applications, Vol. 104, 2000, pp. 301-322.

[7] J.L. Cohon, Multiobjective Programming and Planning, *Academic Press, New York*, 1978.

[8] G.B. Dantzig, Linear Programming and Extensions, *Princeton University Press, Princeton, NJ*, 1963.

[9] S. I. Gass, Linear Programming: Methods and Applications, *McGraw-Hill Book Company, New York*, 1994

[10] J.G. Ecker and I.A. Kouada, *Finding All Efficient Extreme Points for Multiple Objective Linear Programs*, Mathematical Programming, Vol. 14, 1978, pp. 249-261.

[11] J.G. Ecker and N. E. Schoemaker, *Selecting Subsets from the Set of Nondominated Vectors in Multiple Objective Linear Programming*, SIAM Journal on Control and Optimization, Vol. 19, 1981, pp. 505-515.

[12] J.G. Ecker and N.S. Hegner, *On Computing an Initial Efficient Extreme Point*, Journal of the Operational Research Society, Vol. 29, 1978, pp. 1005-1007.

[13] G. Hadley, Linear Programming, *Addison-Wesley, Reading, MA*, 1963.

[14] N.T. Bach Kim, *Finite Algorithm for Minimizing the Product of Two Linear Functions over a Polyhedron*, Journal of Industrial and Management Optimization. 3(1), 2007, pp. 481-487.

[15] N.T.Bach Kim and D.T. Luc, *Normal Cone to a Polyhedral Convex Set and Generating Efficient Faces in Linear Multiobjective Programming*, Acta Mathematica Vietnamica, Vol. 25, No. 1, 2000, pp. 101-124.

[16] D.T.Luc, Theory of Vector Optimization, *Springer-Verlag, Berlin, Germany*, 1989.

[17] T. Matsui, *NP-Hardness of Linear Multiplicative Programming and Related Problems*, Journal of Global Optimization, Vol. 9, 1996, pp. 113-119.

[18] L.D. Muu and B. T. Tam, *Minimizing the Sum of a Convex Function and the Product of Two Affine Functions over a Convex Set*, Optimization, Vol. 24 (1992), pp. 57-62.

[19] R.T. Rockafellar, Convex Analysis, *Princeton University Press, Princeton, New Jersey*, 1970.

[20] M. K. Starr and M. Zeleny (Eds.), Multicriteria Decision Making, *North-Holland Publ. Company, Amsterdam*, 1977.

[21] R. E. Steuer, Multiple Criteria Optimization: Theory, Computation and Application, *Wiley*, 1985.

[22] M. T. Tabucanon, *Multiojective Programming for Industrial Engineers*, in Mathematical Programming for Industrial Engineers, M. Avriel et al. (Eds.), New York, Marcel Dekker, 1996, pp. 487-542.

[23] M. Zeleny, Linear Multi-Objective Programming, *Springer-Verlag, New York, New York*, 1974.

[24] P.L. Yu and M. Zeleny, *The Set of All Nondominated Solutions in Linear Case and a Multicriteria Simplex Method*, J. Math. Anal. Vol. 49,1975, pp. 430-468.

[25] P.L. Yu, Multiple Criteria Decision Making: Concepts, Techniques and Extensions, *Plenum Press, New York*, 1985.