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ON GENERAL GAUSS MAPS OF SURFACES

Dang Van Cuong

Department of Mathematics,
Duy Tan University,
k7/25 Quang Trung street, DaNang city, Vietnam
e-mail: cuongdangvan@gmail.com

Abstract

For studying surfaces of higher dimensions and codimensions, some general Gauss maps were used. In this paper, we introduce some such maps including a new one, $\mathbf{n}_r^{\pm}$-Gauss maps of codimension two spacelike surfaces in the Lorentz-Minkowski space $\mathbb{L}^{n+1}$.

1. Introduction

The classical Gauss map plays an important role in the study of codimension one surfaces both in $\mathbb{R}^n$ and $\mathbb{L}^{n+1}$. It is a map from the surface to the unit sphere. It is well-known that, the derivative of the Gauss map, called the Weingarten map, is self-adjoint.

For studying the behaviour of surfaces of higher dimensions and codimensions, some general Gauss maps were used in the same way as in the classical differential geometry of surfaces. For example, for studying minimal 2-surfaces in $\mathbb{R}^n$, one consider the Gauss map from the surfaces to $G(2,\mathbb{R}^n)$, the Grassmannian of oriented 2-plane in $\mathbb{R}^n$

$$g : S \to G(2,\mathbb{R}^n),$$

where $g(p)$ is the tangent plane to S at p (see [3]).

Marek Kossowski (see [9]) introduced the S^2-valued Gauss maps to study spacelike surfaces of dimension two in $\mathbb{L}^4$, while for studying spacelike surfaces

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of codimension two, Izumiya et. al. (see [4], [6]) used Gauss maps associated with a normal field or ones whose values are in the lightcone.

In this paper, we introduce some general Gauss maps including a new one in the last section.

Let M be a spacelike surface of codimension two. The normal plane of M at $p \in M$, denoted by N_pM, can be viewed as a timelike 2-plane passing the origin. The intersection of this plane and the hyperbolic with center $\nu = (-1, 0, 0, \ldots, 0)$ and radius 1, $H^n_+(\nu, 1)$, is a hyperbola. For any $r > 0$, the hyperplane $\{x_0 = r\}$ meets this hyperbola exactly at two points, denoted by $\mathbf{n}^{\pm}_r(p)$.

This gives two differential maps $p \mapsto \mathbf{n}^{\pm}_r(p)$, which are called $\mathbf{n}^{\pm}_r$-Gauss maps. These maps are used in [2] to study the umbilicity of spacelike surfaces. The derivatives of the $\mathbf{n}^{\pm}_r$-Gauss maps are self-adjoint as in the classical case, and hence we can define the $\mathbf{n}^{\pm}_r$-Weingarten maps, $\mathbf{n}^{\pm}_r$-Gauss-Kronecker curvatures, $\mathbf{n}^{\pm}_r$-mean curvatures and $\mathbf{n}^{\pm}_r$-principal curvatures ....

Throughout this paper, a surface is always assumed smooth, oriented and regular.

2. General Gauss maps of surfaces in $\mathbb{R}^n$

For a parametric surface $X : \Omega \to \mathbb{R}^3$, where Ω is an open domain in $\mathbb{R}^2$,

$$N = \frac{1}{|X_u \wedge X_v|} X_u \wedge X_v$$

stands as the unit normal vector field of S and can be seen as the Gauss map of the surface.

2.1. General Gauss maps of minimal 2-surfaces in $\mathbb{R}^n$

We refer the reader to [1] and [3] for more details about this topic.

Let S be a 2-surface in $\mathbb{R}^n$, $n > 3$. The dimension of the normal space N_pS of the surface at each point p is bigger than 1 and one can not define the Gauss map as above. For solving that problem, first one takes an arbitrary unit normal vector field N and defines the second fundamental form ($b_{ij}(N) := \langle \frac{\partial^2 X}{\partial u_i \partial u_j}, N \rangle,\ i, j = 1, 2$), the principal curvatures $(k_1(N), k_2(N))$ as well as the mean curvature $(H(N))$ of S with respect to N. It is known that $b_{ij}(N)$ and $H(N)$ are linear in N, and therefore there exist a unique vector field $\overrightarrow{H}$ such that

$$H(N) = \overrightarrow{H} \cdot N, \quad \text{for any unit normal vector field } N.$$

The vector field $\overrightarrow{H}$ is called the mean curvature vector field of the surface and S is called minimal if $\overrightarrow{H} = 0$.

Another way to study minimal 2-surfaces S in $\mathbb{R}^n$ is studying the map

$$g : S \to G(2, \mathbb{R}^n),$$

where $G(2, \mathbb{R}^n)$ is the Grassmannian of oriented 2-planes in $\mathbb{R}^n$ and $g(p)$ is the tangent plane to S at p (see [3]).

2.2. Curvature ellipses of 2-surfaces in $\mathbb{R}^4$

Let S be a surface in $\mathbb{R}^3$ and denote by $k_n(x, v)$ the normal curvature of S at $x \in S$ with respect to the unit vector $v \in T_pS$. It is well-known that $k_n(x, v) \in [k_1(x), k_2(x)]$, where $k_1(x),\ k_2(x)$ are principal curvatures of S at x and x is umbilic if the segment $[k_1(x), k_2(x)]$ degenerates into a point, i.e. $k_1(x) = k_2(x)$. This fact can be generalized to a 2-surface in $\mathbb{R}^4$ as follow.

Let $X : M \to \mathbb{R}^4$ be an immersion and $\{e_1, e_2, e_3, e_4\}$ be an orthonormal frame field on S chosen so that at each point $x \in S$, $\{e_1, e_2\}$ is an orthonormal basis of T_xS and $\{e_3, e_4\}$ is an orthonormal basis of N_xS.

The first fundamental form I and the second fundamental form II_{e_i} with respect to $e_i, i = 3, 4$, are defined as in Subsection 2.1. Then, $II := II_{e_3} + II_{e_4}$ is called the second fundamental form of X and $\eta(x, v) := \frac{II_x(v)}{I_x(v)}$ is called the normal curvature vector of S at x with respect to v. When v runs along the unit circle $S^1 \subset T_xS$, $\eta(x, v)$ draws an ellipse, called the curvature ellipse of S at x. It is known that, the center of this ellipse is the mean curvature vector and when the curvature ellipse degenerates into a segment or a point, x is called semiumbilic or umbilic, respectively.

The similar curvature ellipses of a spacelike 2-surface in $\mathbb{L}^4$ are introduced in [4] (see Section 3 for the definition of the Lorentz-Minkowski $\mathbb{L}^4$). We have the following result.

Theorem 2.1 ([4, Theorem 5.3]) *A spacelike surface $M \subset \mathbb{L}^4$ is totally semi-umbilical if and only if M is ν-umbilical for some non-zero normal vector field ν locally defined at each non umbilical point.*

This result implies that, $M \subset H^3(a, r)$ ($M \subset S^3(a, r)$ or $M \subset LC^3(a)$) if and only if every point of M is either spacelike- (timelike- or lightlike-, resp.) semiumbilical or umbilical and the curvature ellipse defines a parallel normal field on M.

3. General Gauss maps of surfaces in Ł$^{n+1}$

The Lorentz-Minkowski $(n + 1)$-space $\mathbb{L}^{n+1}$ is the $(n + 1)$-dimensional vector space $\mathbb{R}^{n+1} = \{(x_0, x_1, \ldots, x_n) : x_i \in \mathbb{R}, i = 0, 1, 2, \ldots, n\}$ endowed a pseudo scalar product defined by

$$\langle x, y\rangle = -x_0y_0 + \sum_{i=1}^{n} x_iy_i,$$

where $x = (x_0, x_1, x_2, \ldots, x_n), y = (y_0, y_1, y_2, \ldots, y_n) \in \mathbb{L}^{n+1}$. Since $\langle , \rangle$ is non-positive defined, $\langle x, x\rangle$ may be zero or negative. We say a nonzero vector $x \in \mathbb{L}^{n+1}$ is spacelike, lightlike or timelike if $\langle x, x\rangle > 0$, $\langle x, x\rangle = 0$ or $\langle x, x\rangle < 0$, respectively. If $\langle x, y\rangle = 0$, we say x, y are pseudo-orthogonal.

For a nonzero vector $\mathbf{n} \in \mathbb{L}^{n+1}$, a hyperplane with pseudo-normal $\mathbf{n}$ is the set

$$HP(\mathbf{n}, c) = \{x \in \mathbb{L}^{n+1} : \langle x, \mathbf{n}\rangle = c,\ c \in \mathbb{R}\}.$$

The hyperplane $HP(\mathbf{n}, c)$ is called spacelike, lightlike or timelike if $\mathbf{n}$ is timelike, lightlike or spacelike, respectively.

A k-surface is call spacelike if its tangent spaces are all spacelike.

3.1 The shape operator associated with a normal field

Let M be a spacelike surface of codimension two in $\mathbb{L}^{n+1}$. Denote by

1. $\mathcal{X}(M)$ the space of all smooth tangent vector fields of M;
2. $\mathcal{N}(M)$ the space of all smooth normal vector fields of M;
3. $\overline{\nabla}$ the pseudo-riemannian connection of $\mathbb{L}^{n+1}$ and ∇ is the reduced one of M;
4. $\overline{X}$ a local extension of X to $\mathbb{L}^{n+1}$ for any $X \in \mathcal{X}(M)$.

The second fundamental map of M is defined as follow

$$\begin{aligned} \alpha : \mathcal{X}(M) \times \mathcal{X}(M) &\to \mathcal{N}(M) \\ (X, Y) &\mapsto \overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y, \end{aligned}$$

and for any vector field $\nu \in \mathcal{N}(M)$

$$II_\nu : T_xM \to \mathbb{R},\ II_\nu(v) = \langle \alpha(v, v), \nu\rangle,$$

is called the second fundamental form of M at x with respect to ν.

The shape operator with respect to the nomal field ν is defined by

$$S_\nu : TM \to TM,\ S_\nu(X) = -\left(\overline{\nabla}_{\overline{X}}\overline{\nu}\right)^T,$$

where $(\ )^T$ stands for the tangent component. This operator has some good properties. For example, it is self-adjoint and satisfies the Weingarten equation

$$\langle S_\nu(X), Y\rangle = \langle \alpha(X, Y), \nu\rangle,\ \forall X, Y \in \mathcal{X}(M).$$

The definitions of ν-principal curvatures, ν-mean curvature, ν-Gauss- Kronecker curvature, ν-umbilic ... are defined as in the classical case.

For some applications of this operator to study ν-umbilical surfaces of codimension two, we refer the reader to [4].

3.2 The lightcone Gauss maps

Let $M = X(U)$ be a spacelike surface of codimension two, defined by an immersion $X : U \to \mathbb{L}^{n+1}$, where U is an open subset of $\mathbb{R}^{n-1}$. For any $p \in M$, the tangent space T_pM is spacelike while the normal space N_pM is timelike. Let $\mathbf{n}^F(u) \in N_pM$ be a future directed (i.e. the first coordinate is possitive) unit timelike normal and set

$$\mathbf{n}^S(u) = \frac{\mathbf{n}^F(u) \wedge X_{u_1}(u) \wedge \cdots \wedge X_{u_{n-1}}(u)}{|\mathbf{n}^F(u) \wedge X_{u_1}(u) \wedge \cdots \wedge X_{u_{n-1}}(u)|}.$$

It is easy to see that, $\mathbf{n}^S$ is a spacelike unit normal vector field, i.e. $\langle \mathbf{n}^S, \mathbf{n}^S \rangle = 1$. Moreover, $\langle \mathbf{n}^F, \mathbf{n}^F \rangle = -1$ and $\langle \mathbf{n}^F, \mathbf{n}^S \rangle = 0$.

Clearly, the vectors $\mathbf{n}^F(u) \pm \mathbf{n}^S(u)$ are lightlike. It is showed that (see [6]), for two different future directed unit timelike normal vectors $\mathbf{n}^F(u)$ and $\overline{\mathbf{n}^F}(u)$, the corresponding lightlike normal vectors $\mathbf{n}^F(u) + \mathbf{n}^S(u)$ and $\overline{\mathbf{n}^F}(u) + \overline{\mathbf{n}^S}(u)$ are parallel.

Therefore, we have the map $p \mapsto \mathbf{n}^F(u) + \mathbf{n}^S(u)$. Its derivative

$$d_p(\mathbf{n}^F + \mathbf{n}^S) : T_pM \to T_p\mathbb{L}^{n+1} = T_pM \oplus N_pM$$

is linear and can be writen as

$$d_p(\mathbf{n}^F + \mathbf{n}^S) = d_p(\mathbf{n}^F + \mathbf{n}^S)^T + d_p(\mathbf{n}^F + \mathbf{n}^S)^N.$$

We call

1. the linear transformation $S_p(\mathbf{n}^F, \mathbf{n}^S) = -d_p(\mathbf{n}^F + \mathbf{n}^S)^T$ the $(\mathbf{n}^F, \mathbf{n}^S)$-shape operator;

2. the linear transformation $d_p(\mathbf{n}^F + \mathbf{n}^S)^N$ the normal connection with respect to $(\mathbf{n}^F, \mathbf{n}^S)$.

By using this shape operator we can define some concepts as in the classical case such as the lightcone principal curvatures, the lightcone Gauss-Kronecker curvature, the lightcone mean curvature ....

For any lightlike vector $x = (x_0, x_1, x_2, \ldots, x_n)$, let $\widetilde{x} = \left(1, \frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \in S_+^{n-1} = \{x = (x_0, x_1, x_2, \ldots, x_n) : \langle x, x \rangle = 0, x_0 = 1\}$.

It is showed that (see [6]), if we choose another future directed unit timelike normal vector $\overline{\mathbf{n}^F}(u)$, then $(\widetilde{\mathbf{n}^F + \mathbf{n}^S}) = \widetilde{\overline{\mathbf{n}^F} + \overline{\mathbf{n}^S}} \in S_+^{n-1}$. Therefore, the lightcone Gauss map of M can be defined

$$\begin{array}{rcl}\widetilde{L}: U & \to & S_+^{n-1} \\ u & \mapsto & (\widetilde{\mathbf{n}^F + \mathbf{n}^S})(u)\end{array}.$$

The following theorem is a result concerning to this Gauss maps.

Theorem 3.1 ([6, Theorem 4.5]) *For a spacelike embedding $X : U \to \mathbb{L}^{n+1}$ (where $U \subset R^{n-1}$), the following conditions are equivalent:*

1. *M is totally lightlike flat.*
2. *The lightcone Gauss map $\widetilde{L}$ is a constant map.*
3. *There exists a lightlike vector v and a real number c such that $M \subset HP(v, c)$.*

4. The $\mathbf{n}_r^{\pm}$-Gauss maps

Let M be a spacelike surface of codimension two, defined by the immersion

$$X : U \subset \mathbb{R}^{n-1} \to \mathbb{L}^{n+1}.$$

Denote

$$HS_r = H_+^n(\nu, 1) \cap \{x_0 = r\},\ r > 0,$$

where $\nu = (-1, 0, \ldots, 0)$ and

$$H_+^n(\nu, 1) = \{x \in \mathbb{L}^{n+1} \mid \langle x - \nu, x - \nu \rangle = -1,\ x_0 \geq 0\}.$$

We can see that $HS_r \cap N_pM = \{\mathbf{n}_r^{\pm}(p)\}$. The vectors $\mathbf{n}_r^{\pm}(p)$ are chosen so that

$$\det(X_{u_1}, X_{u_2}, \ldots, X_{u_{n-1}}, \mathbf{n}_r^-(p), \mathbf{n}_r^+(p)) > 0.$$

Definition 4.1 The maps

$$\begin{array}{rcl}\mathbf{n}_r^{\pm} : M & \to & HS_r \\ p & \mapsto & \mathbf{n}_r^{\pm}(p)\end{array}$$

are called the $\mathbf{n}_r^{\pm}$- Gauss maps of M.

Remark The $\mathbf{n}_r^{\pm}$-Gauss maps of the spacelike surface $M = X(U)$ are the solutions of the following system of equations

$$\begin{cases} \langle n, X_{u_i}\rangle = 0,\ i = 1, 2, \ldots, n-1, \\ \langle n-\nu, n-\nu\rangle = -1, \\ n_0 = r. \end{cases} \tag{1}$$

Proposition 4.2 *The $\mathbf{n}_r^{\pm}$-Gauss maps are differentiable.*

Proof Denote

$$X_{u_i} = (a_{i0}, a_{i1}, \ldots, a_{in}),$$

where $a_{ij},\ i = 1, 2, \ldots, n-1, j = 0, 1, 2, \ldots, n$ are differentiable functions and $\mathbf{n}_r^{\pm} = (r, n_{r,1}^{\pm}, \ldots, n_{r,n}^{\pm})$, then (1) can be written as follow:

$$\begin{cases} -a_{10}r + a_{11}n_{r,1}^{\pm} + \cdots + a_{1n}n_{r,n}^{\pm} &= 0 \\ -a_{20}r + a_{21}n_{r,1}^{\pm} + \cdots + a_{2n}n_{r,n}^{\pm} &= 0 \\ \qquad\vdots & \\ -a_{(n-1)0}r + a_{(n-1)1}n_{r,1}^{\pm} + \cdots + a_{(n-1)n}n_{r,n}^{\pm} &= 0 \\ (n_{r,1}^{\pm})^2 + (n_{r,2}^{\pm})^2 + \cdots + (n_{r,n}^{\pm})^2 &= r^2 + 2r \end{cases} \tag{2}$$

Consider the first $n-1$ equations. This is a system of $n-1$ equations in n unknowns $n_{r,1}, n_{r,2}, \ldots, n_{r,n}$. Since X is an immersion, the rank of the system is $n-1$. Thus, there are $n-1$ unknowns, says $n_{r,1}, n_{r,2}, \ldots n_{r,n-1}$ can be expressed in term of $n_{r,n}$.

Substituting these results into the last equation of (2), we get a quadratic equation in the unknown $n_{r,n}^{\pm}$. This equation has exactly two roots and of course they are differentiable. Thus $\mathbf{n}_r^{\pm}$ are differentiable. □

The derivatives of $\mathbf{n}_r^{\pm}$ at p

$$d\mathbf{n}_r^{\pm}(p)\ :\ T_pM \to T_{\mathbf{n}_r^{\pm}(p)}H_+^n(\nu, 1) = T_pM \oplus N_pM;$$

can be writen as

$$d\mathbf{n}_r^{\pm}(p) = d\mathbf{n}_r^{\pm T}(p) + d\mathbf{n}_r^{\pm N}(p),$$

where $d\mathbf{n}_r^{\pm T}$ and $d\mathbf{n}_r^{\pm N}$ are the tangent and normal components of $d\mathbf{n}_r^{\pm}$, respectively.

We have some definitions.

Definition 4.3

(1) $A_p^{\mathbf{n}_r^{\pm}} := -d\mathbf{n}_r^{\pm T}(p)$ are called the $\mathbf{n}_r^{\pm}$-*Weingarten maps* of M at p.

(2) $K_p^{\mathbf{n}_r^\pm} := \det(A_p^{\mathbf{n}_r^\pm})$ are called the $\mathbf{n}_r^\pm$-Gauss-Kronecker curvatures of M at p.

(3) $H_p^{\mathbf{n}_r^\pm} := \frac{1}{n-1}\mathrm{tr}(A_p^{\mathbf{n}_r^\pm})$ are called the $\mathbf{n}_r^\pm$-mean curvatures of M at p.

(4) The eigenvalues $k_1^{\mathbf{n}_r^\pm}(p), k_2^{\mathbf{n}_r^\pm}(p), \ldots, k_{n-1}^{\mathbf{n}_r^\pm}(p)$ of $A_p^{\mathbf{n}_r^\pm}$ are called the $\mathbf{n}_r^\pm$-principal curvatures of M at p.

(5) $b_{ij}^{\mathbf{n}_r^\pm}(p) := \langle \frac{\partial^2 X}{\partial u_i \partial u_j}(p), \mathbf{n}_r^\pm(p)\rangle,\ i,j = 1,2,\ldots,n-1$ are called the coefficients of the $\mathbf{n}_r^\pm$-second fundamental forms of M at p.

Remark: By the definition

$$K_p^{\mathbf{n}_r^\pm} = k_1^{\mathbf{n}_r^\pm}(p) k_2^{\mathbf{n}_r^\pm}(p) \ldots k_{n-1}^{\mathbf{n}_r^\pm}(p),$$

and

$$H_p^{\mathbf{n}_r^\pm} = \frac{1}{n-1}(k_1^{\mathbf{n}_r^\pm}(p) + k_2^{\mathbf{n}_r^\pm}(p) + \cdots + k_{n-1}^{\mathbf{n}_r^\pm}(p)).$$

Theorem 4.4

(1) The $\mathbf{n}_r^\pm$-Weingarten maps are self-adjoint.

(2) The $\mathbf{n}_r^\pm$-principal curvatures $k_i^{\mathbf{n}_r^\pm}(p), i = 1,2,\ldots,n-1$ of M at p are the solutions of the following equation

$$\det(b_{ij}^{\mathbf{n}_r^\pm}(p) - k g_{ij}(p)) = 0. \tag{3}$$

(3) $K_p^{\mathbf{n}_r^\pm} = \frac{\det(b_{ij}^{\mathbf{n}_r^\pm}(p))}{\det(g_{ij}(p))}$.

Proof Let $p = X(u_1, u_2, \ldots, u_{n-1}) \in M$, we write $\mathbf{n}_r^\pm(u_1, u_2, \ldots, u_{n-1})$ instead of $\mathbf{n}_r^\pm(X(u_1, u_2, \ldots, u_{n-1}))$. Then, we have $d\mathbf{n}_r^\pm(X_{u_i}) = (\mathbf{n}_r^\pm)_{u_i},\ i = 1,2,\ldots,n-1$.

Since $\langle \mathbf{n}_r^\pm, X_{u_i}\rangle = \langle \mathbf{n}_r^\pm, X_{u_j}\rangle = 0$, we have

$$\langle (\mathbf{n}_r^\pm)_{u_j}, X_{u_i}\rangle = -\langle \mathbf{n}_r^\pm, X_{u_i u_j}\rangle = \langle (\mathbf{n}_r^\pm)_{u_i}, X_{u_j}\rangle.$$

But

$$\langle (\mathbf{n}_r^\pm)_{u_i}, X_{u_j}\rangle = \langle d\mathbf{n}_r^\pm(X_{u_i}), X_{u_j}\rangle = \langle d{\mathbf{n}_r^\pm}^T(X_{u_i}), X_{u_j}\rangle = -\langle A_p^{\mathbf{n}_r^\pm}(X_{u_i}), X_{u_j}\rangle,$$

and

$$\langle (\mathbf{n}_r^\pm)_{u_j}, X_{u_i}\rangle = \langle d\mathbf{n}_r^\pm(X_{u_j}), X_{u_i}\rangle = \langle d{\mathbf{n}_r^\pm}^T(X_{u_j}), X_{u_i}\rangle = -\langle A_p^{\mathbf{n}_r^\pm}(X_{u_j}), X_{u_i}\rangle.$$

These give the proof of (1) because $\{X_{u_1}, X_{u_2}, \dots, X_{u_{n-1}}\}$ is a basis of T_pM. Now suppose that

$$A_p^{\mathbf{n}_r^\pm}(X_{u_i}) = \sum_{k=1}^{n-1} a_{ki} X_{u_k}, \ k = 1, 2, \dots, n-1. \tag{4}$$

Then

$$\begin{aligned} b_{ij}^{\mathbf{n}_r^\pm} &= \langle X_{u_i u_j}, \mathbf{n}_r^\pm \rangle = -\langle d\mathbf{n}_r^\pm(X_{u_i}), X_{u_j} \rangle \\ &= \langle A_p^\pm(X_{u_i}), X_{u_j} \rangle = \sum_{k=1}^{n-1} a_{ki} \langle X_{u_k}, X_{u_j} \rangle \\ &= \sum_{k=1}^{n-1} a_{ki} g_{jk}; \end{aligned}$$

where $i = 1, 2, \dots, n-1$; $j = 1, 2, \dots, n-1$.

Thus, $(b_{ij}^{\mathbf{n}_r^\pm}) = (g_{ij})(a_{ij})$, and hence

$$\begin{aligned} \det[(a_{ij}) - kI] &= \det[(g_{ij})^{-1}(b_{ij}^{\mathbf{n}_r^\pm}) - kI] \\ &= \det(g_{ij}) \det[(b_{ij}^{\mathbf{n}_r^\pm}) - k(g_{ij})]. \end{aligned}$$

This proves (2).

The proof of (3) follows from the definition. □

For some applications of this Gauss map, we refer the reader to [2].

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