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# A NOTE ON COVER-AVOIDING PROPERTIES OF FINITE GROUPS

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**Abstract**

A subgroup $H$ of a group $G$ is said to be a $CAP^*$-subgroup of a group $G$ if, for any non-Frattini chief factor $K/L$ of $G$, we have $HK = HL$ or $H \cap K = H \cap L$. In this paper, some new characterizations for finite groups are obtained based on the assumption that some subgroups are $CAP^*$-subgroups of $G$.

## 1. Introduction

All groups considered in this paper are finite. A subgroup $H$ of a group $G$ is said to have the cover-avoiding property in $G$ if $H$ covers or avoids every chief factor of $G$, in short, $H$ is a $CAP$-subgroup of $G$. There has been much interest in the past in investigating the structure of finite groups when some subgroups have the cover-avoiding property, and many interesting results have been made, for example[1, 2, 3, 4, 5, 7, 8, 9, 10, 13, 15, 16, 17].

In [14], Li and Liu introduced the $CAP^*$-subgroup.

**Definition 1.1** *A subgroup $H$ of a group $G$ is said to be a $CAP^*$-subgroup of $G$ if, for any non-Frattini chief factor $K/L$ of $G$, we have $HK = HL$ or*

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$H \cap K = H \cap L$.

The authors had set up some meaningful results under the assumption of some subgroups are $CAP^*$-subgroup. In this paper, some new characterizations are obtained based on the assumption that some subgroups are $CAP^*$-subgroups of $G$.

Recall that a class of groups $\mathcal{F}$ is a formation if $\mathcal{F}$ contains all homomorphic images of group in $\mathcal{F}$, and if $G/M$ and $G/N$ are in $\mathcal{F}$, then $G/(M \cap N)$ is in $\mathcal{F}$ for normal subgroups $M$, $N$ of $G$. A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$ [11, VI, Satz 7.1 and 7.2].

## 2. Basic definitions and preliminary results

Let $K$ and $L$ be normal subgroups of a group $G$ with $K \leq L$. Then $K/L$ is called a normal factor of $G$. A subgroup $H$ of $G$ is said to cover $K/L$ if $HK = HL$. On the other hand, if $H \cap K = H \cap L$, then $H$ is said to avoid $K/L$. If $K/L$ is a chief factor of $G$ and $K/L \leq \Phi(G/L)$ (respectively $K/L \nleq \Phi(G/L)$), then $K/L$ is said to be a Frattini (respectively non-Frattini) chief factor of $G$.

**Lemma 2.1** [14, Lemma 2.1] *Let $N$ be a normal subgroup of a group $G$. If $H$ is a $CAP^*$-subgroup of $G$, then:*

*(1) $HN/N$ is a $CAP^*$-subgroup of $G/N$.*

*(2) $H \cap N$ is a $CAP^*$-subgroup of $G$.*

*(3) If $N \leq \Phi(G)$ or $gcd(|H|, |N|) = 1$, then $HN$ is a $CAP^*$-subgroup of $G$, where $gcd(-,-)$ denotes the greatest common divisor.*

The generalized Fitting subgroup $F^*(G)$ of $G$ is the unique maximal normal quasinilpotent subgroup of $G$.

**Lemma 2.2** *Let $G$ be a group and let $M$ be a subgroup of $G$.*

*(1) If $M$ is normal in $G$, then $F^*(M) \leq F^*(G)$.*

*(2) $F^*(G) \neq 1$ if $G \neq 1$, in fact, $F^*(G)/F(G) = Soc(F(G)C_G(F(G))/F(G))$.*

*(3) $F^*(F^*(G)) = F^*(G) \geq F(G)$; If $F^*(G)$ is solvable, then $F^*(G) = F(G)$.*

*(4) $C_G(F^*(G)) \leq F(G)$.*

*(5) Let $N = Z(E(G))\Phi(F(G))$. Then $F^*(G/N) = F^*(G)/N$, where $E(G)$ is the layer of $G$.*

*(6) $E(G)/Z(E(G))$ is the direct product of non-abelian simple groups.*

**Proof.** By [12, X.13], (1)-(4) and (6) follow. By [6, Proposition 4.10], (5) is obtained. □

**Lemma 2.3** [18, Chapter1, Theorem 7.15] *Let $H$ be a normal subgroup of $G$. If every chief factor of $G$ contained in $H$ is cyclic, then $G/C_G(H)$ is supersolvable.*

**Lemma 2.4** [8, Lemma 3.12] *Let $p$ be the smallest prime dividing the order of a group $G$ and let $P$ be a Sylow $p$-subgroup of $G$. If $|P| \leq p^2$ and $G$ is $A_4$-free, then $G$ is $p$-nilpotent.*

**Lemma 2.5** [8, Lemma 3.16] *Let $\mathcal{F}$ be the class of groups with Sylow tower of supersolvable type. Also let $H$ a normal subgroup of a group $G$ such that $G/H \in \mathcal{F}$. If $G$ is $A_4$-free and all 2-maximal subgroups of every Sylow subgroup of $H$ are $CAP$-subgroups of $G$, then $G$ is in $\mathcal{F}$.*

## 3. Results

**Theorem 3.1** *Let $H$ be a normal subgroup of a group $G$ such that $G/H$ is supersolvable. Suppose that every maximal subgroup of any Sylow subgroups of $F^*(H)$ is a $CAP^*$-subgroup of $G$, then $G$ is supersolvable.*

**Proof.** Suppose that the theorem is false and let $G$ be a counterexample with smallest order. Then:

(1) $Z(E(H)) = 1$, in particular, $E(H)$ is the direct product of non-abelian simple groups.

Otherwise, $Z(E(H)) \neq 1$. Let $N = \Phi(F(H))Z(E(H))$. It is clear that $G/N\big/H/N \cong G/H$ is supersolvable. Let $M/N$ be a maximal subgroup of a Sylow subgroup $PN/N$ of $F^*(H)/N$, where $P$ is a Sylow subgroup of $F^*(H)$. We can see that $M \cap P$ is a maximal subgroup of $P$. By hypothesis, $M \cap P$ is a $CAP^*$-subgroup of $G$. Applying Lemma 2.1, $M/N$ is a $CAP^*$-subgroup of $F^*(H)/N$. Furthermore, $F^*(H)/N = F^*(H/N)$ by Lemma 2.2. It follows that $G/N$ satisfies the hypothesis of our theorem for normal subgroup $H/N$. Thus, by the minimality of $G$, $G/N$ is supersolvable and therefore $G$ is solvable. This implies that $F^*(H) = F(H)$. We can finish the argument by following:

(a) All minimal normal subgroups of $G$ contained in $F^*(H)$ are cyclic of prime order and non-Frattini.

Let $N$ be a minimal normal subgroup of $G$ contained in $F(H)$. Then $N$ is a $p$-group for some prime $p$. If $N \leq \Phi(G)$, then $F(H/N) = F(H)/N$ by [Huppert, III, satz 4.2]. We can see that $G/N$ satisfies the hypothesis of our theorem. By the minimal choice of $G$, $G/N$ is supersolvable and therefore $G$ is supersolvable, a contradiction. Hence we may assume that $N/1$ is a non-Frattini chief factor of $G$. There exists a maximal subgroup $P_1$ of a Sylow $p$-subgroup $P$ of $F(H)$ such that $P_1 \cap N = 1$, this implies that $|N| = p$, as desired.

(b) A contradiction.

Let $P$ be a Sylow $p$-subgroup of $F(H)$ and let $K/L$ be a chief factor of $G$ contained in $P$. We can choose a maximal subgroup $P_1$ of $P$ such that $L \leq P_1$ and $K \nleq P_1$. If $P_1$ covers $K/L$, then $P_1K = P_1$ and so $K \leq P_1$, a contradiction. It follows from $P_1$ avoids $K/L$ that $P_1 \cap K = L$. By comparing the order, we can see that $|K/L| = p$. Hence every chief factor of $G$ under $F(H)$ is cyclic of prime order. On the one hand, by Lemma 2.3, $G/C_G(F(H))$ is supersolvable and therefore $G/H \cap C_G(F(H)) = G/C_H(F(H))$ is supersolvable. On the other hand, $C_H(F(H)) \leq F(H)$, it is clear that $G/F(H)$ is supersolvable. Therefore $G$ is supersolvable, another contradiction. Hence $E(H) = 1$ and $E(H)$ is the direct product of non-abelian simple groups by Lemma 2.2.

(2) $F^*(H) = F(H)$.

Suppose that $E(H) \neq 1$. Let $N$ be a minimal normal subgroup of $G$ contained in $E(H)$, then $N$ is a product of some non-abelian simple groups. It is clear that $N \nleq \Phi(G)$. If every maximal subgroup of Sylow subgroup $P$ of $F^*(H)$ covers $N/1$, then $N \leq \Phi(P)$ and so $N \leq \Phi(G)$, a contradiction. Thus, there exists a maximal subgroup $P_1$ of $P$ such that $P_1 \cap N = 1$ for every Sylow subgroup $P$ of $F^*(H)$. This implies that $N$ is the subgroup with square-free order and therefore $N$ is solvable, a contradiction.

By (1) and (2), we can finish our proof. □

**Corallary 3.2** *Let $H$ be a solvable normal subgroup of a group $G$ such that $G/H$ is supersolvable. Suppose that every maximal subgroup of any Sylow subgroups of $F^*(H)$ is a $CAP^*$-subgroup of $G$, then $G$ is supersolvable.*

**Remark 3.3** The condition "$H$ is solvable" in Corollary 3.2 can not be removed. For example, let $G = H = GL(2,4)$. Then $F(H) \cong Z_3$, where $Z_3$ is a cyclic group of order 3. It is clear that $G$ satisfies the hypothesis of the Corollary 3.2 for normal subgroup $H$, but $G$ is not supersolvable.

If $M$ is a maximal subgroup of $G$ and $H$ is a maximal subgroup of $M$, then we call $H$ a 2-maximal subgroup of $G$. We say the group $G$ is $A_4$-free if there is no subgroup in $G$ for which $A$ is an isomorphic image. We prove the following results.

**Theorem 3.4** *Let $H$ be a normal subgroup of a group $G$ and let $p$ be the smallest prime dividing the order of $H$. If every 2-maximal subgroup of every Sylow $p$-subgroup of $H$ is a $CAP^*$-subgroup of $G$ and $G$ is $A_4$-free, then $H$ is $p$-nilpotent.*

**Proof.** Suppose that the theorem is false and let $G$ be a counterexample with smallest order. Then:

(1) $O_{p'}(H) = 1$.

Otherwise, $O_{p'}(H) \neq 1$. We can see that $G/O_{p'}(H)$ satisfies the theorem for normal subgroup $H/O_{p'}(H)$. By the choice of $G$, $H/O_{p'}(H)$ is $p$-nilpotent and therefore $H$ is $p$-nilpotent, as desired.

(2) Let $N$ be a minimal normal subgroup of $G$, then $N \nleq \Phi(G)$.

It is clear that $G/N$ satisfies the hypothesis of the theorem for normal subgroup $HN/N$. By the minimality of $G$, $HN/N$ is $p$-nilpotent. If $N \nleq H$, then $H \cap N = 1$ and so $H \cong HN/N$ is $p$-nilpotent, as desired. Hence we can see that $N \leq H$ and so $H/N$ is $p$-nilpotent. Since the $p$-nilpotent group classes is saturate, $N \nleq \Phi(G)$. By (1), $N/1$ is a $p$-chief factor.

(3) Final contradiction.

Let $S \in Syl_p(N)$. If $|S| \leq p^2$, then $N$ is $p$-nilpotent by Lemma 2.4, in contradiction to the fact that $N$ is a minimal normal subgroup of $G$. Hence we may assume that $|S| \geq p^3$. If all 2-maximal subgroups cover $N/1$, then $N \leq \Phi(M)$ and so $N \leq \Phi(G)$, where $M$ is a maximal subgroup of $P$, it is impossible. Thus, there exists a 2-maximal subgroup $P_1$ such that $P_1 \cap N = 1$. This implies that $|P_1N|_p = |P_1||S| > |P|$, a final contradiction. □

**Remark 3.5** In Theorem 3.4, we can not remove the assumption that $G$ is $A_4$-free in general. For example, $G = A_4$. It is clear that every 2-maximal subgroup of the Sylow 2-subgroup of $A_4$ is a $CAP^*$-subgroup of $G$. But $G$ is not 2-nilpotent.

**Corollary 3.6** *Let $p$ be the smallest prime dividing the order of a group $G$. If $G$ is $A_4$-free and every 2-maximal subgroup of every Sylow $p$-subgroup of $G$ is a $CAP^*$-subgroup of $G$, then $G$ is $p$-nilpotent.*

**Corollary 3.7** *Let $H$ be a normal subgroup of a group $G$. If $G$ is $A_4$-free and every 2-maximal subgroup of every Sylow subgroup of $H$ is a $CAP^*$-subgroup of $G$, then $H$ is a Sylow tower group of supersolvable type.*

**Proof.** We use induction on $|H|$. Let $p$ be the smallest prime dividing the order of $H$. By Theorem 3.4, $H$ is $p$-nilpotent and so $H$ possesses a normal Hall $p'$-subgroup $K$. It is clear that $G$ satisfies the hypothesis of the corollary for the normal subgroup $K$, by induction, $K$ has the Sylow tower property. Consequently, $H$ has the Sylow tower property. □

**Theorem 3.8** *Let $\mathcal{F}$ be the class of groups with Sylow tower of supersolvable type and let $H$ be a normal subgroup of a group $G$ such that $G/H \in \mathcal{F}$. If $G$ is $A_4$-free and every 2-maximal subgroup of every Sylow subgroup of $H$ is a $CAP^*$-subgroup of $G$, then $G \in \mathcal{F}$.*

**Proof.** Suppose that the theorem is not true and let $G$ be a minimal counterexample. Then by corollary 3.7, we can see that $H$ has a Sylow tower of supersolvable type. Let $p$ be the largest prime in $\pi(H)$ and $P \in Syl_p(H)$. Then $P$ is a normal subgroup of $G$ and every 2-maximal subgroup of $P$ is a $CAP^*$-subgroup of $G$. It is easy to see that all 2-maximal subgroups of every Sylow subgroup of $H/P$ are $CAP^*$-subgroups of $G/P$ and $G/P$ is $A_4$-free. Thus, by the minimality of $G$, we have $G/P \in \mathcal{F}$.

Let $N$ be a minimal normal subgroup of $G$ contained in $P$, it is clear that $G/N$ satisfies the hypothesis for normal subgroup $H/N$ and $G/N \in \mathcal{F}$. If $N \leq \Phi(G)$, then $G \in \mathcal{F}$, a contradiction. It follows that $N \nleq \Phi(G)$. If every 2-maximal subgroup of $P$ cover $N/1$, then $N \leq \Phi(G)$, a contradiction. Then there exits a 2-maximal subgroup $P_1$ such that $P_1 \cap N = 1$, this implies that $|N| \leq p^2$. By Lemma 2.5, $G \in \mathcal{F}$, a final contradiction. □

**Remark 3.9** In Theorem 3.8, the group $G$ is not necessary supersolvable. For example, let $H$ be a direct product of two copies of a cyclic group of order 3. There exist elements $a$, $b$ in $H$ such that $a^3 = b^3 = [a, b] = 1$ and let $H = \langle a, b\rangle$. The group $H$ has an automorphism of order 4 such that $a^c = b^{-1}$ and $b^c = a$. If we write $K = \langle c\rangle$, let $G$ be the semidirect product $G = H \rtimes K$. Clearly, $G$ satisfies the hypothesis of the Theorem 3.8 for normal subgroup $H$, but $G$ is not supersolvable.

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