East-West J. of Mathematics: Vol. 13, No 2(2011) pp. 200-206

NOTE ON JORDAN TRIPLE $(\alpha, \beta)^*$-DERIVATIONS IN H^*-ALGEBRAS

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Abstract

The main purpose of this paper is to prove the following result. Let R be a 2-torsion free semiprime *-ring and α, β are endomorphisms of R. Then any Jordan triple $(\alpha, \beta)^*$-derivation on R is a Jordan $(\alpha, \beta)^*$-derivation. As an application of this result, we establish that any linear Jordan triple $(\alpha, \beta)^*$-derivation on a semisimple H^*-algebra is a linear Jordan $(\alpha, \beta)^*$-derivation.

1 Introduction

This research is inspired by our earlier work [1] and the work of M. Fošner and D. Ileševic [5]. Throughout, R will denote an associative ring and A will represent a $*$-algebra over the field F. Let $n \geq 2$ be an integer. A ring R is said to be n-torsion free if for $x \in R$, $nx = 0$ implies $x = 0$. Recall that R is prime if $aRb = \{0\}$ implies $a = 0$ or $b = 0$. A ring R is called semiprime if $aRa = \{0\}$ implies $a = 0$. An additive mapping $x \mapsto x^*$ satisfying $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$ is called an involution. A ring equipped with an involution is called a $*$-ring or a ring with involution. If R is an algebra we assume additionally that $(\lambda x)^* = \bar{\lambda}x^*$ for all $x, y \in R$ and $\lambda \in F$, where $\bar{\lambda}$ denotes the complex conjugate of λ. An algebra equipped with an involution is called a $*$-algebra or algebra with involution. The radical of A, denoted by $rad(A)$, is the intersection of all maximal left(or right) ideals of A. An algebra

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This research is partially supported by the research grant from UGC (Grant No. $39 - 37/2010$(SR)), India.
Key words: Semiprime $*$-ring, H*-algebra, $(\alpha, \beta)^*$-derivation, Jordan $(\alpha, \beta)^*$-derivation, Jordan triple $*$-derivation and Jordan triple $(\alpha, \beta)^*$-derivation.
2000 AMS Mathematics Subject Classification: 16N60, 16N10, 16W25, 47B47.

A is called semisimple if $rad(A) = 0$. A Banach algebra is a linear associative algebra which, as a vector space, is a Banach space with norm $\|\cdot\|$ satisfying the multiplicative inequality; $\|xy\| \leq \|x\|\|y\|$ for all x and y in A. A $*$-algebra which is also a Banach algebra is called a Banach $*$-algebra. Let us recall that a semisimple H*-algebra is a semisimple Banach $*$-algebra whose norm is a Hilbert space norm such that $(x, yz^*) = (xz, y) = (z, x^*y)$ is fulfilled for all $x, y, z \in A$ (see [2] for details).

An additive mapping $d : R \to R$ is called a derivation (resp. Jordan derivation) if $d(xy) = d(x)y + xd(y)$ (resp. $d(x^2) = d(x)x + xd(x)$) holds for all $x, y \in R$. An additive mapping $d : R \to R$ is called a Jordan triple derivation if $d(xyx) = d(x)yx + xd(y)x + xyd(x)$ holds for all $x, y \in R$. Of course, any derivation is a Jordan triple derivation. Moreover, if R is a 2-torsion free, one can easily prove that any Jordan derivation is a Jordan triple derivation, but converse is not true in general. A classical result due to Brešar ([3], Theorem 4.3) asserts that a Jordan triple derivation on 2-torsion free semiprime ring is a derivation.

Let R be a $*$-ring, and let $\alpha,\ \beta$ be endomorphisms of R. An additive mapping $d : R \to R$ is said to be a $*$-derivation (resp. Jordan $*$-derivation) if $d(xy) = d(x)y^* + xd(y)$ (resp. $d(x^2) = d(x)x^* + xd(x)$) holds for all $x, y \in R$. Following [5], an additive mapping $d : R \to R$ is called a Jordan triple $*$-derivation if $d(xyx) = d(x)y^*x^* + xd(y)x^* + xyd(x)$ holds for all $x, y \in R$. One can easily prove that every Jordan $*$-derivation on a 2-torsion free $*$-ring is a Jordan triple $*$-derivation (see the proof of [[4], Lemma 2]) but not conversely. In [7], P. Šemrl has proved that if R is a real Banach $*$-algebra with identity then the converse also holds. Further, Vukman [8] established that any Jordan triple $*$-derivation on a 6-torsion free semiprime *-ring is a Jordan $*$-derivation. In the year 2008, M. Fošner and D. Iliševic [5] generalized this result for 2-torsion free semiprime $*$-rings.

In [1], the notion of Jordan triple $*$-derivation was extended as follows: an additive mapping $d : R \to R$ is called a Jordan triple $(\alpha, \beta)^*$-derivation if $d(xyx) = d(x)\alpha(y^*x^*) + \beta(x)d(y)\alpha(x^*) + \beta(xy)d(x)$ holds for all $x, y \in R$, where α and β are endomorphisms of R. In any $*$-ring with automorphisms α and β, the mapping $x \mapsto a\alpha(x^*) - \beta(x)a$, where a is fixed element in R, is a Jordan triple $(\alpha, \beta)^*$-derivation on R. Note that for I_R, the identity map on R, a Jordan triple $(I_R, I_R)^*$-derivation is just a Jordan triple $*$-derivation. Using similar approach as in Lemma 2 of [4], it can be easily seen that any Jordan $(\alpha, \beta)^*$-derivation on a 2-torsion free $*$-ring is a Jordan triple $(\alpha, \beta)^*$-derivation, but not conversely(cf.; [[1], Example 2.4]). Most recently, the author together with A. Fošner [1] proved that on a 6-torsion free semiprime $*$-ring R, every Jordan triple $(\alpha, \beta)^*$-derivation on R is a Jordan $(\alpha, \beta)^*$-derivation.

The main goal of this paper is to improve the above mentioned result by removing 3-torsion free restriction and prove that any Jordan triple $(\alpha,\beta)^*$-derivation on 2-torsion free semiprime $*$-ring is a Jordan $(\alpha,\beta)^*$-derivation. As consequence of this result, it was shown that any linear Jordan triple $(\alpha,\beta)^*$-derivation on a semisimple H^*-algebra is a linear Jordan $(\alpha,\beta)^*$-derivation.

2 The main results

The following theorem is a generalization of [[5], Theorem 5.2] and [[8], Theorem 1].

Theorem 1. *Let R be a 2-torsion free semiprime $*$-ring, and let α,β be surjective endomorphisms of R. Let $d : R \to R$ be an additive mapping. Then the following conditions are equivalent:*

(i) d is a Jordan $(\alpha,\beta)^*$-derivation;

(ii) $d(xyx) = d(x)\alpha(y^*x^*) + \beta(x)d(y)\alpha(x^*) + \beta(xy)d(x)$ for all $x, y \in R$.

In order to prove above theorem, first we establish the following technical lemma which extended the result of [[6], Section 2, p-5].

Lemma 1. *Let R be a semiprime $*$-ring, and let α,β be surjective endomorphisms of R. If there exists element $x \in R$ such that $\beta(y)x\alpha(y^*) = 0$ for all $y \in R$ or $\alpha(y)x\beta(y^*) = 0$ for all $y \in R$, then $x = 0$.*

Proof. We consider the relation $\beta(y)x\alpha(y^*) = 0$ for all $y \in R$. Replacing y by $y^* + z$ we obtain

$$\beta(y^*)x\alpha(y)+\beta(z)x\alpha(y)+\beta(y^*)x\alpha(z^*)+\beta(z)x\alpha(z^*) = 0 \text{ for all } y, z \in R. \quad (1)$$

This implies that

$$\beta(z)x\alpha(y) + \beta(y^*)x\alpha(z^*) = 0, \quad \text{for all } y, z \in R. \quad (2)$$

Replace z^* by y in (2) to get

$$\beta(z)x\alpha(y) + \beta(y^*)x\alpha(y) = 0 \text{ for all } y, z \in R. \quad (3)$$

and hence in view of our hypothesis we obtain

$$\beta(z)x\alpha(y) = 0 \text{ for all } y, z \in R.$$

Using the last relation, we find that

$$\begin{aligned}(x\beta(z)x)\alpha(y)(x\beta(z)x) &= x(\beta(z)x\alpha(y))(x\beta(z)x)\\ &= 0 \quad \text{for all } y, z \in R. \qquad (4)\end{aligned}$$

Therefore, we find that $(x\beta(z)x)\alpha(y)(x\beta(z)x) = 0$ for all $y, z \in R$. Since α is a surjective endomorphism of R, so we have $(x\beta(z)x)R(x\beta(z)x) = \{0\}$ for all $z \in R$. The semiprimeness of R forces that $x\beta(z)x = 0$ for all $z \in R$. Since R is semiprime and β is surjective endomorphism of R, we conclude that $x = 0$.

Using similar arguments one can prove that if $\alpha(y)x\beta(y^*) = 0$ for all $y \in R$, then $x = 0$. The proof is complete. □

We are now ready to complete the proof of Theorem 1.

Proof of Theorem 1. We proceed to prove (i) implies (ii). Suppose that d is a Jordan $(\alpha, \beta)^*$-derivation i.e.,

$$d(x^2) = d(x)\alpha(x^*) + \beta(x)d(x)) \text{ for all } x \in R. \qquad (5)$$

The linearization of the above relation yields that

$$\begin{aligned}d(xy + yx) \ &= \ d(x)\alpha(y^*) + \beta(x)d(y) + d(y)\alpha(x^*)\\ &\quad + \beta(y)d(x) \text{ for all } x, y \in R. \qquad (6)\end{aligned}$$

Replacing y by $xy + yx$ in (6), then on one hand we find that

$$\begin{aligned}d(x(xy + yx) + (xy + yx)x) &= d(x)\alpha(x^*y^*) + d(x)\alpha(y^*x^*) + \beta(x)d(x)\alpha(y^*))\\ &\quad + \beta(x^2)d(y) + \beta(x)d(y)\alpha(x^*) + \beta(xy)d(x)\\ &\quad + d(x)\alpha(y^*x^*) + \beta(x)d(y)\alpha(x^*) + d(y)\alpha(x^{*^2})\\ +\beta(y)d(x)\alpha(x^*) &\quad + \beta(xy)d(x) + \beta(yx)d(x) \text{ for all } x, y \in R. \quad (7)\end{aligned}$$

On the other hand, we have

$$\begin{aligned}d(x(xy + yx) + (xy + yx)x) &= d(x^2y + yx^2) + 2d(xyx)\\ &= d(x)\alpha(x^*y^*) + \beta(x)d(x)\alpha(y^*) + \beta(x^2)d(y)\\ &\quad + d(y)\alpha((x^{*^2}) + \beta(y)d(x)\alpha(x^*) + \beta(yx)d(x)\\ &\quad + 2d(xyx) \quad \text{for all } x, y \in R. \qquad (8)\end{aligned}$$

Combining (7) and (8), we obtain

$$2d(xyx) = 2\{d(x)\alpha(y^*x^*) + \beta(x)d(y)\alpha(x^*) + \beta(xy)d(x)\} \text{ for all } x, y \in R.$$

Since R is 2-torsion free, the last expression forces that

$$d(xyx) = \quad d(x)\alpha(y^*x^*) + \beta(x)d(y)\alpha(x^*) + \beta(xy)d(x) \quad \text{for all } x, y \in R; \quad (9)$$

and hence d is Jordan triple $(\alpha,\beta)^*$-derivation on R.

Let us prove the converse part *i.e.*, (ii) implies (i). Suppose relation (9) holds. Replace x by $x+z$ in (9) to get

$$\begin{aligned} d((x+z)y(x+z)) &= d(x+z)\alpha(y^*)\alpha(x^*+z^*)+\beta(x+z)d(y)\alpha(x^*+z^*)\\ &\quad+\beta(x+z)\beta(y)d(x+z)\\ &= d(x)\alpha(y^*x^*)+d(z)\alpha(y^*x^*)+d(x)\alpha(y^*z^*)+d(z)\alpha(y^*z^*)\\ &\quad+\beta(x)d(y)\alpha(x^*)+\beta(z)d(y)\alpha(x^*)+\beta(x)d(y)\alpha(z^*)\\ &\quad+\beta(z)d(y)\alpha(z^*)+\beta(xy)d(x)+\beta(zy)d(x)+\beta(xy)d(z)\\ &\quad+\beta(zy)d(z)\quad\text{for all } x,y,z\in R. \end{aligned} \tag{10}$$

On the other hand, we have

$$\begin{aligned} d((x+z)y(x+z)) &= d(xyx)+d(zyz)+d(xyz+zyx)\\ &= d(x)\alpha(y^*x^*)+\beta(x)d(y)\alpha(x^*)+\beta(xy)d(x)\\ &\quad+d(z)\alpha(y^*z^*)+\beta(z)d(y)\alpha(z^*)+\beta(zy)d(z)\\ &\quad+d(xyz+zyx)\quad\text{for all } x,y,z\in R. \end{aligned} \tag{11}$$

Comparing (10) and (11), we arrive at

$$\begin{aligned} d(xyz+zyx) &= d(x)\alpha(y^*z^*)+\beta(x)d(y)\alpha(z^*)+\beta(xy)d(z)\\ &\quad+d(z)\alpha(y^*x^*)+\beta(z)d(y)\alpha(x^*)\\ &\quad+\beta(zy)d(x)\ \text{for all } x,y,z\in R. \end{aligned} \tag{12}$$

Since d is additive, so for any $x,y\in R$, we have

$$d((xy)^2)=d(xyxy)=d(xy(xy)+(xy)yx-xy^2x)=d(xy(xy)+(xy)yx)-d(xy^2x).$$

Application of (9) and (12) yields that

$$\begin{aligned} d((xy)^2) &= d(x)\alpha(y^*)\alpha(y^*x^*)+\beta(x)d(y)\alpha(y^*x^*)+\beta(xy)d(xy)\\ &\quad+d(xy)\alpha(y^*x^*)+\beta(xy)d(y)\alpha(x^*)+\beta(xy)\beta(y)d(x)\\ &\quad-d(x)\alpha({y^*}^2)\alpha(x^*)-\beta(x)d(y^2)\alpha(x^*)\\ &\quad-\beta(xy^2)d(x)\quad\text{for all } x,y\in R. \end{aligned} \tag{13}$$

For any $x,y\in R$, the above relation implies that

$$\begin{aligned} &d(xy)^2-d(xy)\alpha(y^*x^*)-\beta(xy)d(xy)\\ &\quad+\beta(x)(d(y^2)-d(y)\alpha(y^*)-\beta(y)d(y))\alpha(x^*)=0. \end{aligned} \tag{14}$$

This can be rewritten as

$$\delta(xy) + \beta(x)\delta(y)\alpha(x^*) \;=\; 0 \;\text{ for all } x, y \in R; \tag{15}$$

where $\delta(x) = d(x^2) - d(x)\alpha(x^*) - \beta(x)d(x)$ for all $x \in R$. Using equation (15) three times, we find that

$$\begin{aligned} 2\beta(zy)\delta(x)\alpha(y^*z^*) &= \beta(z)(\beta(y)\delta(x)\alpha(y^*))\alpha(z^*) + \beta(zy)\delta(x)\alpha(y^*z^*) \\ &= \beta(z)(-\delta(yx)\alpha(z^*)) - \delta((zy)x) \\ &= -\beta(z)\delta(yx)\alpha(z^*) - \delta(zyx) \\ &= -\beta(z)\delta(yx)\alpha(z^*) - \delta(zyx) \\ &= \delta(z(yx)) - \delta(zyx) \\ &= 0 \;\text{ for all } x, y, z \in R. \end{aligned} \tag{16}$$

This implies that

$$2\beta(zy)\delta(x)\alpha(y^*z^*) \;=\; 0 \;\text{ for all } x, y, z \in R. \tag{17}$$

Since R is 2-torsion free, the above expression forces that $\beta(zy)\delta(x)\alpha(y^*z^*) = 0$ for all $x, y, z \in R$ i.e., $\beta(z)(\beta(y)\delta(x)\alpha(y^*))\alpha(z^*) = 0$ for all $x, y, z \in R$. Application of Lemma 1 yields that $\beta(y)\delta(x)\alpha(y^*) = 0$ for all $x, y \in R$. Again, using Lemma 1, we obtain $\delta(x) = 0$ *i.e.,* $d(x^2) = d(x)\alpha(x^*) + \beta(x)d(x)$ for all $x \in R$. Hence, d is a Jordan $(\alpha, \beta)^*$-derivation on R. This completes the proof of our theorem.

Following are the immediate consequences of Theorem 1.

Corollary 1. *Let R be a 2-torsion free semisimple $*$-ring, and let α, β be surjective endomorphism of R. Let $d : R \to R$ be an additive mapping. Then the following conditions are equivalent:*

(i) d is a Jordan $(\alpha, \beta)^*$-derivation;

(ii) $d(xyx) = d(x)\alpha(y^*x^*) + \alpha(x)d(y)\alpha(x^*) + \beta(xy)d(x)$ for all $x, y \in R$.

Proof. As a consequence of Theorem 1 and of the fact that every simple $*$-ring is a semiprime $*$-ring.

Corollary 2. *([5], Theorem 5.2) Let R be a 2-torsion free semiprime $*$-ring, and let $d : R \to R$ be an additive mapping. Then the following condition, are mutually equivalent:*

(i) d is a Jordan $*$-derivation;

(ii) $d(xyx) = d(x)y^*x^* + xd(y)x^* + xyd(x)$ *for all* $x, y \in R$.

Corollary 3. *([8], Theorem 1). Let* R *be a* $6-$*torsion free semiprime* $^*-$*ring and let* $d : R \rightarrow R$ *be an additive mapping satisfying the relation*

$$d(xyx) = d(x)y^*x^* + xd(y)x^* + xyd(x)$$

for all $x, y \in R$. *Then* d *is a Jordan* $^*-$*derivation on* R.

Finally, we prove another theorem in the spirit of Theorem 1, that is,

Theorem 2. *Let* A *be a semisimple* H^**-algebra. Suppose that* α *and* β *are surjective homomorphisms of* A. *Let* $d : A \rightarrow A$ *be a linear mapping. Then the following conditions are equivalent:*

(i) d *is a Jordan* $(\alpha, \beta)^*$*-derivation;*

(ii) $d(xyx) = d(x)\alpha(y^*x^*) + \beta(x)d(y)\alpha(x^*) + \beta(xy)d(x)$ *for all* $x, y \in A$.

Proof. By the structure theorem of semisimple H*-algebra (see [2]), every semisimple H*-algebra is a semiprime $*$-ring and hence proof is complete by Theorem 1.

Corollary 4. *Let* A *be a semisimple* H^**-algebra. Let* $d : A \rightarrow A$ *be a linear mapping. Then the following conditions are equivalent:*

(i) d *is a Jordan* $*$*-derivation;*

(ii) $d(xyx) = d(x)y^*x^* + xd(y)x^* + xyd(x)$ *for all* $x, y \in A$.

Acknowledgment. The author is greatly indebted to the referee for his valuable suggestions which improved the paper immensely.

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